FACTORIAL TRANSFORMATION FOR SOME CLASSICAL COMBINATORIAL SEQUENCES

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Предположим, что мы искали решение задачи, и нашли его в виде формального степенного ряда. Однако ряд всюду расходится. Значит ли это что

а) решение не существует

б) м.б. и существует, но не аналитично

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Эта цитата

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Borel-Ritt theorem

Let $S \subset (\mathbb{C}, 0)$ be an open sector $\{0 < |z| < \rho,\ |\theta_- < \arg z < \theta_+\}$ with the opening angle $\theta_+ - \theta_-$ less than 2π on the complex plane with the vertex at the origin, and $\{c_k :\ k = 0, 1, 2, \dots\}$ a sequence of complex numbers

Then there exists a function f holomorphic in S, for which the formal series $\sum c_k z^k$ is an asymptotic series

$$\forall m \in \mathbb{N} \quad \lim_{z \to 0} \frac{1}{z^m} \left(f(z) - \sum_0^m c_k z^k \right) = 0 \qquad \text{as } z \to 0,\ z \in S.$$

$$y(x) = x - x^2 + 2x^3 - 6x^4 + 24x^5 - 120x^6 + 720x^7 - 5040x^8 + 40320x^9 + O(x^{10})$$

$$x^2 \left(\frac{d}{dx} y(x)\right) + y(x) = x$$

$$y(x) = \sum_{n=1}^{\infty} (-1)^{(n-1)} (n-1)!\, x^n = \mathbf{e}^{\left(\frac{1}{x}\right)} \operatorname{Ei}\left(1, \frac{1}{x}\right) = \mathbf{e}^{\left(\frac{1}{x}\right)} \int_1^{\infty} \frac{\mathbf{e}^{\left(-\frac{t}{x}\right)}}{t}\, dt$$

$$y(x) = x - x^2 + 2\,x^3 - 6\,x^4 + 24\,x^5 - 120\,x^6 + 720\,x^7 - 5040\,x^8 + 40320\,x^9 + O(x^{10})$$

$$x^2\left(\frac{d}{dx}\,y(x)\right) + y(x) = x$$

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$$y(x) = \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{\frac{1}{2} + \cfrac{x}{1 + \cfrac{x}{\frac{1}{3} + \cfrac{x}{1 + \cfrac{x}{\frac{1}{4} + \cfrac{x}{1 + \cfrac{x}{\frac{1}{5} + \cfrac{x}{1 + \cfrac{x}{\frac{1}{6} + \cfrac{x}{1 + \dots}}}}}}}}}}}}}$$

```
a:=proc(m,n) option remember: global c:
     if m=0 then c(n) elif n=0 then 1/a(m-1,1) else
       -add(a(m,n-j)*a(m-1,j+1),j=1..n)/a(m-1,1)
     fi
end:

c:=n->`if`(n=0,0,(-1)^(n-1)*(n-1)!);

seq(a(m,0),m=0..20);
```

$$0, 1, 1, 1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, 1, \frac{1}{6}, 1, \frac{1}{7}, 1, \frac{1}{8}, 1, \frac{1}{9}, 1, \frac{1}{10}$$

$$1+\tanh(x)=\sum_{n=0}^{\infty}\frac{T(n)\,x^n}{n!} \qquad \frac{1}{\cosh(x)}=\sum_{n=0}^{\infty}\frac{E(n)\,x^n}{n!}$$

$T(n)$ = 1, 1, 0, -2, 0, 16, 0, -272, 0, 7936, 0, -353792, 0, 22368256, 0, -1903757312 ...

$E(n)$ = 1, 0, -1, 0, 5, 0, -61, 0, 1385, 0, -50521, 0, 2702765, 0, -199360981, 0, 19391512145 ...

$$T(n)=\frac{\left(4^{(n+1)}-2^{(n+1)}\right)B(n+1)}{n+1} \qquad B(1)=\frac{1}{2}$$

$$T_g(x)=\sum_{n=0}^{\infty}T(n)\,x^n \qquad E_g(x)=\sum_{n=0}^{\infty}E(n)\,x^n$$

$$T_g(x)=1+\cfrac{x}{1+\cfrac{x^2}{\frac{1}{2}+\cfrac{x^2}{\frac{1}{3}+\cfrac{x^2}{\frac{1}{4}+\cfrac{x^2}{\frac{1}{5}+\cfrac{x^2}{\frac{1}{6}+\cfrac{x^2}{\frac{1}{7}+\cdots}}}}}}}$$

$$E_g(x)=1+\cfrac{x^2}{-1+\cfrac{x^2}{-\frac{1}{5}+\cfrac{x^2}{-\frac{25}{36}+\cfrac{x^2}{-\frac{36}{445}+\cfrac{x^2}{-\frac{7921}{14400}+\cfrac{x^2}{-\frac{1600}{33909}+\cdots}}}}}}$$

$$\mathrm{E_g}(x) = \cfrac{1}{1 + \cfrac{x^2}{1 + \cfrac{4x^2}{1 + \cfrac{9x^2}{1 + \cfrac{16x^2}{1 + \cfrac{25x^2}{1 + \cfrac{36x^2}{1 + \cfrac{49x^2}{1 + \cfrac{64x^2}{1 + \cfrac{81x^2}{1 + \cfrac{100x^2}{1 + \cdots}}}}}}}}}}}$$

$$\mathrm{T_g}(x) = \cfrac{1}{1 - \cfrac{x}{1 + \cfrac{x}{1 - \cfrac{2x}{1 + \cfrac{2x}{1 - \cfrac{3x}{1 + \cfrac{3x}{1 - \cfrac{4x}{1 + \cfrac{4x}{1 - \cfrac{5x}{1 + \cfrac{5x}{1 - \cfrac{6x}{1 + \cdots}}}}}}}}}}}}$$

$$(1)\quad \beta_1\left(\frac{x}{1-x}\right) - \beta_1(x) = x^2, \qquad \beta_1(x) = \sum_{n=0}^{\infty} B(n)\, x^{n+1}$$

$$\sum_{m=0}^{n-1} C(n,m)\, B(m) = 0, \quad n > 1,$$

$$B(1) = 1/2$$

$$\frac{x}{\exp(x)-1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!}\, x^n, \qquad \frac{x\,\exp(x)}{\exp(x)-1} = \sum_{n=0}^{\infty} \frac{B(n)}{n!}\, x^n$$

$$B(0) = 1, \quad B(1) = -1/2, \quad B(2) = 1/12, \quad B(3) = 0, \ldots$$

Proposition 1. *Let $f(x) = f(x+p)$ be any p-periodic function, $p \in \mathbb{C}$. Then the function*

$$\tilde{\beta}_1(x) = \beta_1(x) + f\left(\frac{p}{x}\right)$$

is the general solution to the functional equation (1). $+ \exp(2\pi\, i/x)$

$$\beta_1(x) - \beta_1\left(\frac{x}{1+x}\right) = \frac{x^2}{(1+x)^2} \qquad \beta_1(x) = x^2 \sum_{n=1}^{\infty} \frac{1}{(1+n\,x)^2} = \Psi\left(1, 1+\frac{1}{x}\right)$$

$$\beta_k\left(\frac{x}{1-x}\right)-\beta_k(x)=k\,x^{k+1},\qquad \beta_k(x)=\frac{(-1)^{k-1}}{(k-1)!}\,\Psi(k,1+1/x),\quad k\in\mathbb{N}$$

$$\beta_k(x)=\sum_{n=0}^{\infty}C(n+k-1,n)\,B(n)\,x^{n+k};\qquad \log\Gamma(t)-\log\Gamma(t+1)=-\log t$$

$$\sum_{k=m}^{n}f(k)=\int_m^n f(x)\,dx+\frac{1}{2}\,(f(m)+f(n))+\sum_{j=1}^{N}\frac{B(2\,j)}{(2\,j)!}\,\left(f^{(2\,j-1)}(n)-f^{(2\,j-1)}(m)\right)$$
$$+\frac{1}{(2\,N+1)!}\int_m^n B(2\,N+1,x-[x])\,f^{(2\,N-1)}(x)\,dx.$$

$$\gamma=\frac{1}{2}+\sum_{j=1}^{\infty}\frac{B(2\,j)}{2\,j}.\ \Big|\ \zeta(2)=1+\sum_{j=0}^{\infty}B(j)=1+\beta_1(1)=1+\Psi(1,1+1),\ \Big|\ \zeta(3)=1+\frac{1}{2}\sum_{j=1}^{\infty}(2\,j+1)\,B(2\,j).$$

$$\zeta_n(x)=1+\frac{(-1)^{n+1}}{n!\,x^n}\,\Psi\left(n,1+\frac{1}{x}\right)=1+\frac{1}{n!}\,\sum_{j=0}^{\infty}\left(\prod_{k=1}^{n-1}(j+k)\right)B(j)\,x^j,$$

$$\zeta_n(1)=\zeta(n+1),\ n\in\mathbb{N},$$

$$\tilde{\zeta}_2(x)=\frac{x}{2}-\frac{1}{2}+\zeta_2(x)=1+\frac{1}{2}\,x^2-\frac{1}{12}\,x^4+\frac{1}{2}\,x^6-\frac{3}{20}\,x^8+\frac{5}{12}\,x^{10}+\ldots,$$

$$\tilde{\zeta}_2(x) = 1 + \cfrac{x^2}{4 + \cfrac{4\,x^2}{3 + \cfrac{2\,x^2}{1 + \cfrac{6\,x^2}{5 + \dots}}}} = b_0 + \frac{a_1\,x^2}{b_1 +}\;\frac{a_2\,x^2}{b_2 +}\;\dots = b_0 + \mathop{\mathbf{K}}_{n=1}^{\infty} \frac{a_n\,x^2}{b_n},$$

$$b_n = 1, 4, \mathbf{3}, \mathbf{1}, \mathbf{5}, \mathbf{1}, \mathbf{7}, \mathbf{1}, \mathbf{9}, \mathbf{1}, \mathbf{11}, \mathbf{1}, \mathbf{13}, \dots,$$

$$a_n = 1, 4, \mathbf{2}, \mathbf{6}, \mathbf{9}, \mathbf{18}, \mathbf{24}, \mathbf{40}, \mathbf{50}, \mathbf{75}, \mathbf{90}, \mathbf{126}, \mathbf{147}, \mathbf{196}, \mathbf{224}, \mathbf{288}, \mathbf{324}, \mathbf{405}, \dots,$$

$$a(n) = \frac{1}{2}\left[\frac{n}{2}\right]\left[\frac{n-1}{2}\right]\left[\frac{n-2}{2}\right] = \{0, 0, 0, 1, \mathbf{2}, \mathbf{6}, \mathbf{9}, \mathbf{18}, \mathbf{24}, \mathbf{40}, \mathbf{50}, \dots\},\ n \in \mathbb{N}.$$

$$\zeta(3) = 1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{12 + \cfrac{8}{1 + \cfrac{8}{20 + \cfrac{27}{1 + \cfrac{27}{28 + \cfrac{64}{1 + \cfrac{64}{36 + \cfrac{125}{1 + \cdots}}}}}}}}}}$$

Factorial transformation

$$(x)_a = \Gamma(x+a)/\Gamma(x), \quad x, a \in \mathbb{C}, \qquad (x)_n = \prod_{k=0}^{n-1} (x+k), \quad n \in \mathbb{N}.$$

$$(x)_n = \sum_{m=0}^{n} (-1)^{n+m} S_1(n,m)\, x^m, \quad x^n = \sum_{m=0}^{n} (-1)^{n+m} S_2(n,m)\, (x)_m$$

$$Q(x,n) = (-1)^n \Gamma\left(1+\frac{1}{x}\right) / \Gamma\left(n+1+\frac{1}{x}\right) = \frac{(-1)^n}{x} \left(\frac{1}{x}\right)_{n+1}^{-1}.$$

Theorem 2. *For any formal PS, both series*

$$\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} Q(x,n) \sum_{m=0}^{n} (-1)^m S_1(n,m)\, a_m,$$

have the same asymptotic expansion as $x \to 0$, $\mathrm{Re}(x) > 0$.

N. Nielsen, *Die Gammafunktion* (Teubner, Leipzig, Berlin, 1906) = (Chelsea, New York, 1965).

G. N. Watson, "The transformation of an asymptotic series into a convergent series of inverse factorials", Rend. Circ. Mat. Palermo, **34**, 41-88, (1912).

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{S_2(m,n)}{n!}(-1)^m x^m = \sum_{n=0}^{\infty}\frac{Q(x,n)}{n!} = F([\,],[1+1/x],-1) = \Gamma\left(1+\frac{1}{x}\right)J_{1/x}(2)$$

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}S_2(m,n)\,n!\right)(-1)^m x^m = \sum_{n=0}^{\infty}Q(x,n)\,n! = F([1,1],[1+1/x],-1)$$

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}S_2(n,m)\right)x^n = \sum_{n=0}^{\infty}\frac{1}{n!}\,\mathrm{B}(n)\,x^n = \exp(\exp(x)-1)$$

$$U(x) = \sum_{n=0}^{\infty}\mathrm{B}(n)\,x^n = 1 + \sum_{n=1}^{\infty}Q(x,n)\sum_{m=1}^{n}(-1)^m S_1(n,m)\,\mathrm{B}(m)$$

$$U(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{1+x\,(n+1)}\,Q(x,n)\,n!\,L_n(-1)$$

$$W(x) = \sum_{n=0}^{\infty}(-1)^n\,\mathrm{B}(n)\,x^n = \sum_{n=0}^{\infty}Q(x,n)\sum_{m=0}^{n}S_1(n,m)\,\mathrm{B}(m) = \sum_{n=0}^{\infty}Q(x,n)$$

$$W(x) = \exp(-1)\,F\left(\left[\frac{1}{x}\right],\left[1+\frac{1}{x}\right],1\right) = \frac{1}{x}\int_0^1 \exp(t-1)\,t^{1/x-1}\,dt$$

$$G(x) = \sum_{n=0}^{\infty} (-1)^n \, B(n) \, x^n; \qquad \beta_1(x) = \sum_{n=0}^{\infty} B(n) \, x^{n+1}$$

$$G(x) = \sum_{n=0}^{\infty} Q(x,n) \sum_{m=0}^{n} S_1(n,m) \, B(m) = \sum_{n=0}^{\infty} Q(x,n) \, a_0(n).$$

$$\{a_1(n)\} = \{(n+1)! \, a_0(n)\}; \; \{a_2(n)\} = \left\{ \frac{a_1(n+1)}{a_1(n)} = -(n+1)^2 \right\}, \; n \in \mathbb{N}_0$$

$$a_0(n) = \sum_{m=0}^{n} S_1(n,m) \, B(m) = (-1)^n \, \frac{n!}{n+1}, \qquad B(n) = \sum_{m=0}^{n} S_2(n,m) \, (-1)^m \, \frac{m!}{m+1}$$

$$G(x) = F([1,1,1],[2,1+\frac{1}{x}],1) = \frac{1}{x} \, \Psi\left(1, \frac{1}{x}\right)$$

$$\beta_1(x) = \Psi\left(1, 1+\frac{1}{x}\right) = x - \frac{x^2}{2\,(1+x)} \, F\left([1,1,2], \left[3, 2+\frac{1}{x}\right], 1\right)$$

$$G(x) = x + \beta_1(x)/x = x + \Psi(1, 1+1/x)/x, \qquad G(-x) \neq \beta_1(x)/x$$

$$\sum_{n=0}^{\infty} \frac{E(n)}{n!} x^n = \frac{1}{\cosh(x)}, \quad \sum_{n=0}^{\infty} \frac{T(n)}{n!} x^n = 1 + \tanh(x), \quad |x| < \frac{\pi}{2}.$$

$$T(n) = \frac{4^{n+1} - 2^{n+1}}{n+1} B(n+1), n \in \mathbb{N}_0, \; B(1) = 1/2, \quad \exp(-x)\,(1 + \tanh(x)) = \frac{1}{\cosh(x)}$$

Proposition 4. *The sequences of Euler and Tangent numbers are binomial transforms of each other, i.e., for $n \in \mathbb{N}_0$, we have*

$$T(n) = \sum_{m=0}^{n} C(n,m)\,(-1)^m\,E(m), \quad E(n) = \sum_{m=0}^{n} C(n,m)\,(-1)^m\,T(m).$$

Proposition 5. *Let the sequences $\{g_n\}$ and $\{h_n\}$ be binomial transforms of each other, and let $g(x)$ and $h(x)$ be their respective formal GFs, i.e.,*

$$g(x) = \sum_{n=0}^{\infty} g_n\, x^n, \quad h(x) = \sum_{n=0}^{\infty} h_n\, x^n, \qquad \textit{then} \quad g(x) = \frac{1}{1-x}\, h\left(\frac{-x}{1-x}\right),$$

considered as formal PS transformation.

$$E_g(x) = \sum_{n=0}^{\infty} E(n)\, x^n = \sum_{n=0}^{\infty} Q(x,n) \sum_{m=0}^{n} (-1)^m\, S_1(n,m)\, E(m) = \sum_{n=0}^{\infty} Q(x,n)\, a_0(n),$$

$$\left\{\frac{a_0(n)}{n!}\right\} = \left\{1, 0, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{4}, 0, \frac{1}{8}, \frac{-1}{8}, \frac{1}{16}, 0, \frac{-1}{32}, \frac{1}{32}, \frac{-1}{64}, 0, \ldots\right\}, \; n \in \mathbb{N}_0,$$

$$g\colon (x,k) \longrightarrow \frac{(k+1)\,x+1}{4\,x}, \frac{(k+2)\,x+1}{4\,x}, \frac{(k+3)\,x+1}{4\,x}, \frac{(k+4)\,x+1}{4\,x},$$

$$f\colon (x,k) \longrightarrow F\left([1, g(1,k-1)], [g(x,k)], -1/4\right),$$

$$E_g(x) = 1 - Q(x,2)\,f(x,2) + 3\,Q(x,3)\,f(x,3) - 6\,Q(x,4)\,f(x,4).$$

$$T_g(x) = 1 - Q(x,1)\,f(x,1) + Q(x,2)\,f(x,2) - 6\,Q(x,4)\,f(x,4).$$

$$Q(x,2)\,f(x,2) = \frac{x^2}{(x+1)\,(2\,x+1)} F\left(\left[\frac{3}{4}, 1, 1, \frac{5}{4}, \frac{3}{2}\right], \left[\frac{3\,x+1}{4\,x}, \frac{4\,x+1}{4\,x}, \frac{5\,x+1}{4\,x}, \frac{6\,x+1}{4\,x}\right], \frac{-1}{4}\right)$$

$$E_g^-(x) = \sum_{n=0}^{\infty} (-1)^n\,E(n)\,x^n,\; T_g^-(x) = \sum_{n=0}^{\infty} (-1)^n\,T(n)\,x^n,\; E_g^-(x) = E_g(x),\;\; T_g(x) + T_g^-(x) = 2$$

$$E_g(x) = \frac{1}{1+x}\,T_g\left(\frac{x}{1+x}\right), \quad T_g^-(x) = \frac{1}{1+x}\,E_g^-\left(\frac{x}{1+x}\right),$$

$$E_g(x) = \frac{2}{1+x} - \frac{1}{1+2\,x}\,E_g\left(\frac{x}{1+2\,x}\right),\; T_g(x) = 2 - \frac{1}{1+2\,x}\,T_g\left(\frac{x}{1+2\,x}\right).$$

$$E_g(x) = \frac{-\log 2}{x} + \frac{1}{x}\,\Psi\left(\frac{1+x}{2\,x}\right) - \frac{1}{x}\,\Psi\left(\frac{1+x}{4\,x}\right), \qquad T_g(x) = \frac{-\log 2}{x} + \frac{1}{x}\,\Psi\left(\frac{1}{2\,x}\right) - \frac{1}{x}\,\Psi\left(\frac{1}{4\,x}\right)$$

$$Q(x,n)=\sum_{j=0}^{\infty}(-1)^j S_2(j,n)\,x^j. \qquad x\,(1+n\,x)\,Q'(x,n)+x^2\,Q'(x,n-1)=Q(x,n),\;\; n\in\mathbb{N}$$

$$Y_m(x)=\sum_{n=0}^{\infty} S_1(n,m)\,x^n,\;\; m\in\mathbb{N}_0. \qquad Y_0(x)\equiv 1, \qquad Y_m(x)\asymp x^m.$$

$$S_1(n,1)=(-1)^{n-1}\,(n-1)!,\; n>0, \qquad x^2\,Y_1'(x)+Y_1(x)=x,$$

$$Y_1(x)=\exp\left(\frac{1}{x}\right)\mathrm{Ei}\left(1,\frac{1}{x}\right)=\exp\left(\frac{1}{x}\right)\int_1^{\infty}\frac{1}{t}\,\exp\left(\frac{-t}{x}\right)dt, \quad \mathrm{Re}(x)>0,$$

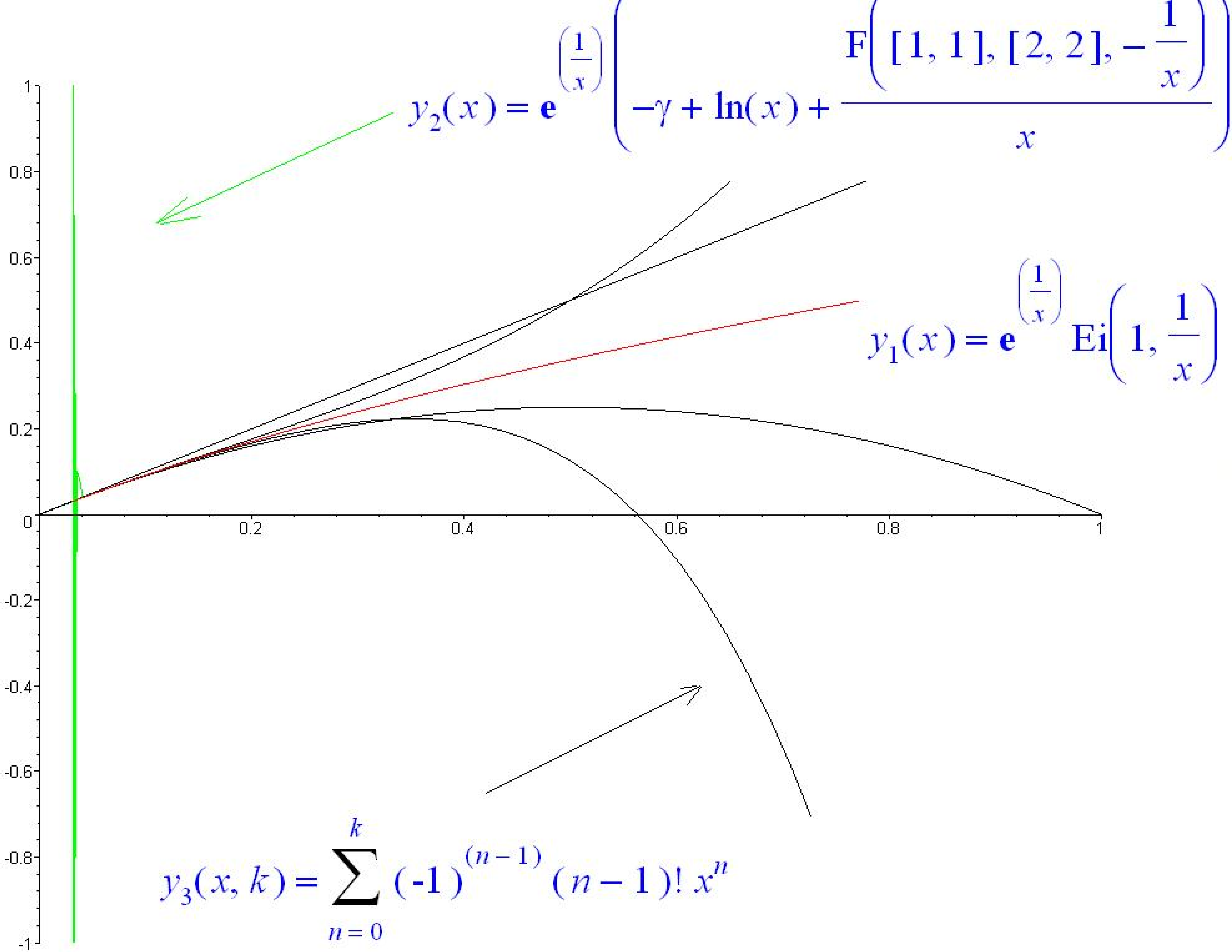
Proposition 5. *The GFs $Y_m(x)$ satisfy the following recurrent ODEs*

$$x^2\,Y_m'(x)+Y_m(x)=x\,Y_{m-1}(x), \quad m\in\mathbb{N}.$$

$$Y_1(x)=\exp\left(\frac{1}{x}\right)\left(\int\frac{1}{x}\exp\left(\frac{-1}{x}\right)dx+C\right)=\exp\left(\frac{1}{x}\right)\left(\int\frac{1}{x}\,F\left([\,],[\,],\frac{-1}{x}\right)dx+C\right)$$
$$=\exp\left(\frac{1}{x}\right)\left(C+\log x+\frac{1}{x}\,F\left([1,1],[2,2],\frac{-1}{x}\right)\right), \quad C=-\gamma,$$

$$Y_2(x)=\mathbf{e}^{1/x}\left(\frac{\pi^2}{12}+\frac{\gamma^2}{2}-\gamma\log x+\frac{\log^2 x}{2}-\frac{1}{x}\,F\left([1,1,1],[2,2,2],\frac{-1}{x}\right)\right),$$

$$Y_m(x)=\mathbf{e}^{1/x}\left(C_m+R_m(\log x)+\frac{(-1)^{m-1}}{x}\,F\left([1]_{m+1},[2]_{m+1},\frac{-1}{x}\right)\right)$$



y_2(x) = e^(1/x) (−γ + ln(x) + F([1, 1], [2, 2], −1/x) / x)
y_1(x) = e^(1/x) Ei(1, 1/x)
y_3(x, k) = Σ_{n=0}^{k} (-1)^(n−1) (n − 1)! x^n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0.2
0.4
0.6
0.8
1

Theorem 3. *For any formal PS, both series*

$$\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} Y_n(x) \sum_{m=0}^{n} S_2(n,m)\, a_m,$$

have the same asymptotic expansion as $x \to 0$, $\mathrm{Re}(x) > 0$.

$$Y_m(x) = \frac{1}{m!} \int_0^{\infty} \exp(-t) \log^m(1 + t\,x)dt, \quad m \in \mathbb{N}_0.$$

$$\frac{1}{x} \exp\left(\frac{1}{x}\right) \mathrm{Ei}\left(1, \frac{1}{x}\right) = \int_0^{\infty} \frac{\exp(-t)}{1 + t\,x}\, dt.$$

$x^2\, Y_m'(x) + Y_m(x) = x\, Y_{m-1}(x), \quad m \in \mathbb{Z}.$	$Y_{-m}(x) = x\, Y_{1-m}'(x) + \frac{1}{x} Y_{1-m}(x),\ m \in \mathbb{N}_0$

Proposition 7. *The functions* $Y_{-m}(x)$ *have the following forms*

$$Y_{-m}(x) = \mathbf{e}^{1/x} \sum_{k=0}^{\infty} \frac{(-1)^{m+k}\, k^m}{k!\, x^k} = \sum_{k=0}^{m} \frac{(-1)^{m+k}\, S_2(m,k)}{x^k}, \quad m \in \mathbb{N}_0.$$

$$Y_m(x) = \sum_{n=0}^{\infty} S_1(n,m)\, x^n,\ m \in \mathbb{N}_0.$$