Discrete orthogonal polynomials, asymptotics of solutions of special second-order linear recurrencies with polynomial coefficients, and boundary effects of polynomial filters (in memory of Professor M. Petkovšek)

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Seminar on Computer Algebra, MSU & CCAS
`http://www.ccas.ru/sabramov/seminar/doku.php`

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Theory and details:

S.P.Tsarev, A.A.Kytmanov, `https://arxiv.org/abs/2004.00414`
+ a paper in preparation for JSC.

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Julia code available at:

`https://github.com/sptsarev/high-deg-polynomial-fitting`

Applied Motivation. Object of study

Final GPS Satellite Ephemerides (final orbits) offered by IGS (International GNSS Service) :

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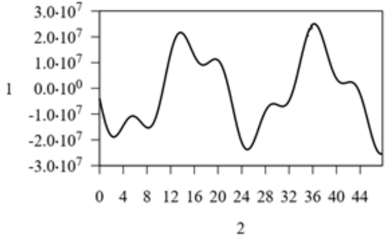

— A typical graph of one of the coordinates of a GPS satellite (terrestrial rotating Cartesian coordinate system). Horizontal axis: time (in hours). Vertical axis: coordinate (in meters).

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More detailed statistics of anomalies for 2010-2018 is also available.

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- This is due to smoothness of the data itself and the relative rarity of these abnormal values.
- It is essential to find stable numerical methods for constructing discrete orthogonal polynomials of high degrees.
- An important effect is theoretically proved — *fast attenuation of the residual of approximation near the boundary of the studied interval*. Experimentally discovered:
  A.F.Nikiforov, M.V.Skachkov, Orthogonal Hahn polynomials in regression models, Matem. Mod., 17:4 (2005), pp. 125–128. (in Russian).

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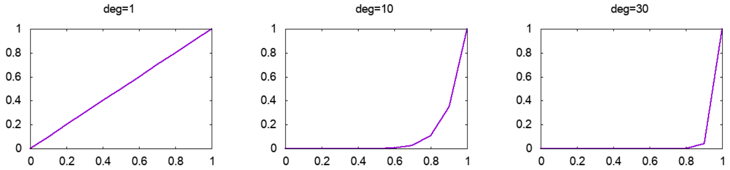

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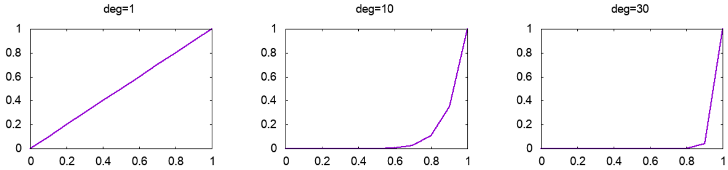

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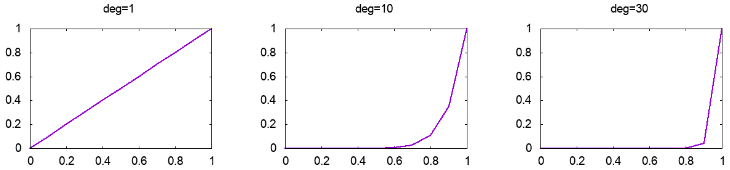

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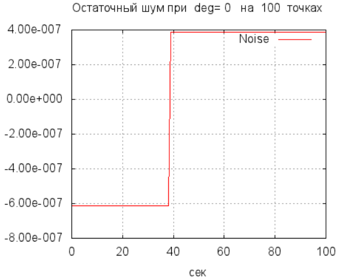

“Stitching” two data series with a small shift = “jump”

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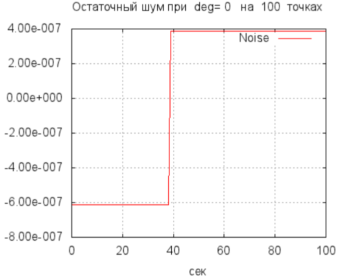
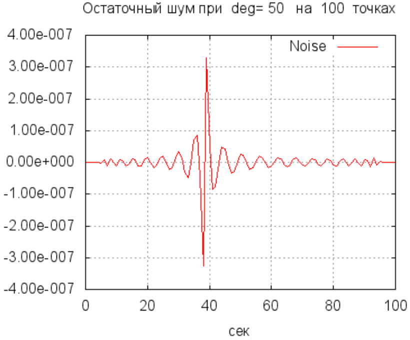
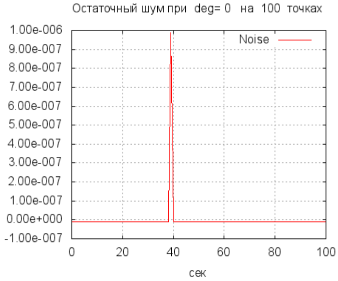


After calculating LSPF (in this case for degree 50) and calculating the difference with the source data we get the approximation residue:



Outlier in a series of smooth data = “spike”

We take the outlier at one point (1 mm) in a series of 100 zero data:



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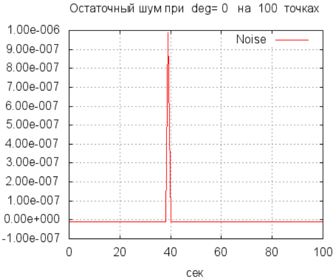
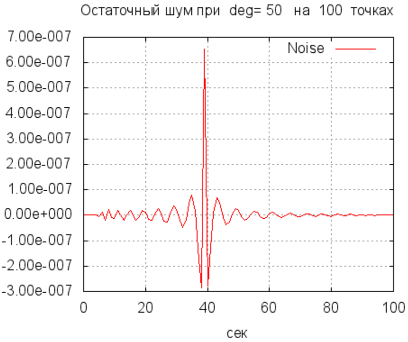
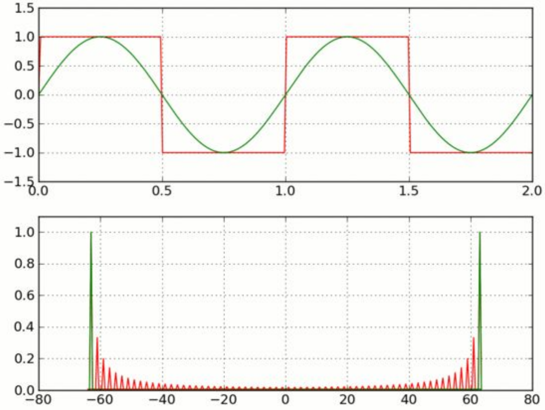


Graph of residuals when approximated by a polynomial of the 50th degree:



Gibbs phenomenon 1

Both cases ("jumps" and "outliers") illustrate the well-known "Gibbs phenomenon", widely known for approximation by the Fourier series:



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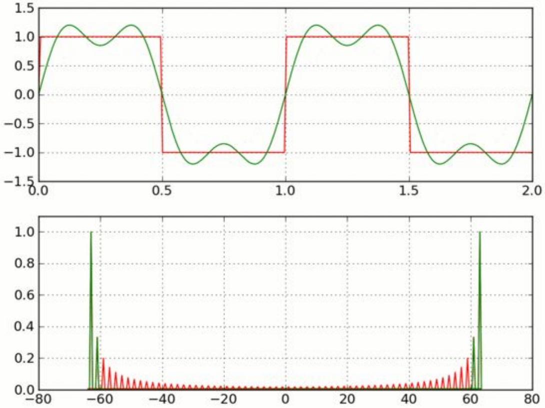

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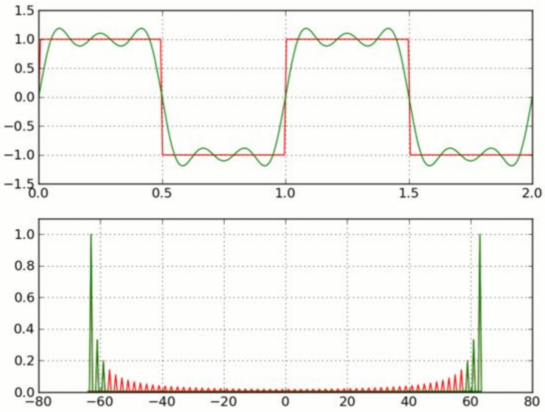

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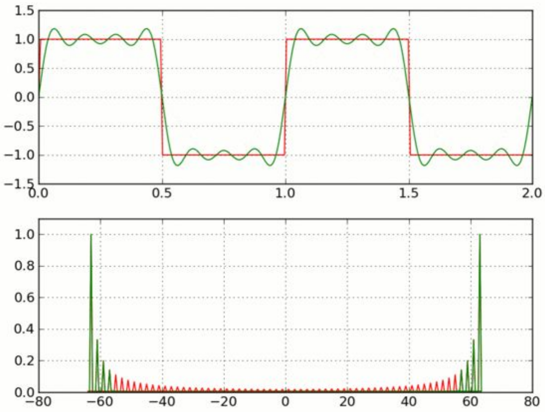

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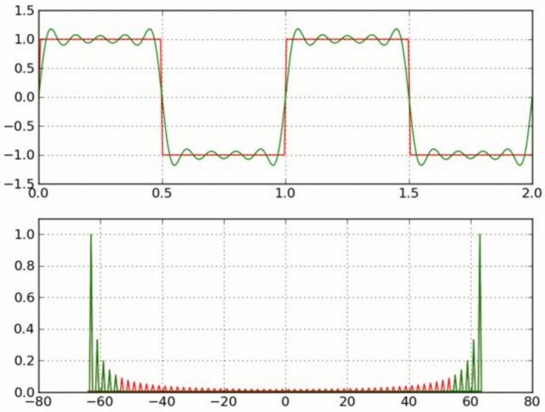

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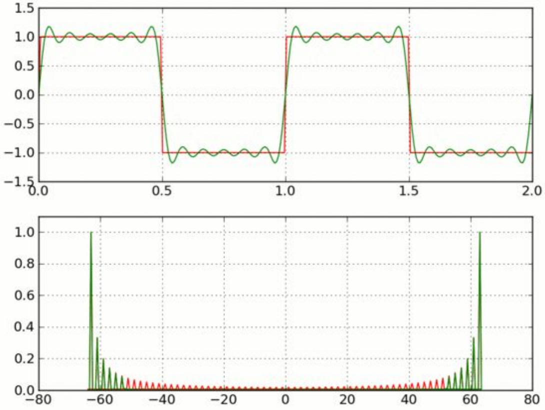

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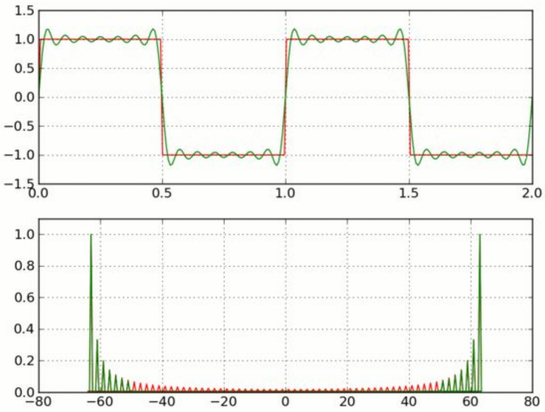

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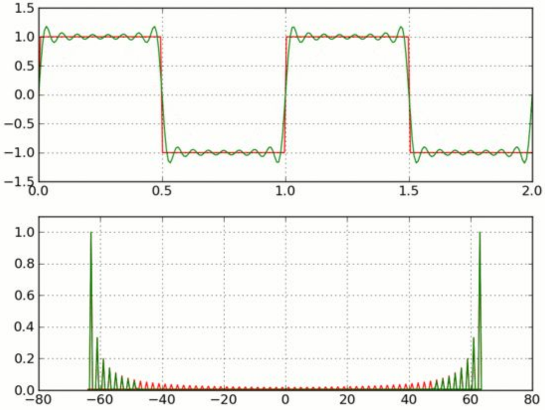

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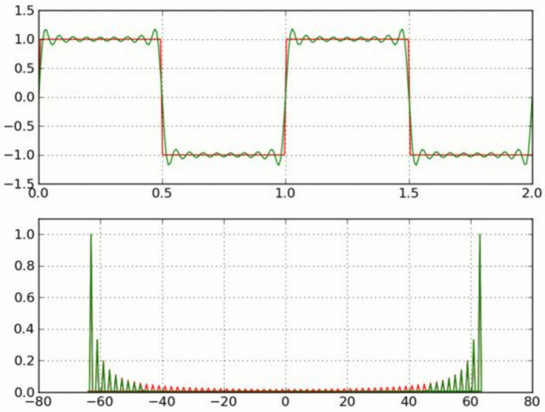

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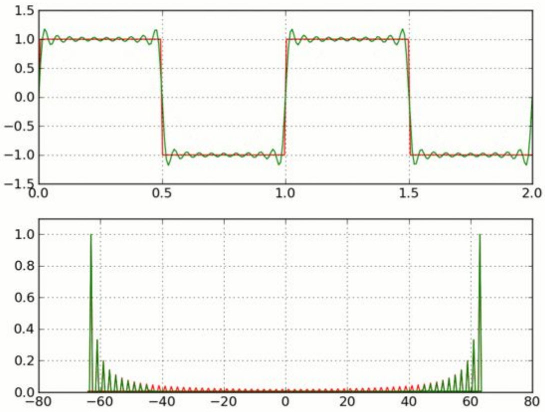
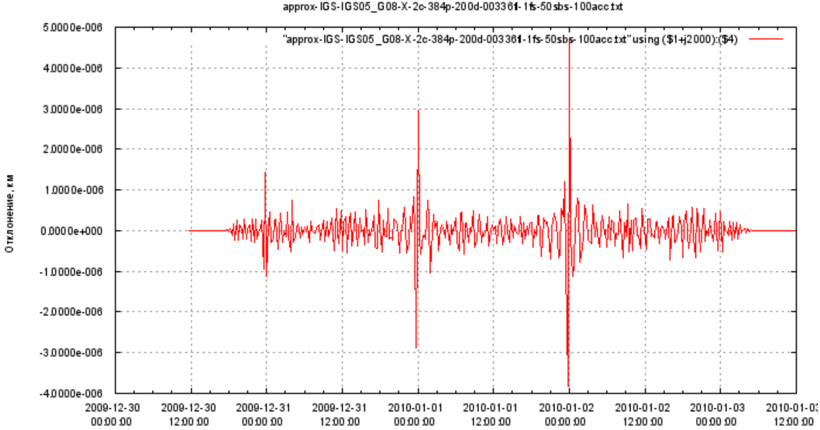
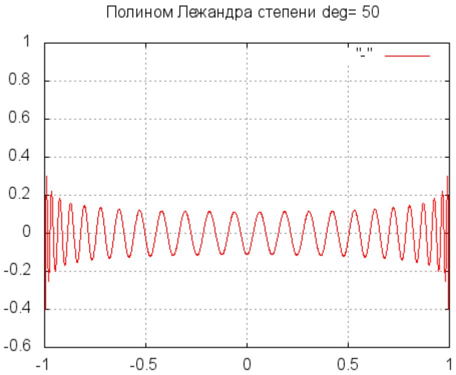


A typical example of jumps of the order of 1 cm at the day boundaries

The approximation residue for LSPF of 200th degree and the X-coordinate of the final orbit of satellite G08 from 12:00:00 30-12-2009 to 12:00:00 03-01-2010 (4 days) at 386 points from SP3 files:



Legendre polynomial of degree 50:



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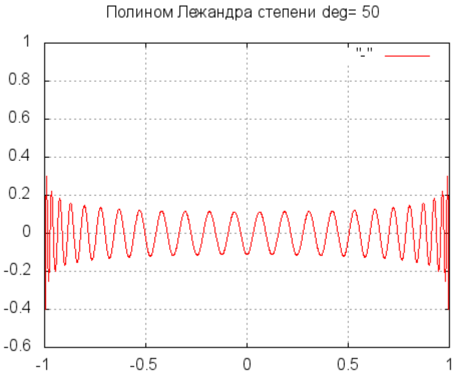
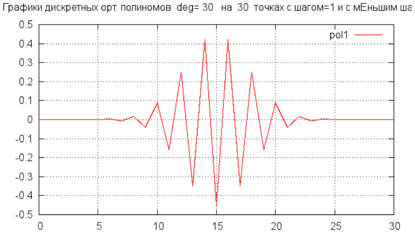


Hahn (Chebyshev) polynomial $p_{30}(x)$ of degree 30 on a lattice of 31 points:



Values near the ends:

x	$p_{30}(x)$
0	$2.9079 \cdot 10^{-9}$
1	$-8.7236 \cdot 10^{-8}$
2	$1.2649 \cdot 10^{-6}$
3	-0.000011806

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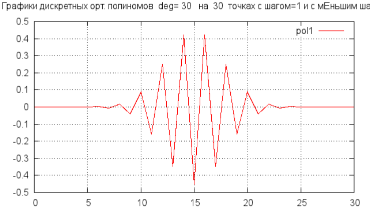

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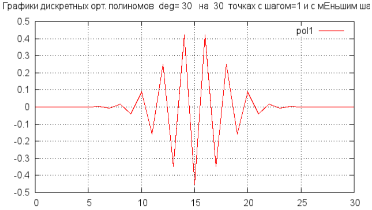
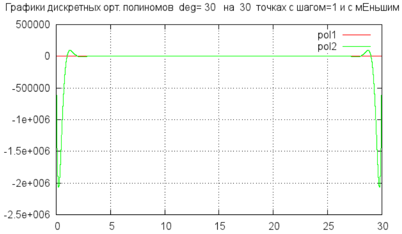
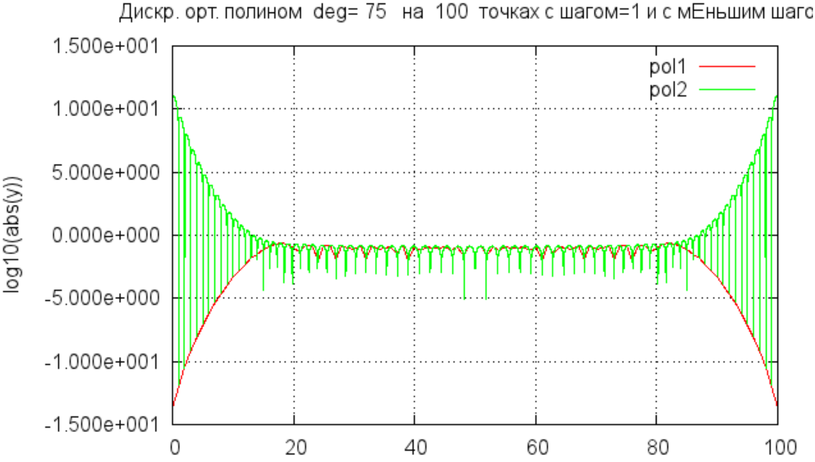


The same polynomial at intermediate points:



Discrete orthogonal polynomials in log-scale

Hahn (Chebyshev) polynomial of degree 75 on a lattice of 101 points:



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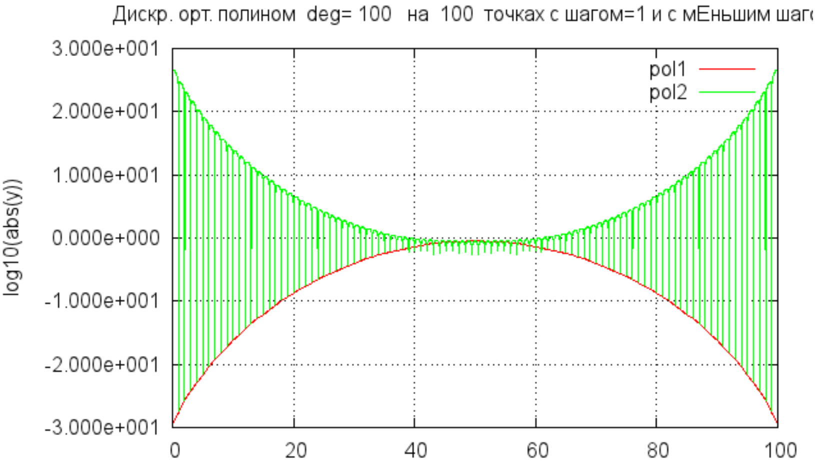
Hahn (Chebyshev) polynomial of degree 100 on a lattice of 101 points:



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$$p_m^N(j) = \sum_{k=0}^{\min\{j,m\}} \frac{(-1)^k (m)_{-k} (m+1)_k (j)_{-k}}{(k!)^2 (N)_{-k}}, \tag{2}$$

where $(a)_k$ are the Pochhammer symbols:

$$(a)_k = \begin{cases} a(a+1)\dots(a+k-1), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ a(a-1)\dots(a+k+1), & \text{if } k < 0. \end{cases} \tag{3}$$

Hahn polynomials: the 2nd-order recurrency etc.

1. *Baik, J., Kriecherbauer, T., McLaughlin, K. D. R., Miller, P. D.* Discrete Orthogonal Polynomials. Princeton University Press, 2007, 170 p.
2. *Olver, F. W. J., Lozier, D. W., Boisvert, R. F., Clark, C. W.* NIST Handbook of Mathematical Functions. Cambridge University Press, New York, NY, 2010.

More general case:

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right), \tag{4}$$

$n = 0, 1, \ldots, N$, with weights $w(x) = \frac{(\alpha+1)_x(\beta+1)_{N-x}}{x!(N-x)!}$ for $x = 0, 1, \ldots, N$.

Hahn polynomials: the 2nd-order recurrency etc.

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We need $\alpha = \beta = 0$ for $w(x) \equiv 1$, then $p_n^N(x) = Q_n(x, 0, 0, N)/||Q_n||$.

Hahn polynomials: the 2nd-order recurrency etc. (2)

With $p_n(x) = Q_n(x; \alpha, \beta, N)$,

$$-xp_n(x) = A_n p_{n+1}(x) - (A_n + C_n)p_n(x) + C_n p_{n-1}(x), \qquad (5)$$

where

$$A_n = \frac{(n+\alpha+\beta+1)(n+\alpha+1)(N-n)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)},$$

$$C_n = \frac{n(n+\alpha+\beta+N+1)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

For a fixed x we have a 2nd-order recurrency in the parameter n. Since we have the known "starting" $p_0(x) = const$, $p_1(x) = ax + b$ for some definite constants, we obtain all (non-normalized!) Hahn polynomials. Their norms are:

$$\sum_{j=0}^{N} (p_n^N)^2 = h_n = \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n (-N)_n N!}$$

Practical construction of Hahn-Chebyshev DOP of high degrees

Possible methods:

- By explicit formulas (2)

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There is a tested code in **Julia** programmimg language.

It is possible to reliably construct Hahn-Chebyshev DOP using standard accuracy (doule, 8-byte) on thousands of points, for degrees up to several hundred.

It is possible to build DOP for lattices with a (slightly) uneven step (missing values of the time series, etc.)

DOP - known asymptotic results

- *Nikiforov A.F., Suslov S.K., Uvarov V.B.*, Classical orthogonal polynomials of a discrete variable. Springer, 1991.
- *Baik, J., Kriecherbauer, T., McLaughlin, K.D.R., Miller, P.D.* Discrete Orthogonal Polynomials. Princeton University Press, 2007, 170 p.
- *Sharapudinov, I.I.* Asymptotic properties of orthogonal Hahn polynomials of a discrete variable, Mat. Sb., (1989), p. 1259-1277. (*Asymptotics is limited by degrees m compared to the number of lattice points N: $m < \alpha\sqrt{N}$*)
- *Aptekarev, A.I., Van Assche, W.* Asymptotics of discrete orthogonal polynomials and the continuum limit of the Toda lattice. Journal of Physics A: Mathematical and General, 2001, v. 34 (48), No.10627.

High-degree DOP behavior near the boundary for equidistant lattices with equal weights

Theorem

Hahn (Chebyshev) polynomials of high degree take values at lattice points that are close to zero when approaching the ends of the lattice.

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Theorem

The roots of high degree Hahn (Chebyshev) polynomials close to the ends of the lattice are separated from the lattice points by a distance of the order of ϵ^2.

(here ϵ is the value of the polynomial at the nearest lattice point).

Is it possible to avoid suppression of anomalies near the boundary using for example:

- DOPs with suitably selected weights w_j:

$$\sum_{j=0}^{N} p_m^N(j) p_s^N(j) w_j = \delta_{ms} \quad ? \tag{6}$$

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From the above resuts on Hahn polynomials it is easy to obtain a general result on suppression of anomalies near the boundaries for arbitrary stable linear polynomial filters.
Wavelets also have boundary effects.

Linear polynomial filters
Example: Hahn-Chebyshev polynomial filters

Let $f_j = f(t_j)$, $j = 0, \ldots, N$ be a time series of (scalar) data.

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Consider its expansion in Hahn-Chebyshev polynomials:

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Make a matrix $M = (m_{jk})$, $0 \leqslant j, k \leqslant N$, with $m_{jk} = p_k^N(j)$. Then

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Taking the first $m+1$ $(m < N)$ coefficients $(c_0, c_1, \dots, c_m)$, we obtain

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— 'smoothed series' f_j
and 'filtered perturbation' = the residue $\widetilde{f} = \bar{f} - \widehat{f}$.

Linear polynomial filter matrix

Denote by D_m the diagonal $(N+1)\times(N+1)$ matrix with first $m+1$ zeros on the diagonal and ones on the rest of the diagonal. Then the 'filtered perturbation' of the time series is

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The fast decay of the Hahn-Chebyshev polynomials near the end points implies the fast decay of the 'filtered perturbation' near the end points for $m \sim N$.

In terms of the filter matrix $F_m = MD_mM^{-1}$ this means *smallness (of the order of ϵ) of its first few and last few rows and columns.*

General linear polynomial filters.
Stable filters

Let $f_j = f(t_j)$, $j = 0, \ldots, N$ be a time series of (scalar) data

General linear polynomial filters. Stable filters

Let $f_j = f(t_j)$, $j = 0, \dots, N$ be a time series of (scalar) data and $P_k^N(t_j)$, $k = 0, \dots, N$ — *any* system of polynomials of degrees $k = 0, \dots, N$, respectively.

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$$M = (m_{jk}), \quad 0 \leqslant j, k \leqslant N \quad \text{c} \quad m_{jk} = P_k^N(t_j),$$

Call the matrix

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a linear polynomial filter matrix of degree m corresponding to the system of polynomials P_k^N.

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a linear polynomial filter matrix of degree m corresponding to the system of polynomials P_k^N.
For the subfamily $\left\{P_k^N\right\}_{k=0}^m$ we obviously have:

$$F_mP_k^N \equiv 0 \quad \text{for} \quad k \leqslant m$$

General linear polynomial filters. Stable filters

So the rows of F_m are orthogonal to $P_k^N(t_j)$, $k \leqslant m$ (with unit weights). Because any polynomial of degree $\leq m$ is a linear combination of $\left\{P_k^N\right\}_{k=0}^m$

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If c_s are not too large, for sufficiently large m the rows of F_m have elements close to zero near the ends.
Since c_s are scalar products of the rows of F_m and the Hahn-Chebyshev polynomials, it is reasonable to introduce the following definition:

Definition

A linear polynomial filter with the matrix F_m is called stable if all the elements of F_m do not exceed some small constant.

General linear polynomial filters. Stable filters

The stability of the filter guarantees:

- filter F_m is practically applicable to data containing small random noise (otherwise small deviations will be amplified by the filter F_m);

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Our main result:

Theorem

Any stable high-degree polynomial filter has the property of suppressing a signal near the ends of the data segment, similar to Hahn-Chebyshev filters.

Note. For unstable linear polynomial filters this statement may not be true. For example, for a monomial basis $P_k^N(t) = t^k$ already for $N = 20$ and $m = 17$ the first and last columns of F_m have elements of the order of 10^9, other elements can reach the order of 10^{14}.

Linear polynomial filters. Extra infos

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The fast decay of several border columns of F_m (this matrix is not symmetric in the general case) guarantees only suppression of the original signal near the edges of the interval, the values of the 'filtered perturbation' $\widetilde{f} = \bar{f} - \widehat{f}$ may not be small near the edge. Example: a stable filter for the basis of discrete Chebyshev polynomials (non-orthogonal!).

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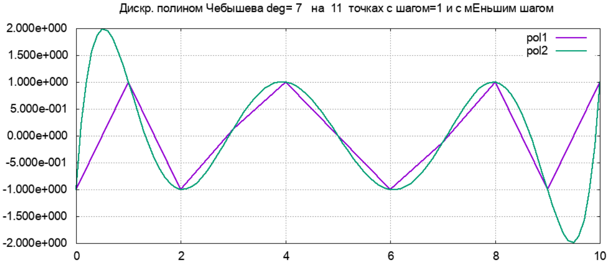
- Is it possible to choose a polynomial basis in such a way that *several border* (not all!) columns of an unstable polynomial filter are not too large?

Digression — discrete Chebyshev polynomials

Discrete Chebyshev polynomials are polynomials constructed for a discrete (finite) lattice and satisfying the *Chebyshev-Markov alternance property* on this lattice:



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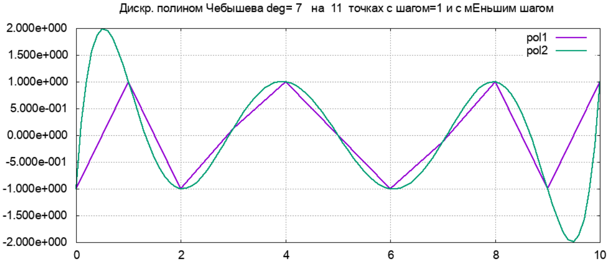

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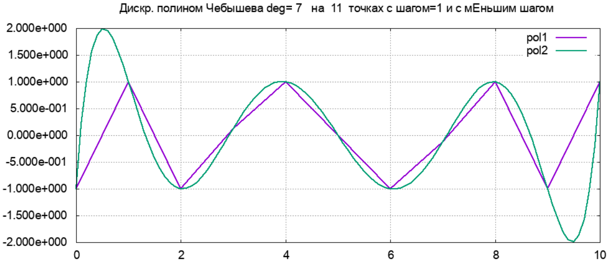

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A remarkable property (apparently not noted earlier): polynomial filters based on them are *stable* — the matrix M^{-1} has elements that do not exceed $1/2$.

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Examples of such stable filters without suppression of small perturbations near the boundary of the domain of the time series:
my talk "The concept of free interpolation for big data: how to increase the accuracy 100 times with a simple formula".

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Linear free extrapolation allows us to identify the required small anomalies near the boundary of the interval!

Nonlinear free interpolation allows us to identify anomalies in the orbits of satellites using just 3 points (adaptability to this class of orbits)!

Thank you for your attention!