# Two-Parameter Analysis of Rician Data: Basics of a Theory and Computer Simulation by Wolfram Mathematica

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The Rice statistical distribution characterizes a value of an amplitude, or envelope of the complex signal composed as a sum of any determined initial signal and a Gaussian noise.

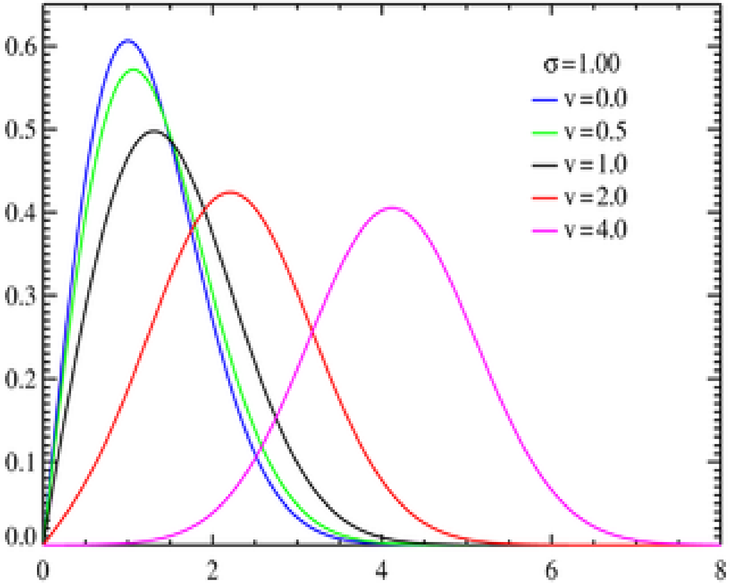
The fields of application:

- *magnetic-resonance visualization,*
- *radio signals reception and processing,*
- *radar signals analysis,*
- *analysis of the sonar signals,*
- *optical medium's properties measurements, etc.*

# Main points of the Speech:

- The applicability of the Rice statistical model at solving many typical tasks of data processing;
- Brief review of the problem development in the science history;
- The Rice statistical distribution’s basic features;
- Interconnection between distributions of Rice, Gauss and Rayleigh;
- Solving of two-parameter task by the maximum likelihood method and variants of method of moments .

# The Rice distribution properties

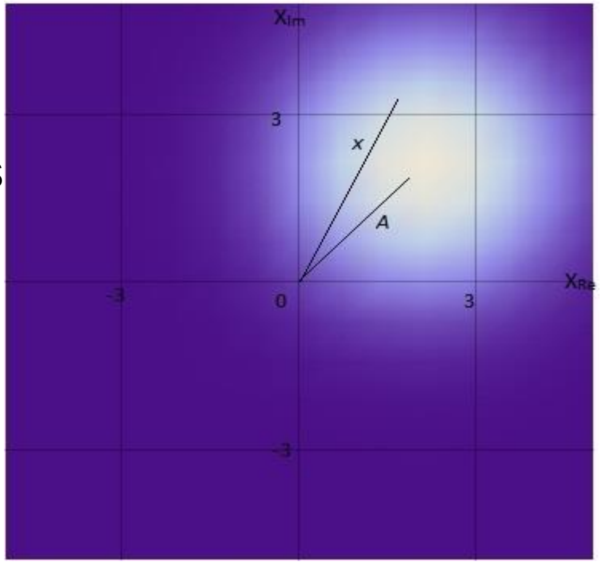



$\vec{X}(x,\varphi)=\vec{A}+\vec{r}$ $\quad \vec{r}=\vec{r}(r_{\mathrm{Re}},r_{\mathrm{Im}})$ - Gaussian noise

$x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2}$ $\quad \overline{r_{\mathrm{Re}}{}^2}=\overline{r_{\mathrm{Im}}{}^2}=\sigma^2 \quad \overline{r}_{\mathrm{Re}}=\overline{r}_{\mathrm{Im}}=0$

$A=\nu$

$\nu,\sigma^2$ – Rician parameters




$$P\left(x\middle|\nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$

$$\overline{x}=\sigma\cdot\sqrt{\pi/2}\cdot L_{1/2}\left(-\nu^2/2\sigma^2\right)$$

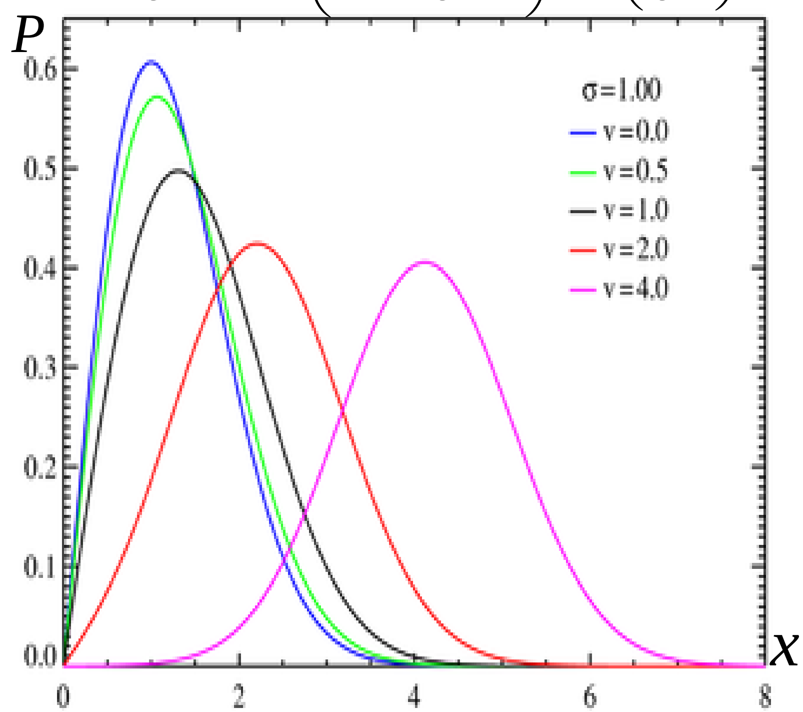
$$\sigma_x^2=\overline{x^2}-\overline{x}^2=2\sigma^2+\nu^2-\frac{\pi\sigma^2}{2}\cdot L_{1/2}^2\left(-\nu^2/2\sigma^2\right) \quad L_{1/2}(z)=e^{z/2}\left[\left(1-z\right)I_0\left(\frac{-z}{2}\right)-zI_1\left(\frac{-z}{2}\right)\right]$$

# FORMULATION OF THE PROBLEM OF RANDOM RICIAN VALUE TWO-PARAMETER ANALYSIS

Initially determined complex value with amplitude $\nu$ is distorted by a Gaussian noise, $\Longrightarrow$ Amplitude $x=\sqrt{x_{\mathrm{Re}}{}^2+x_{\mathrm{Im}}{}^2}$ of the resulting signal obeys the Rice statistical distribution with parameters $\nu$ and $\sigma^2$ .

$x_{\mathrm{Re}}$ , $x_{\mathrm{Im}}$ – independent Gaussian values with dispersion $\sigma^2$

$$P\left(x\middle|\nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$




***The task*** consists in ***<u>reconstruction of useful signal</u>*** $\nu$ *against the noise background* ***<u>by joint computing the both Rician parameters</u>*** *on the basis of measurements of the summary signal* $x_i$ ( $i=1,\ldots,n$ ), without any a priory assumptions concerning value $\sigma^2$ .

# *Principle stages of the development of Rician data analysis theory*

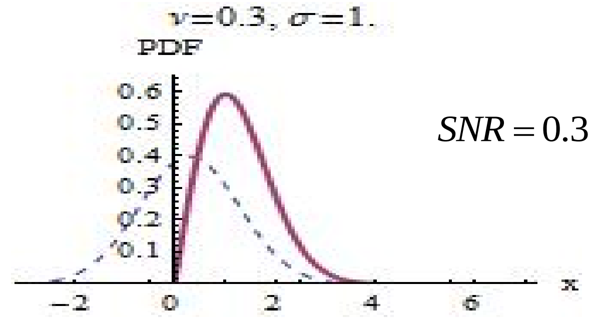
| | |
|---|---|
| **1880** | Rayleigh distribution, introduced by **Lord J. Rayleigh, Strutt** as a distribution of value $Z=\sqrt{X^2+Y^2}$ where $X$ and $Y$ are Gaussian values with zero mean |
| **1944** | Generalization of the Rayleigh distribution onto the case of nonzero mean of values $X$ and $Y$ (**Stephan O. Rice**) |
| **1967** | Estimations of the Rician parameters' standard deviations on the basis of Cramer-Rao inequality (**T.R. Benedict, T. T. Soong**) |
| **1991/ 1998** | Elaboration of one-parameter approximation for the Rician signals' analysis by the method of moments (**K.K. Talukdar, W.D. Lawing**)/by the maximum likelihood technique (**J. Sijbers, et al**) |
| **2008** | A first theoretical consideration of the task of joint signal and noise estimation (**K.Carobbi, M. Cati**) |
| **since 2013** | Development of a strict theory of the two-parameter Rician signal's analysis as a tool for efficient data processing (**T. Yakovleva, N. Kulberg**) |

# Comparison of the Rice, Gauss and Rayleigh statistical distributions

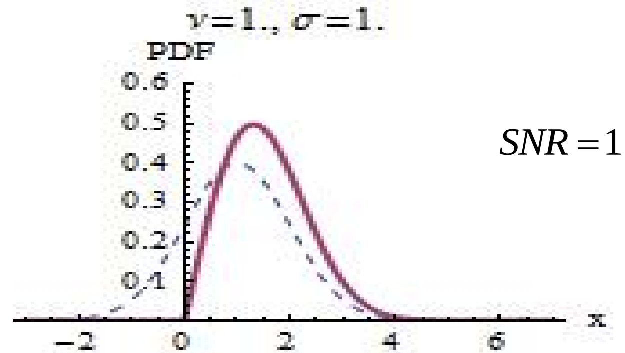
$\nu, \sigma$ - the signal and noise parameters

$x$ - value to be analyzed

$$SNR = \nu / \sigma$$

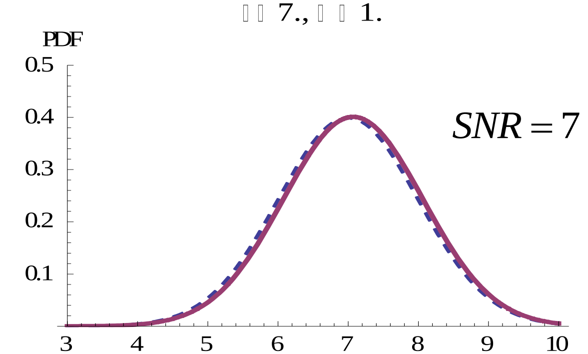



$$P\left(x \middle| \nu, \sigma^2\right) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2 + \nu^2}{2\sigma^2}\right) \cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$




$$P_G\left(x \middle| \nu, \sigma^2\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right)$$

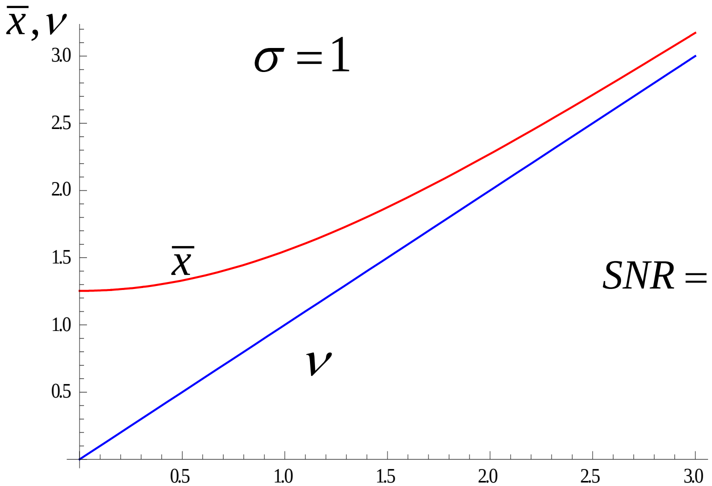
$$I_0(z) \underset{z \gg 1}{\approx} \frac{e^z}{\sqrt{2\pi z}} \left\{1 + O\left(\frac{1}{z}\right)\right\} \qquad P\left(x \middle| \nu, \sigma^2\right) \underset{\frac{\nu}{\sigma} \gg 1}{\rightarrow} P_G\left(x \middle| \nu, \sigma^2\right)$$




*__The difference between the Gauss and Rice distributions is negligible only at sufficiently large values of the signal-to-noise ratio__*

$$P_R\left(x \middle| \sigma^2\right) = \frac{x}{\sigma^2} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

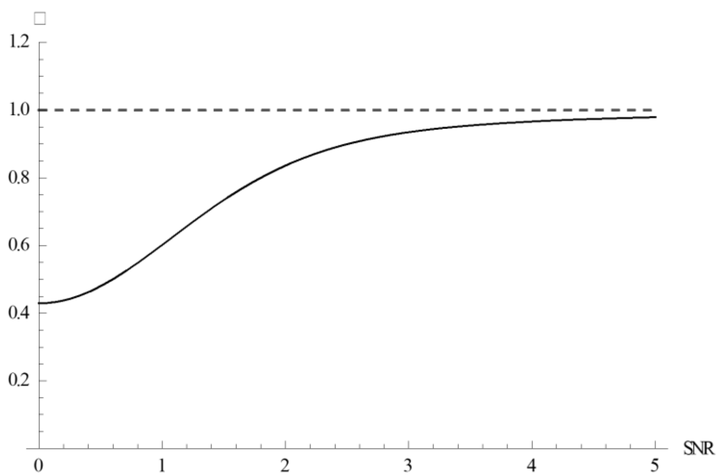
# Nonlinear properties of the Rice distribution

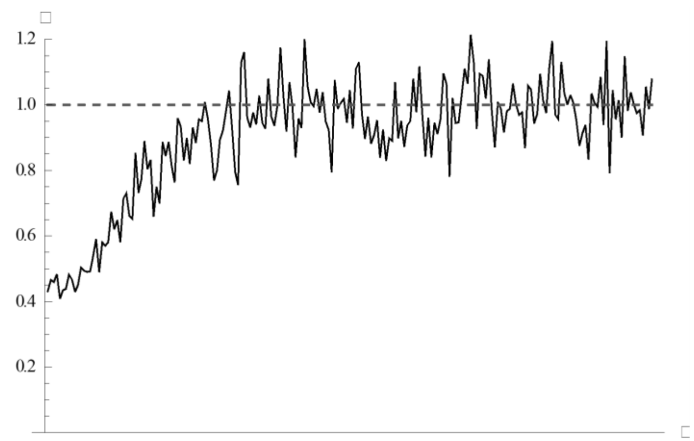



$$SNR = \nu / \sigma$$

$$\bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L_{1/2}\left(-\nu^2 / 2\sigma^2\right)$$

$$\sigma_x^2 = \overline{x^2} - \bar{x}^2 = 2 \cdot \sigma^2 + \nu^2 - \sigma^2 \cdot \frac{\pi}{2} \cdot L_{1/2}^2\left(-\nu^2 / 2\sigma^2\right)$$



***(a) - theory***

***(b) – numerical experiment***

## SOLVING OF THE TWO-PARAMETER TASK BY THE MAXIMUM LIKELIHOOD TECHNIQUE

$\nu$ - a sough-for amplitude of un-noised signal $x$ ;

$\sigma^2$ - a dispersion of the Gaussian noise distorting components $x_{\mathrm{Re}}$ and $x_{\mathrm{Im}}$ of signal $x=\sqrt{x_{\mathrm{Re}}^2+x_{\mathrm{Im}}^2}$

$$P\left(x \middle| \nu,\sigma^2\right)=\frac{x}{\sigma^2}\cdot\exp\left(-\frac{x^2+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x\nu}{\sigma^2}\right)$$

***The likelihood function*** (i.e. a joint probability density function of the events resulting in $x=x_i$ $(i=1,\dots,n)$ , :

$$L\left(\vec{x}\middle|\nu,\sigma^2\right)=\prod_{i=1}^{n}P\left(x_i\middle|\nu,\sigma\right)=\prod_{i=1}^{n}\frac{x_i}{\sigma^2}\cdot\exp\left(-\frac{x_i^{\,2}+\nu^2}{2\sigma^2}\right)\cdot I_0\left(\frac{x_i\nu}{\sigma^2}\right)$$

$$\ln L\left(\vec{x}\middle|\nu,\sigma^2\right)=\sum_{i=1}^{n}\ln P\left(x_i\middle|\nu,\sigma\right)=\sum_{i=1}^{n}\left\{-2\cdot\ln\sigma-\frac{x_i^2+\nu^2}{2\cdot\sigma^2}+\ln I_0\left(\frac{x_i\nu}{\sigma^2}\right)\right\}$$

# MAXIMUM LIKELIHOOD (ML) TECHNIQUE FOR SOLVING THE TASK OF TWO-PARAMETER RICIAN DATA ANALYSIS

The key ML equations' system for parameters $\nu$ and $\sigma^2$:

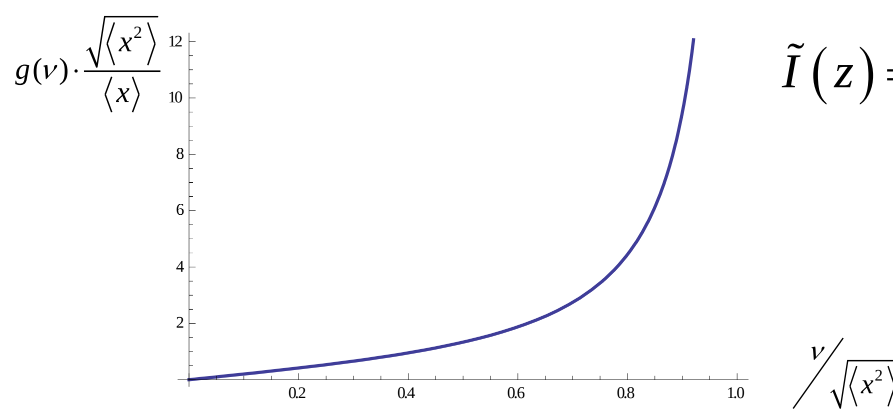
$$\begin{cases} \dfrac{\partial}{\partial \nu} \ln L\left(\vec{x} \middle| \nu, \sigma^2\right) = 0 \\ \dfrac{\partial}{\partial \sigma} \ln L\left(\vec{x} \middle| \nu, \sigma^2\right) = 0 \end{cases} \Longrightarrow \begin{cases} \nu = \dfrac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{x_i \nu}{\sigma^2}\right) \\ \sigma^2 = \dfrac{1}{2 \cdot n} \sum_{i=1}^{n} \left(x_i^2 + \nu^2\right) - \dfrac{\nu}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{x_i \nu}{\sigma^2}\right) \end{cases} \quad \begin{cases} \nu = \dfrac{1}{n} \sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\dfrac{2 \cdot x_i \cdot \nu}{\left\langle x^2 \right\rangle - \nu^2}\right) \\ \sigma^2 = \dfrac{1}{2} \cdot \left(\left\langle x^2 \right\rangle - \nu^2\right) \end{cases}$$

***THEOREM: <u>Solution of the ML equation' system exists and is a unique one</u>***

$$g(\nu) = \frac{2 \cdot \langle x \rangle \cdot \nu}{\left\langle x^2 \right\rangle - \nu^2}$$

$$\nu = \xi\left(g(\nu)\right)$$

$$\xi(g) = \frac{1}{n} \sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i}{\langle x \rangle} g\right)$$




$$\tilde{I}(z) = \frac{I_1(z)}{I_0(z)}$$

# Analysis of the second derivative's sign for determination of the character of the likelihood function extremum

$$\frac{\partial^2}{\partial \nu^2}\ln L\left(\vec{x}\,\middle|\,\nu,\sigma^2\right)=\frac{n}{\sigma^2}\left\{\frac{\langle x^2\rangle}{\sigma^2}-1-\frac{1}{n\cdot\sigma^2}\sum_{i=1}^{n}\left[x_i^{\,2}\cdot\frac{\tilde{I}\left(\frac{x_i\nu}{\sigma^2}\right)}{\left(\frac{x_i\nu}{\sigma^2}\right)}+\tilde{I}^{\,2}\left(\frac{x_i\nu}{\sigma^2}\right)\right]\right\}$$

At $\nu \ll 1$: $\dfrac{\partial^2}{\partial \nu^2}\ln L\left(\vec{x}\,\middle|\,\nu,\sigma^2\right)\approx\dfrac{n}{\sigma^2}\cdot\left(\dfrac{\langle x^2\rangle}{2\sigma^2}-1\right)$

1. Not Rayleigh distribution: $\langle x^2\rangle > 2\sigma^2 \quad\Longrightarrow\quad \dfrac{\partial^2}{\partial \nu^2}\ln L\left(\vec{x}\,\middle|\,\nu,\sigma^2\right)>0$

The trivial solution corresponds to the minimum value of likelihood function

2. Rayleigh distribution: $\langle x^2\rangle \le 2\sigma^2 \quad\Longrightarrow\quad \dfrac{\partial^2}{\partial \nu^2}\ln L\left(\vec{x}\,\middle|\,\nu,\sigma^2\right)\le 0$

The trivial solution corresponds to the maximum value of likelihood function.

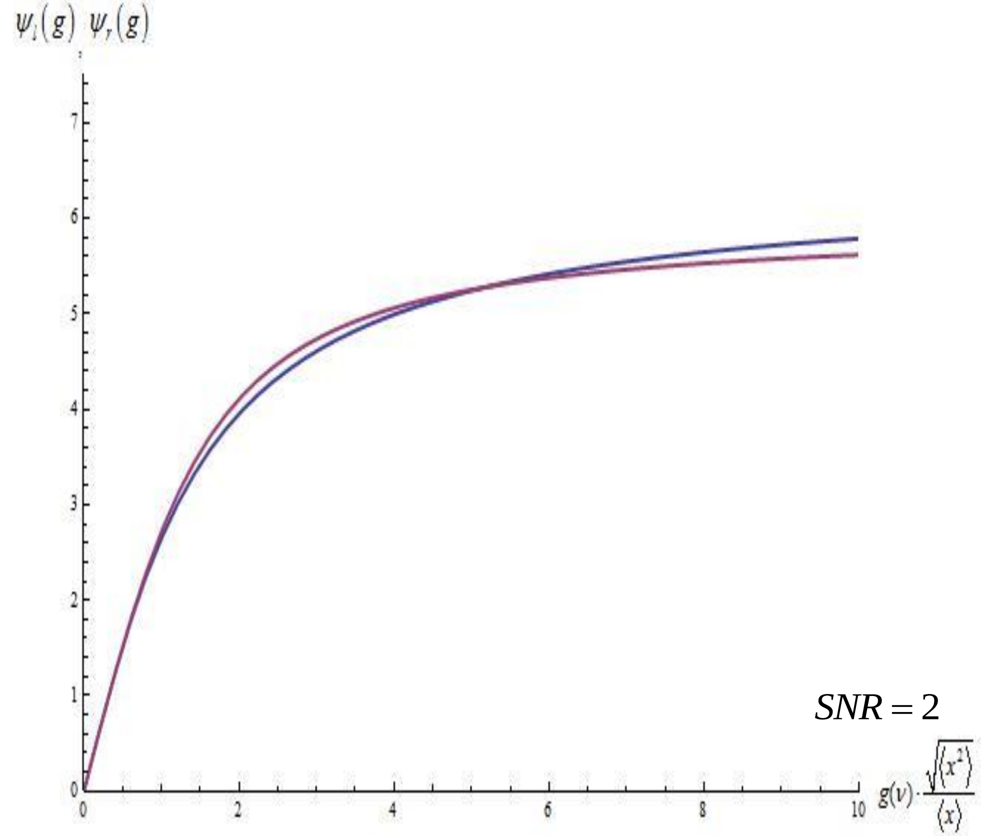
# *On proving the theorem concerning the key equation solution*

Equation for $\nu$ is transformed to equation

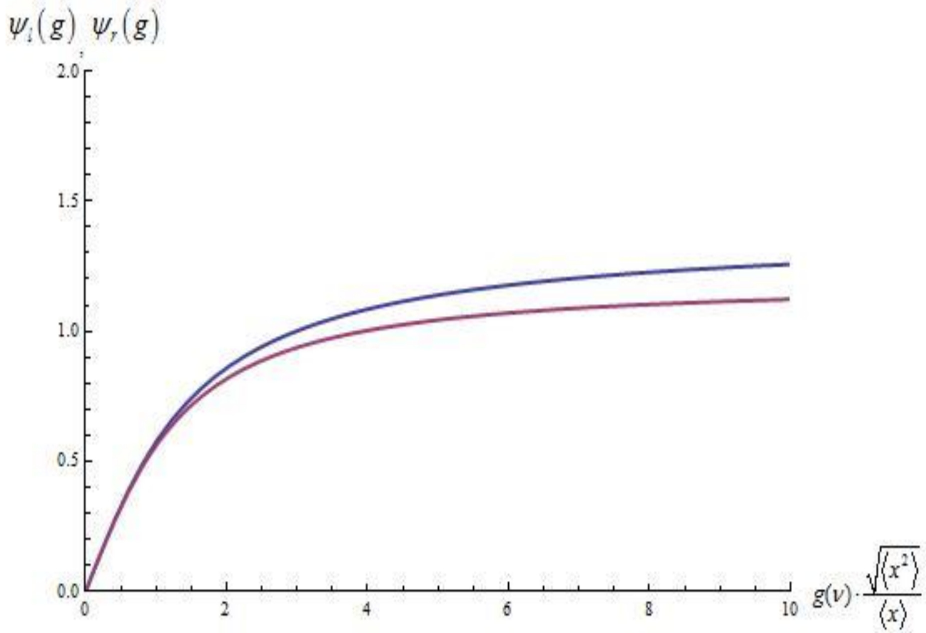
$$\xi(g) = \frac{1}{n}\sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i}{\langle x\rangle} g\right) \qquad (*)$$

for variable $g(\nu) = \frac{2\cdot\langle x\rangle\cdot\nu}{\langle x^2\rangle - \nu^2}$:

$$\psi_l(g) = \xi(g) \qquad \psi_r(g) = \frac{1}{n}\sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i}{\langle x\rangle} g\right)$$




The common case of Rice distribution

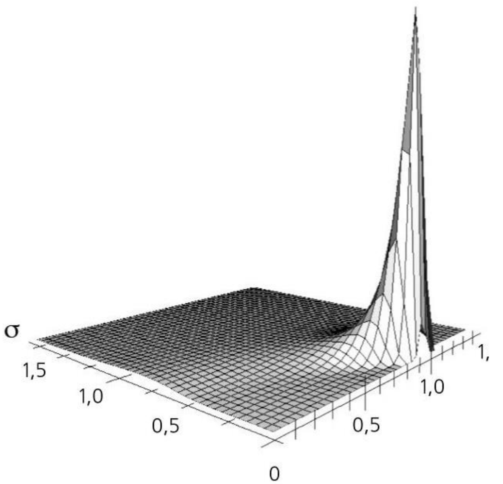


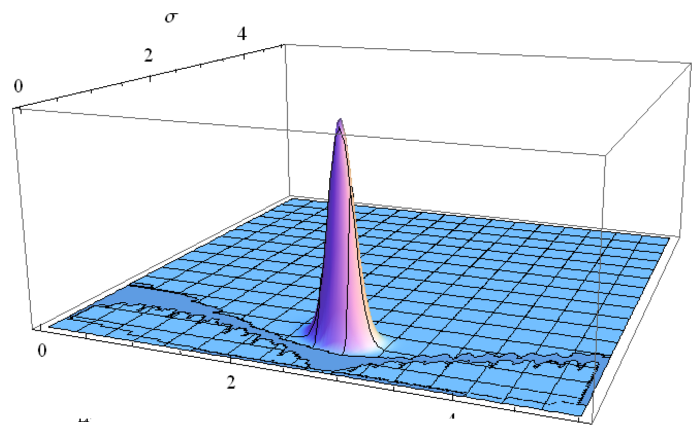

The particular case of Rayleigh distribution

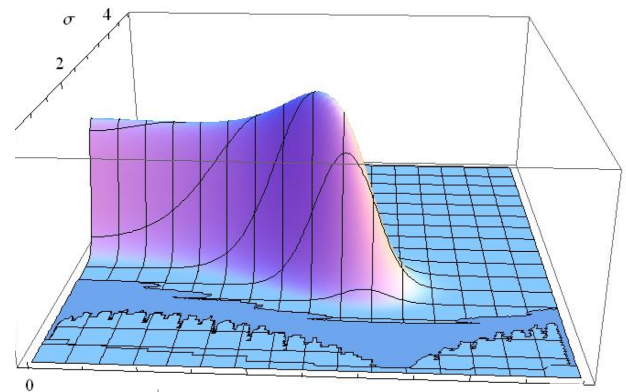
$$\nu = 0$$

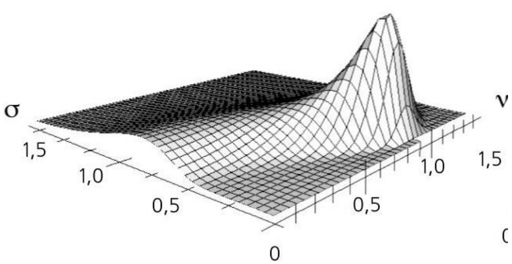
$$\sigma^2 = \frac{1}{2}\cdot\langle x^2\rangle$$

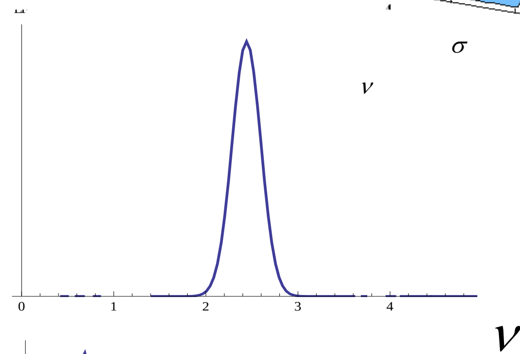
# RICIAN LIKELIHOOD FUNCTION AS DEPENDENT UPON THE SIGNAL–TO-NOISE RATIO



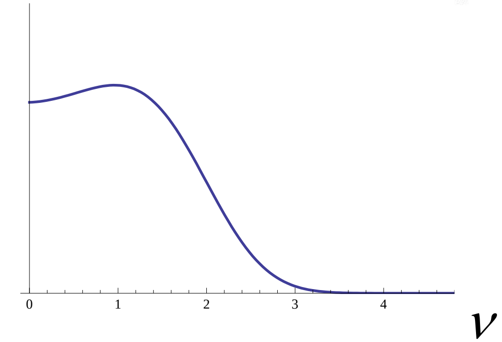



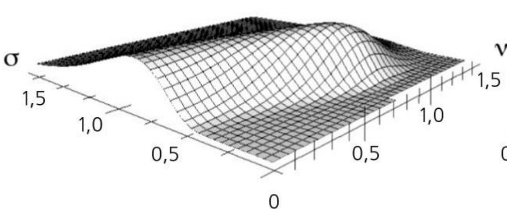



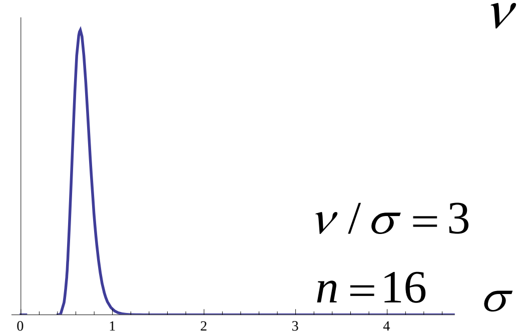




$\sigma = const$



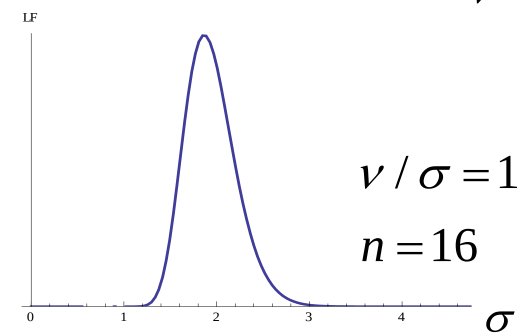




$\nu / \sigma = 3$

$n = 16$

$\nu = const$



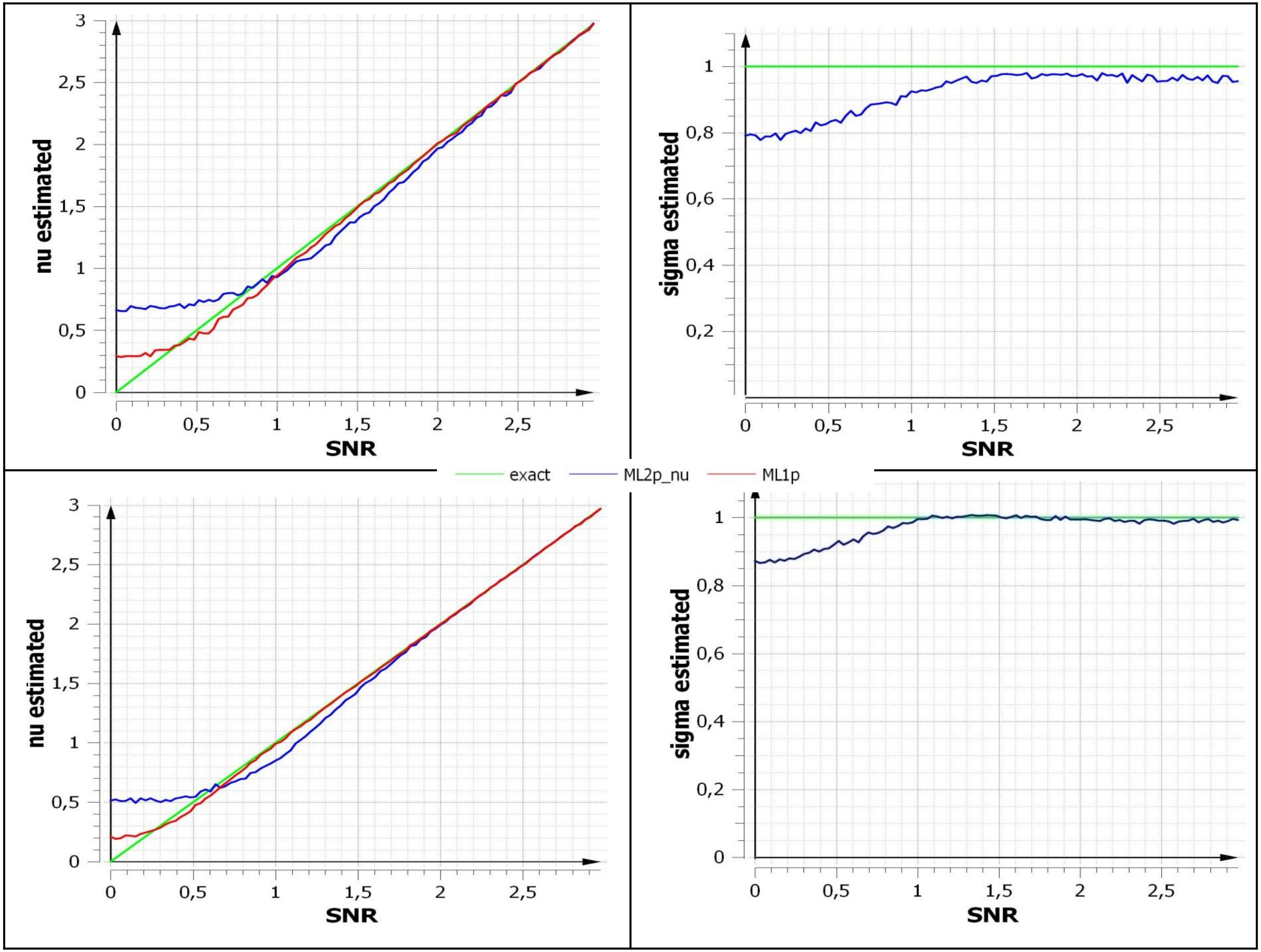

$\nu / \sigma = 1$

$n = 16$

$\nu / \sigma = 10; 4; 2$

$n = 3$

# РЕЗУЛЬТАТЫ ЧИСЛЕННОГО МОДЕЛИРОВАНИЯ РЕШЕНИЯ ЗАДАЧИ МЕТОДОМ МАКСИМУМА ПРАВДОПОДОБИЯ




n=16

n=64

$\sigma = 1$

$N_{aver} = 1000$

# Variants of the Method of Moments at solving the Task of Two-parameter Analysis of Rician Data

1. The method of lower even-numbered moments (MM24);

2. The method of lower moments (MM12).

## Two-parameter method MM24

$$\begin{cases} \overline{x^2} = 2\cdot\sigma^2 + \nu^2 \\ \overline{x^4} = 8\cdot\sigma^4 + 8\cdot\sigma^2\cdot\nu^2 + \nu^4 \end{cases} \Longrightarrow \quad \begin{aligned} \nu^2 &= \langle x^2 \rangle \sqrt{1-t} \\ \sigma^2 &= \frac{\langle x^2 \rangle}{2}\left(1-\sqrt{1-t}\right) \end{aligned}$$

$$t = \frac{\langle x^4 \rangle}{\left(\langle x^2 \rangle\right)^2} - 1 \qquad 0 < t \le 1$$

At $\nu = 0 \quad t = 1 \qquad \sigma^2 = \dfrac{\langle x^2 \rangle}{2}$

# Two-parameter method MM12

1-st moment of rician value:

$$\bar{x} = \sigma \cdot \sqrt{\pi / 2} \cdot L_{1/2}\left(-\nu^2 / 2\sigma^2\right)$$

$L_{1/2}$ - Laguerre polynomial:

$$L_{1/2}(z) = e^{z/2}\left[\left(1-z\right) I_0\left(\frac{-z}{2}\right) - z I_1\left(\frac{-z}{2}\right)\right]$$

$$\begin{cases} \sigma \cdot \sqrt{\pi / 2} \cdot e^{-\frac{\nu^2}{4\sigma^2}}\left[\left(1+\frac{\nu^2}{2\sigma^2}\right) I_0\left(\frac{\nu^2}{4\sigma^2}\right) + \frac{\nu^2}{2\sigma^2} I_1\left(\frac{\nu^2}{4\sigma^2}\right)\right] = \bar{x} \\ 2\sigma^2 + \nu^2 = \overline{x^2} \end{cases}$$

$$\begin{cases} \sqrt{\frac{\pi}{2} \cdot \sigma^2} \cdot e^{-\frac{r}{2}} \cdot \left[\left(1+r\right) I_0\left(\frac{r}{2}\right) + r I_1\left(\frac{r}{2}\right)\right] = \left\langle x \right\rangle \\ 2\sigma^2\left(1+r\right) = \left\langle x^2 \right\rangle \end{cases}$$

$$r = \frac{\nu^2}{2\sigma^2}$$

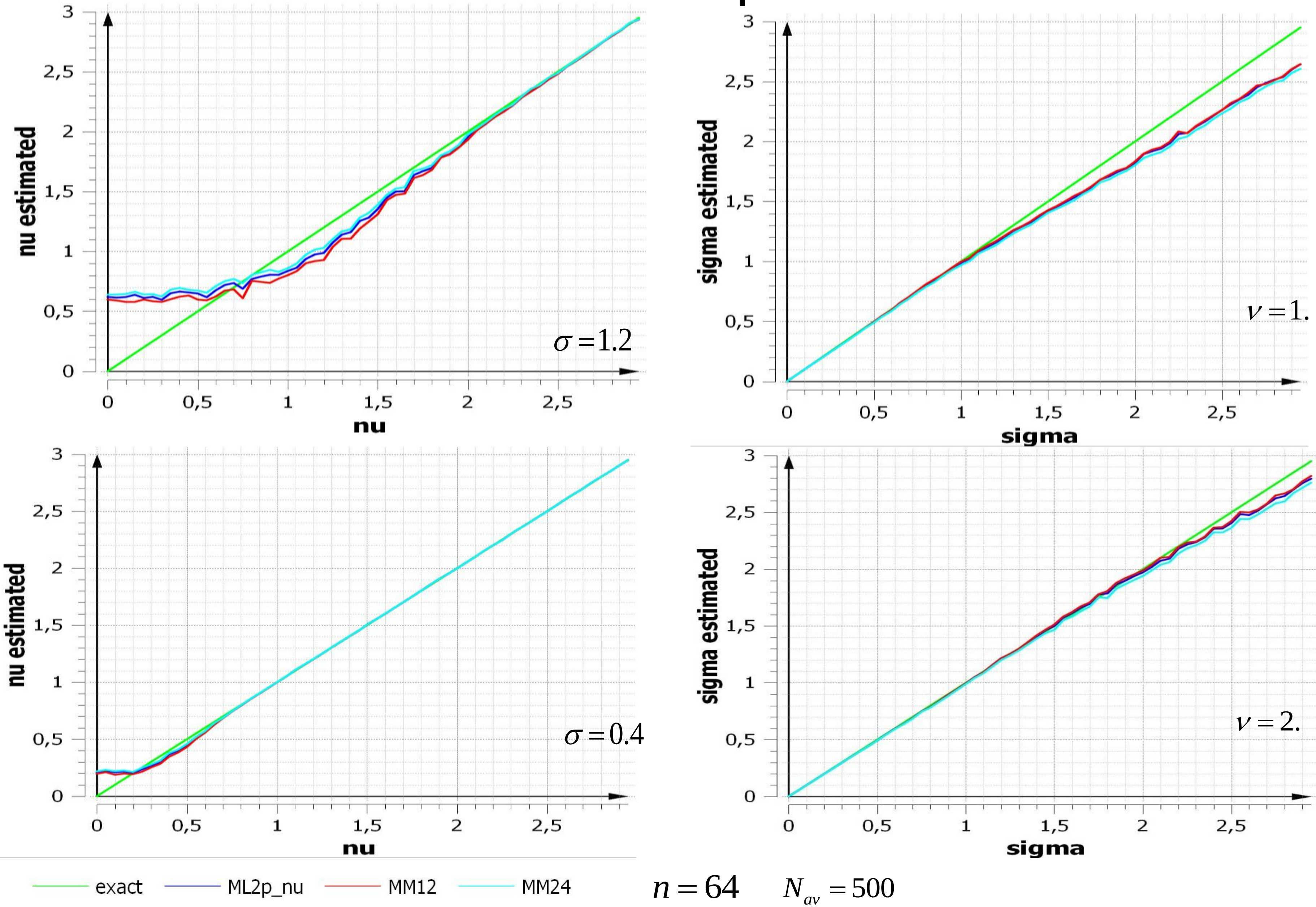
**MM12 – method: equation for** $r$

$$\sqrt{\frac{\pi}{4} \cdot \left\langle x^2 \right\rangle} \cdot \sqrt{1+r} \cdot e^{-\frac{r}{2}} \cdot I_0\left(\frac{r}{2}\right)\left[1 + \frac{r}{(1+r)} \cdot \tilde{I}\left(\frac{r}{2}\right)\right] = \left\langle x \right\rangle$$

$$\nu = \sqrt{\frac{r}{1+r}} \sqrt{\left\langle x^2 \right\rangle}$$

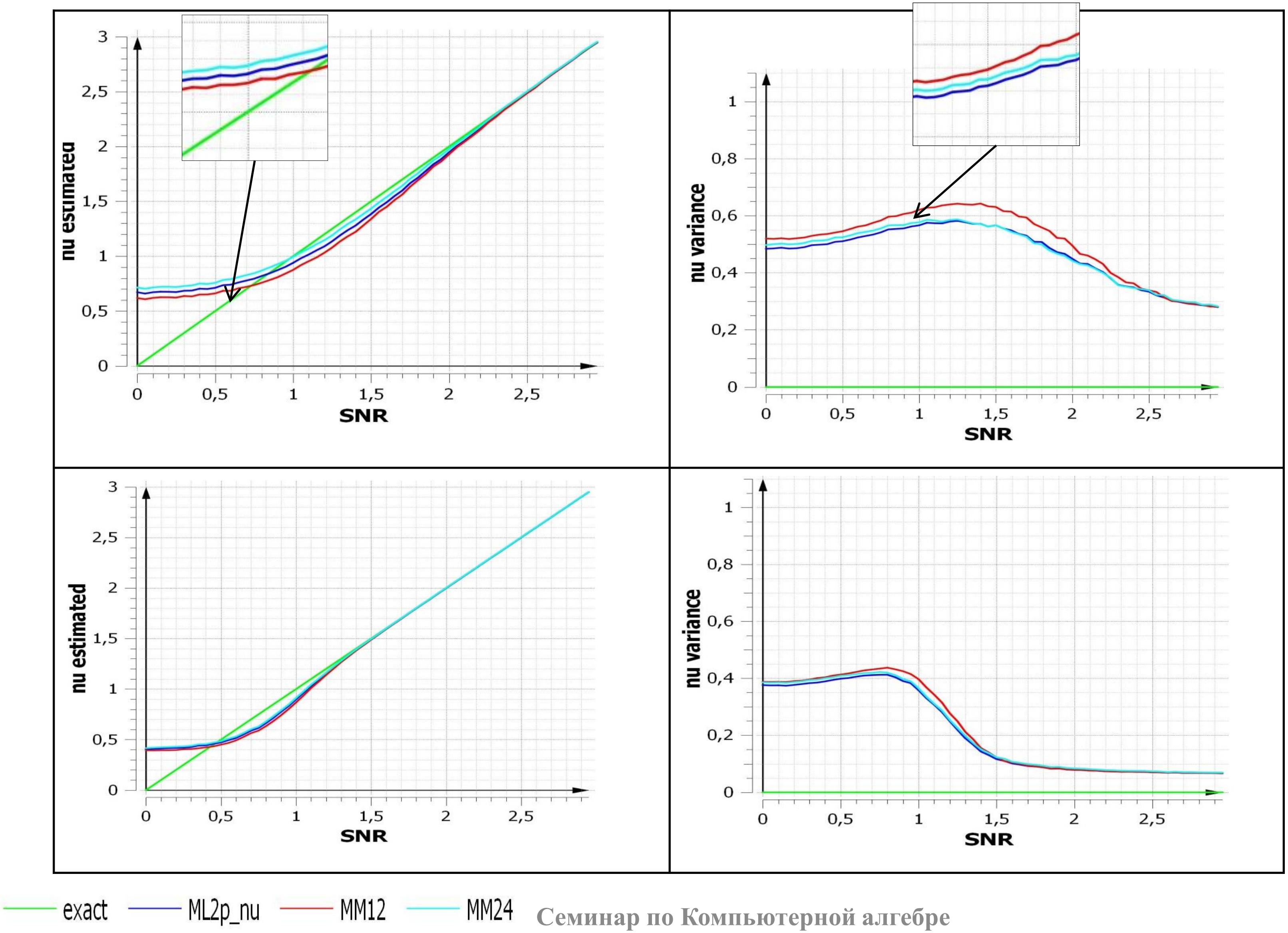
$$\sigma^2 = \frac{\left\langle x^2 \right\rangle}{2\left(1+r\right)}$$

# Results of the task digital solution by the elaborated techniques




$n = 64 \qquad N_{av} = 500$

**The plots of parameter $\nu$ (left) and its standard deviation (right), calculated by methods ML, MM12, MM24**

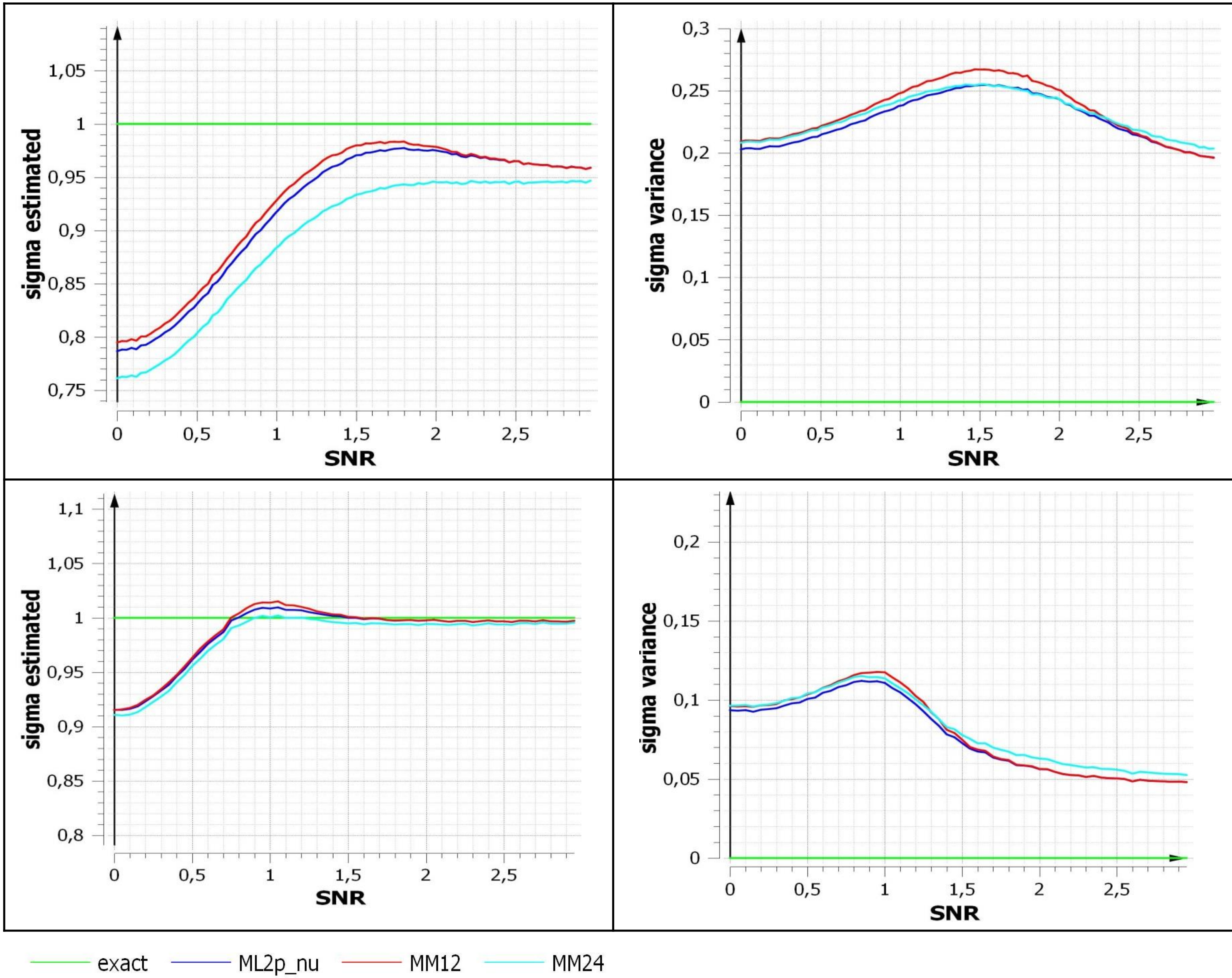



n=16

n=256

$\sigma = 1 \quad N_{av} = 10^4$

# The plots for calculated parameter $\sigma$ (on the left) and their mean square deviations (on the right)




n=16

n=256

$N_{av} = 10^4$

## TWO-PARAMETER COMBINED ML-MM METHOD

$\nu, \sigma^2 -$ Rician parameters

$$\begin{cases} \nu = \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \tilde{I}\left(\frac{x_i \cdot \nu}{\sigma^2}\right) \\ \frac{1}{n}\sum_{i=1}^{n} x_i = \sigma\sqrt{\frac{\pi}{2}} \cdot L_{1/2}\left(-\frac{\nu^2}{2\sigma^2}\right) \end{cases} \Longrightarrow \quad 2\sqrt{r/\pi} \cdot \frac{\langle x \rangle}{L_{1/2}(-r)} = \frac{1}{n}\sum_{i=1}^{n} x_i \tilde{I}\left(\frac{x_i\sqrt{\pi \cdot r}}{\langle x \rangle} L_{1/2}(-r)\right)$$

$L_q(z)$ - the Laguerre polynomial $\qquad r = \frac{\nu^2}{2\sigma^2} \qquad \nu = \sqrt{2\sigma^2 r}$

## TWO-PARAMETER METHOD OF MOMENTS MM13

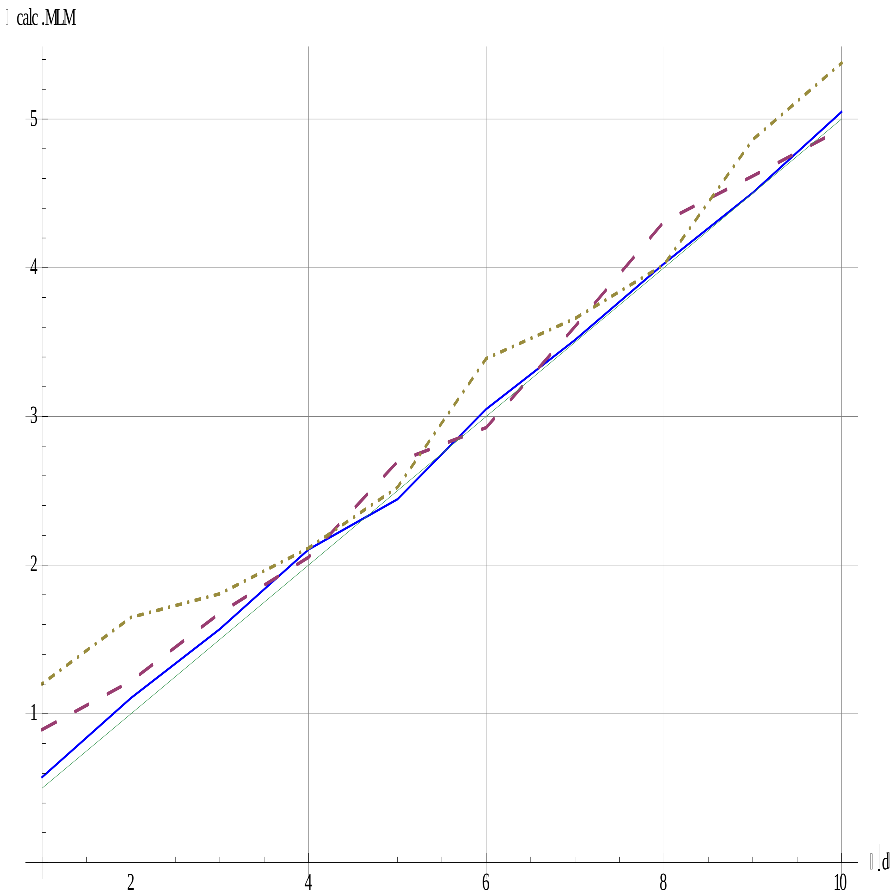
$$\overline{x} = \sigma \cdot \sqrt{\pi/2} \cdot L_{1/2}\left(-\nu^2/2\sigma^2\right),$$
$$\overline{x^3} = 3 \cdot \sigma^3 \cdot \sqrt{\pi/2} \cdot L_{3/2}\left(-\nu^2/2\sigma^2\right).$$

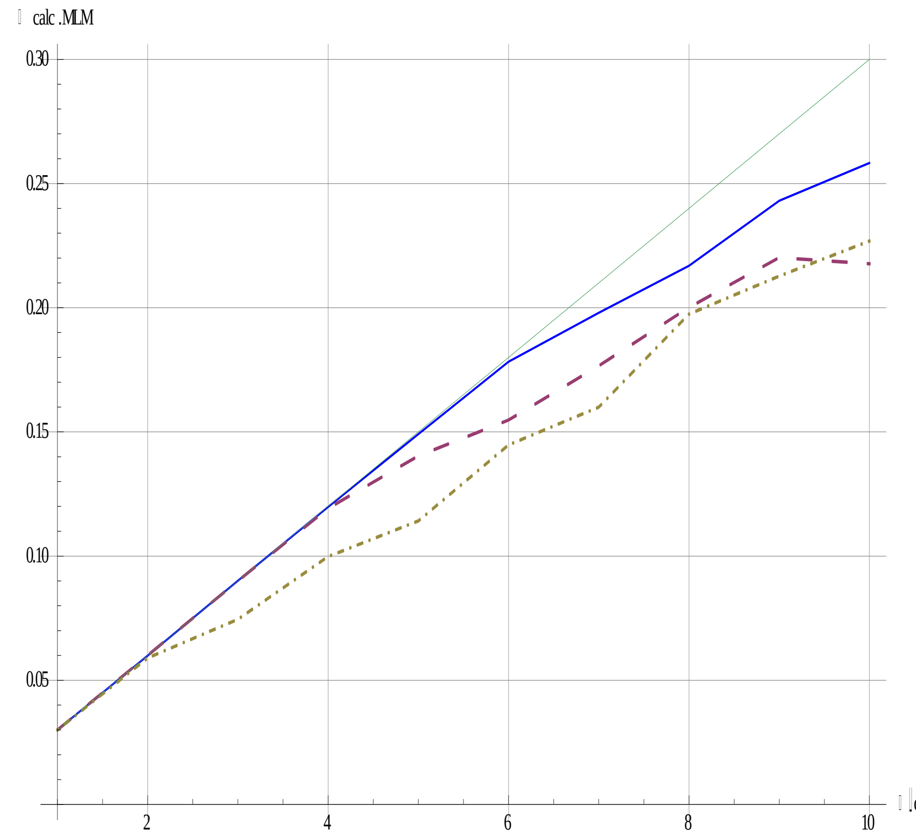
$$\Longrightarrow \langle x \rangle^3 \cdot {}_1F_1\left(-\frac{3}{2};1;-r\right) = \langle x^3 \rangle \cdot \frac{\pi}{6} \cdot {}_1F_1^3\left(-\frac{1}{2};1;-r\right),$$

$$\sigma = \langle x \rangle \sqrt{\frac{2}{\pi}} \, / \, {}_1F_1\left(-\frac{1}{2};1;-r\right)$$

${}_1F_1$ -confluent hypergeometric function of the 1-st order, or Kummer’s function

# Computer simulation results for ML-MM method



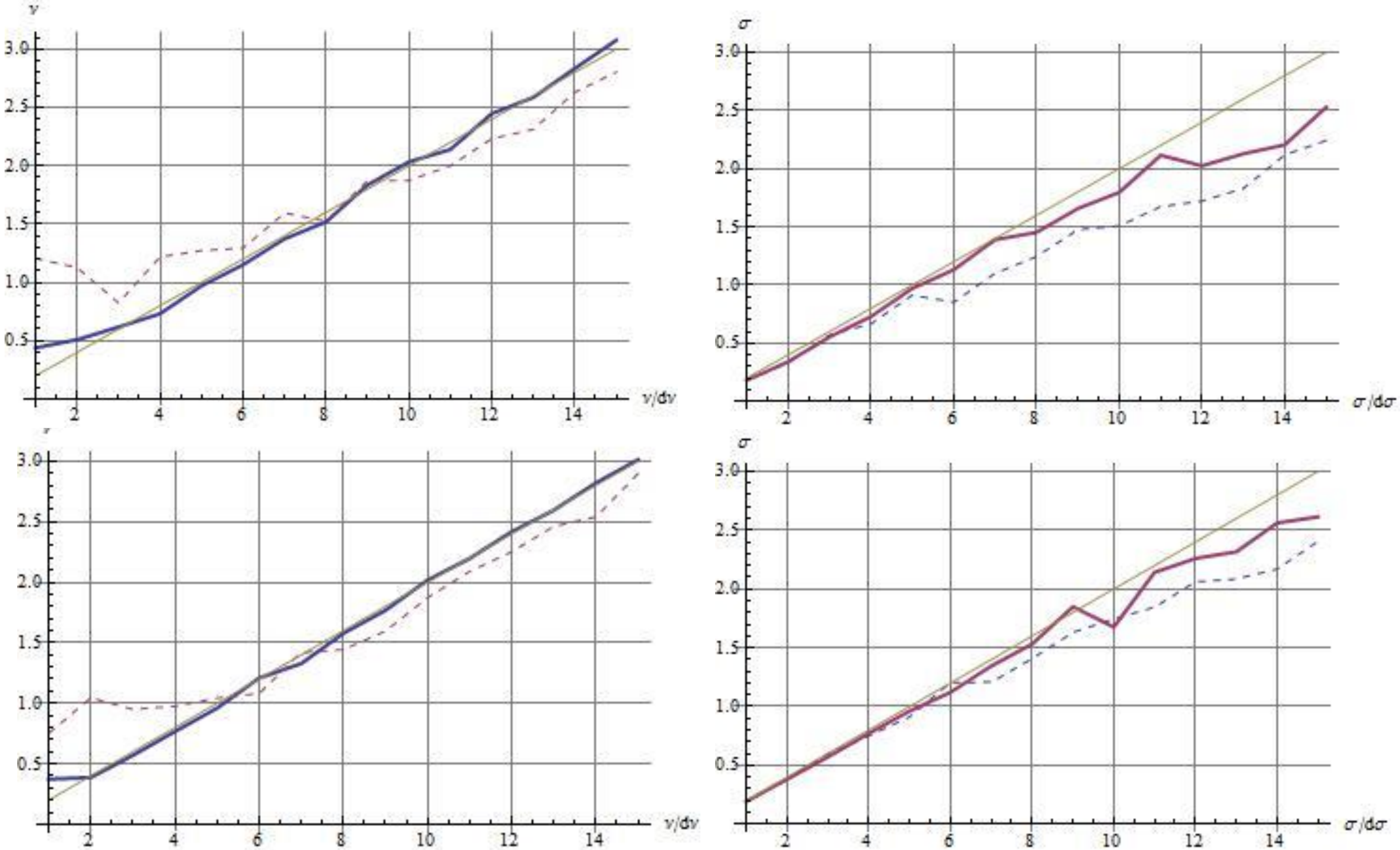




$\sigma$= 0,5 (solid line); 1 (dashed line); 1,5 (dotted-dashed line)

$\nu$=3.0 (solid line); 2.0 (dashed line); 1.0 (dotted-dashed line)

n=4

# Computer simulation results for MM13 method




n=8

n=16

σ= 0,6 (solid line); 1,5 (dashed line)

ν=3.0 (solid line); 1,5 (dashed line).

## The meaningful peculiarities of the proposed methods of two-parameter analysis

- The possibility of computing the both sought-for parameters of the task by computer algebra techniques without any additional calculative capacities if compared with the traditional one-parameter approximation.

- The applicability of the elaborated techniques of the rician data analysis for solving the nonlinear tasks in the absence of any restrictions connected with the *a-priori* suppositions on the parameters of the task.

- The elaborated techniques may be applied in data processing systems with the priority of operation in a real time mode.

# Conclusion

- Concept and methods of the ***two-parameter analysis based upon the both Rician parameters joint computing*** have been developed ***<u>to optimize the computer algebra techniques for solving the task</u>*** of separation of the informative and the noise signal's components, namely:

  solving ***the system of two nonlinear equations with two unknown variables can be reduced to solving just one equation for only one unknown variable;***

- The proposed approach provides a new efficient tool for ***accurate reconstruction of the Rician signal's informative component against the noise background***.

*Thank you very much for your attention*