Structure of the lattice of right divisors of a linear ordinary differential operator

(joint work with S.V. Larin)

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СИБИРСКИЙ ФЕДЕРАЛЬНЫЙ УНИВЕРСИТЕТ
SIBERIAN FEDERAL UNIVERSITY

MSU, Computer algebra seminar, 14 Dec. 2011

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Factorization of linear ordinary differential operators:

$$L = D^n + a_1(x)D^{n-1} + a_2(x)D^{n-2} + \ldots + +a^{n-1}(x)D + a_n(x),$$

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Algorithms for factorization, $\mathbb{F} = \bar{Q}(x)$: E.Beke (1894), M.Bronstein, F.Schwarz, M.van Hoej, ...

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Algorithm for enumeration of all possible factorization of a given LODO: S.Ts. (ISSAC-1996)

Main open question: can we mathematically describe the structure of the answers, given by the enumeration algorithm?

Basic facts about LODO factorization

Theorem (E.Landau, 1902, A.Loewy, 1902–1903) *Any two different decompositions of a given LODO L into products of irreducible LODOs* $L = L_1 \cdot \ldots \cdot L_k = \overline{L}_1 \cdot \ldots \cdot \overline{L}_p$ *have the same number of factors* ($k = p$) *and the factors are pairwise similar (in some transposed order).*

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Definition. *Two (irreducible for simplicity) operators L and M are called* similar *(or* operators of the same type*) if one can find operators A and B such that* $\mathrm{ord}(A) = \mathrm{ord}(B) < \mathrm{ord}(L) = \mathrm{ord}(M)$ *and* $A \cdot L = M \cdot B$.

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Case 1: $L = L_1 \cdot L_2$,
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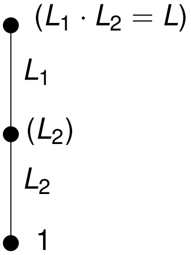

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In this case L_1 and L_2 are called *not interchangeable*.

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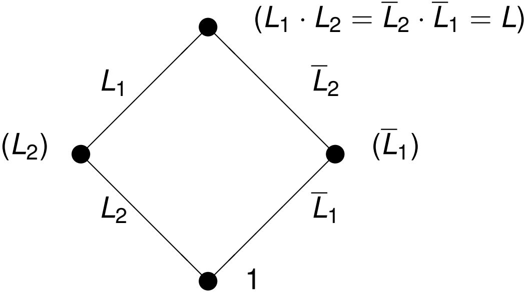

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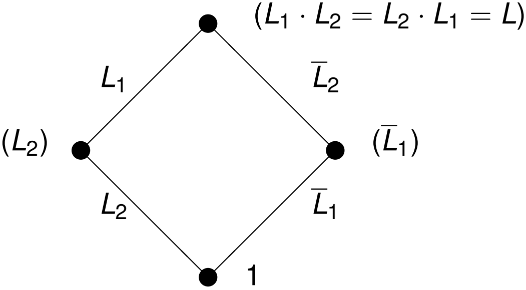

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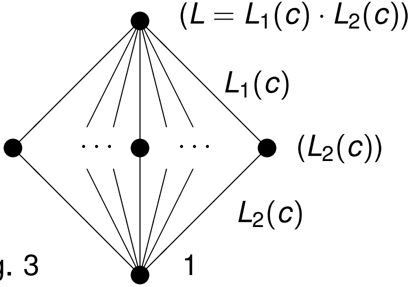

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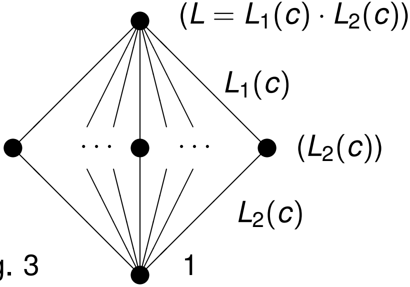

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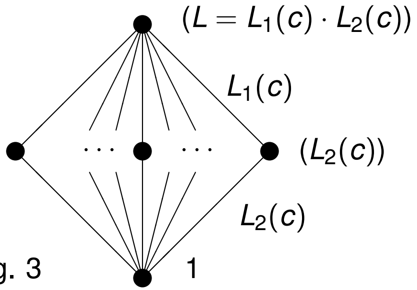

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Theorem (A.Loewy, 1903) *If the subfield of constants of $\mathbb{F}$ is algebraically closed, then the cases 1–3 are the only possible cases for a LODO which is factorizable into two irreducible factors.*

Loewy-Ore theory

Loewy, A. Über reduzible lineare homogene Differentialgleichungen. *Math. Annalen* (1903-1906).

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Right (left) division:
for any LODOs L, M, there exist unique LODOs Q, R, Q_1, R_1, such that:

$$L = Q \cdot M + R, \qquad L = M \cdot Q_1 + R_1.$$

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$rGCD(L, M) = G \Longleftrightarrow L = L_1 \cdot G$, $M = M_1 \cdot G$
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$rLCM(L, M) = K$
$\iff \langle Sol(L), Sol(M) \rangle = \{u + v | Lu = 0, Mv = 0\} = Sol(K)$

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Q.: How one can find out if two given LODOs are similar?

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Landau theorem: for $L = L_1 \cdots L_k = \overline{L}_1 \cdots \overline{L}_r$ we have two maximal chains of ascending left principal ideals
$|L\rangle \subset |L_2 \cdots L_k\rangle \subset |L_3 \cdots L_k\rangle \subset \ldots \subset |L_k\rangle \subset |1\rangle = k[D]$
and $|L\rangle \subset |\overline{L}_2 \cdots \overline{L}_r\rangle \subset |\overline{L}_3 \cdots \overline{L}_r\rangle \subset \ldots \subset |\overline{L}_r\rangle \subset |1\rangle = k[D]$
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L_1 divides L on the right $\Leftrightarrow |L\rangle \subset |L_1\rangle$

Landau theorem: for $L = L_1 \cdots L_k = \overline{L}_1 \cdots \overline{L}_r$ we have two maximal chains of ascending left principal ideals
$|L\rangle \subset |L_2 \cdots L_k\rangle \subset |L_3 \cdots L_k\rangle \subset \ldots \subset |L_k\rangle \subset |1\rangle = k[D]$
and $|L\rangle \subset |\overline{L}_2 \cdots \overline{L}_r\rangle \subset |\overline{L}_3 \cdots \overline{L}_r\rangle \subset \ldots \subset |\overline{L}_r\rangle \subset |1\rangle = k[D]$
$\Longrightarrow k = r$

Jordan-Hölder theorem

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Irreducibility of factors $\Leftrightarrow$ *maximality* of this chain.

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This partially ordered set $\mathfrak{L}$ has the following two properties:

(a) $\forall A, B \in \mathfrak{L}$ one can find a unique $C = \sup(A, B)$:
$C \geq A$, $C \geq B$, and $\forall X \in \mathfrak{L}$, $(X \geq A, X \geq B) \Rightarrow X \geq C$.
Analogously there exist a unique $D = \inf(A, B)$.
$\sup(A, B)$ and $\inf(A, B)$ correspond to the *(left) least common multiple* and the *(right) greatest common divisor* of the corresponding right factors of L.
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If $\forall A, B, C \in \mathfrak{L}$, $(A + B) \cdot C = A \cdot C + B \cdot C$, it is called **distributive lattice**.

RDLs of height 2

Fig. 1

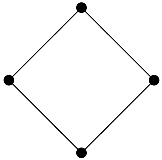

Fig. 2

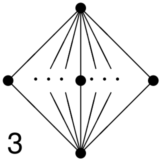

Fig. 3

RDLs of height 2

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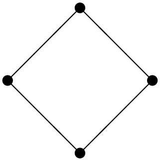

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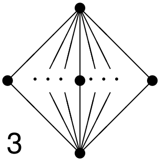

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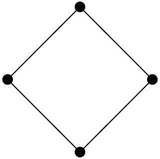

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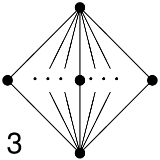

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Definition. $\mathcal{RDL}$ is the class of all possible RDLs (given $\mathbb{F}$).

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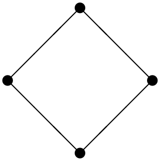
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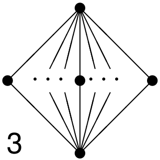
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Definition. $\mathcal{LOEWY}$ is the class of all modular lattices of finite height, such that all subintevals $[A, B] = \{C : A \leq C \leq B\}$ of height 2 are given on Fig. 1–3 (with the constant in $L_1(c)$ in *const*$(\mathbb{F})$).

Note: on Fig. 3 one should fix a correspondence between the elements of height 1 and the constant c in the parametric factorizations $L = L_1(c) \cdot L_2(c)$.

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How large is the class of distributive lattices of height n? How can one describe this class?

Finite distributive lattices

Theorem *There exists a natural functorial isomorphism between the category FDISTR of finite distributive lattices and the category FPOSET of all finite posets. The height n of a distributive lattice from FDISTR is equal to the cardinality of the corresponding element in FPOSET.*

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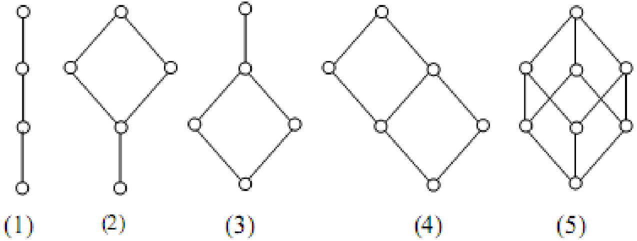

Figure: Distributive lattices of height 3

RDL of Loewy blocks

Definition. *The (first) Loewy block of a LODO L is the lLCM of all possible irreducible right factors of L:*
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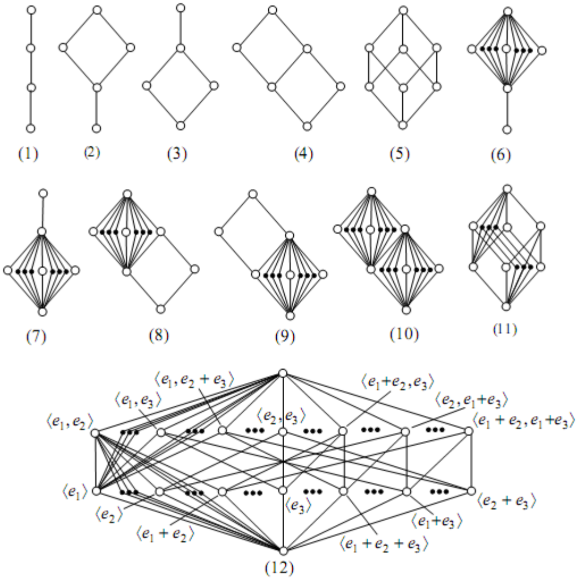
Theorem (A.V.Purgin, 2007) *RDL of the Loewy block is direct product of a finite number of 2-chains (for non-similar divisors) and the lattices of subspaces of k_i-dimensional linear space over const$(\mathbb{F})$.*

RDLs of height 3



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Reformulate lattice-theoretic results about RDLs into the language of factorization of LODOs.