Series and Sequences Defined by Quadratic Differential Equations

Seminar on Computer Algebra, CMC faculty of MSU & CCAS

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VERSITÄT

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Outline

1. Introduction
2. The class of δ_2-finite functions
3. Computing quadratic differential equations
4. Normal forms for δ_2-finite power series
5. Guessing

1. Introduction

Motivation

What do Computer Algebra systems give as power series formulas for

$$\tan(z);\ \sec(z);\ \csc(z);$$
$$\mathrm{sech}(z);\ \tanh(z);\ \frac{z}{\exp(z)-1};$$
$$\frac{1}{\log(1+z)};\ \frac{1}{\sin(z)+\cos(z)}, \cot(z)\ldots? \tag{1}$$

We investigate this question and propose an algorithm with implementations in Maple and Maxima.

Motivation

Let us have a look at the current power series formulas given by Maple 2021.1 and Maxima 5.44.

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- Bernoulli numbers

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0,\ n \geqslant 2,\ B_0 = 1 \tag{2}$$

- Euler numbers

$$\sum_{k=0,\ 2|k}^{n} \binom{n}{k} E_k = 0,\ n \geqslant 2,\ E_0 = 1 \tag{3}$$

Problem and Perspective

1. The non-holonomic character of power series defined by the Bernoulli and Euler numbers (see [Stanley, 1980]).

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2. The usual derivative operator $\frac{d}{dz}$ presents quadratic differential equations as an appropriate target of differential equations to consider.

$$\frac{d}{dz}\left(\frac{1}{f(z)}\right) = -\frac{f'(z)}{f(z)^2} \tag{4}$$

$$\frac{d}{dz}\tan(z) = 1 + \tan(z)^2 \tag{5}$$

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3. We extend the computations for holonomic functions to a larger class of functions (see [Teguia Tabuguia, 2020, Chapter 8]).

Perspective

We deduce power series representations in normal forms (see [Geddes et al., 1992]) as for example

```
> FPS(1/log(1+z),z,n)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\, z^{n-1}, A(n+3) = -\frac{(n+1)\,A(n+2) + \left(\sum_{\_k=1}^{n+2} A(\_k)\, A(n+3-\_k)\right)}{n+4}\right],\right.$$
$$\left.\{A(n)\}, \left\{A(0) = 1, A(1) = \frac{1}{2}, A(2) = -\frac{1}{12}\right\}, INFO\right), \tag{6}$$

given by our Maple implementation.

Perspective

Our method consists of three main steps. Given a function f,

Representing non-holonomic power series

1. find a quadratic differential equation (QDE) (can be holonomic) satisfied by f;
2. convert that QDE into a *quadratic recurrence equation* (QRE) for the power series coefficients of f;
3. write the power series in a recursive form with enough initial values.

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3. write the power series in a recursive form with enough initial values.

This process defines a normal function which enables automatic proofs for non-holonomic identities like (see [Geddes et al., 1992, Chapter 3], [Koepf, 2021, Chapter 9])

$$\log\left(\tan\left(\frac{z}{2}\right)+\sec\left(\frac{z}{2}\right)\right)=\operatorname{arcsinh}\left(\frac{\sin(z)}{1+\cos(z)}\right). \tag{7}$$

2. The class of δ_2-finite functions

Quadratic differential equation

Let $\mathbb{K}$ a field of characteristic zero (generally $\mathbb{K} = \mathbb{Q}$), and assume

$$\frac{d^{-1}}{dz} f = f^{(-1)} = 1, \quad \text{and} \quad \frac{d^0}{dz} f = f^{(0)} = f, \tag{8}$$

Definition (Homogeneous quadratic differential equation)

Let d be a non-negative integer. A differential equation of order d in the dependent variable y is said to be homogeneous quadratic over $\mathbb{K}$, if there exist polynomials P_0, P_1,..., P_r, $r \geqslant d$, such that

$$P_r(z) y^{(d)} y^{(d)} + P_{r-1}(z) y^{(d)} y^{(d-1)} + \cdots + P_{r-d-1}(z) y^{(d)} + \cdots + P_1(z) y^2 + P_0(z) y = 0, \tag{9}$$

and $P_r, \ldots, P_{r-d}, P_{r-d-2}, \ldots, P_1$ are not all zero.

δ_2-finite functions

Like for D-finite functions we define δ_2-finite functions to "naturally" extend the class of holonomic functions. The derivative operator to be considered is $\delta_{2,z}$, which computes the product of two derivatives (square included) of f according to the ordering below

$$\begin{array}{ccccc} (1)\ 1, & & & & \\ (2)\ f, & (3)\ f^2, & & & \\ (4)\ f', & (5)\ f'f, & (6)\ (f')^2, & & \\ (7)\ f'', & (8)\ f''f, & (9)\ f''f', & (10)(f'')^2, & \\ (11)\ f''', & (12)\ f'''f, & (13)\ f'''f', & (14)\ f'''f'', & (15)(f''')^2, \\ \ldots & & & & \end{array} \tag{10}$$

δ_2-finite functions

We obtain

$$\delta_{2,z}^{k}(f) = \frac{d^{i-2}}{dz^{j-2}} f \cdot \frac{d^{j-2}}{dz^{i-2}} f, \text{ with } (i,j) = \nu(k), \tag{11}$$

where

$$\nu(k) = (i,j) = \begin{cases} (l,l) & \text{if } N = k \\ (l+1, k-N) & \text{otherwise} \end{cases}, \; l = \left\lfloor \sqrt{2k + \frac{1}{4}} - \frac{1}{2} \right\rfloor, \; N = \frac{l(l+1)}{2}. \tag{12}$$

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Definition (δ_2-finite functions)

A function f is called δ_2-finite if there exist polynomials $P_0, \cdots, P_d$, not all zero, such that

$$P_d(z)\delta_{2,z}^{d+2}(f(z)) + \cdots + P_2(z)\delta_{2,z}^{4}(f(z)) + P_1(z)\delta_{2,z}^{3}(f(z)) + P_0(z)f(z) = 0. \tag{13}$$

3. Computing quadratic differential equations

Computing QDEs

We use the method of ansatz with undetermined coefficients by replacing the usual derivative operator by $\delta_{2,z}$.

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Computing QDEs

1. For each positive integer N,
2. seek rational functions $C_{N-1}, \ldots, C_0$ over $\mathbb{K}$ such that

$$\delta_{2,z}^{N+2}(f(z)) + C_{N-1}(z)\delta_{2,z}^{N+1}(f(z)) + \cdots + C_0(z)f(z) = 0. \tag{14}$$

3. If successful, clear the denominators and deduce the δ_2-finite equation sought.
4. Otherwise increment N and repeat the process until a certain upper bound $N_{\max}$.

QDE for $f(z) := 1/\log(1+z)$

1. $N = 1$, seek $C_0 \in \mathbb{Q}(z)$ such that

$$\delta_{2,z}^3(f(z)) + C_0(z)f(z) = \frac{1}{\log(1+z)^2} + C_0\frac{1}{\log(1+z)} = 0. \tag{15}$$

But $-\delta_{2,z}^3(f(z))/f(z) = -1/\log(1+z) \notin \mathbb{Q}(z)$.

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   But $-\delta_{2,z}^3(f(z))/f(z) = -1/\log(1+z) \notin \mathbb{Q}(z)$.

2. $N = 2$, seek $C_0, C_1 \in \mathbb{Q}(z)$ such that

$$\delta_{2,z}^4(f(z)) + C_1\delta_{2,z}^3(f(z)) + C_0 f(z) = \frac{-1}{((1+z)\log(1+z)^2)} + \frac{C_1}{\log(1+z)^2} + \frac{C_0}{\log(1+z)} = 0, \tag{16}$$

   which is equivalent to

$$\frac{C_0\,(1+z)\log(1+z) + C_1(1+z) - 1}{\log(1+z)^2(1+z)} = 0. \tag{17}$$

   We force the numerator to vanish by equating the coefficient in $\mathbb{Q}(z)[\log(1+z)]$ to zero.

QDE for $f(z) := 1/\log(1+z)$

2. We get

$$\left\{\left(C_0 = 0, C_1 = \frac{1}{1+z}\right)\right\}. \tag{18}$$

3. Thus we have the differential equation

$$\delta_{2,z}^4(y(z)) + \frac{1}{(1+z)}\delta_{2,z}^3(y(z)) = \frac{d}{dz}y(z) + \frac{1}{(1+z)}y(z)^2 = 0, \tag{19}$$

and after clearing the denominators we finally get

$$(1+z)\frac{d}{dz}y(z) + y(z)^2 = 0, \tag{20}$$

with polynomial coefficients.

Computing QDEs

Some computations in Maple

4. Normal forms for δ_2-finite power series

QDE to QRE

Assume the power series of $f(z)$ is given by $\sum_{n=0}^{\infty} a_n z^n$. For any constant x and a non-negative integer k, $(x)_0 = 1$ and $(x)_k = x \cdot (x+1) \cdots (x+k-1)$ denotes the Pochhammer symbol.

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When expanding the left-hand side of the QDE obtained one finds two types of differential monomials:

$$z^p \cdot f^{(j)} \ \text{ and } \ z^p \cdot f(z)^{(i)} \cdot f(z)^{(j)}, \tag{21}$$

where i, j, p are positive integers.

In the D-finite case, the transformation is

$$z^p \cdot f^{(j)} \longrightarrow (n+1-p)_j \cdot a_{n+j-p}. \tag{22}$$

QDE to QRE

For all non-negative integers i, we have

$$f(z)^{(i)} = \sum_{n=0}^{\infty} (n+1)_i \cdot a_{n+i} \cdot z^n, \tag{23}$$

therefore

$$\begin{aligned} f(z)^{(i)} \cdot f(z)^{(j)} &= \left(\sum_{n=0}^{\infty} (n+1)_i \cdot a_{n+i} \cdot z^n\right) \cdot \left(\sum_{n=0}^{\infty} (n+1)_j \cdot a_{n+j} \cdot z^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (k+1)_i \cdot a_{k+i} \cdot (n-k+1)_j \cdot a_{n-k+j}\right) \cdot z^n \end{aligned} \tag{24}$$

by application of the Cauchy product formula which introduces the dummy variable k.

Rewrite rule to convert QDEs into QREs

$$z^p \cdot f(z)^{(i)} \cdot f(z)^{(j)} \longrightarrow \left(\sum_{k=0}^{n-p} (k+1)_i \cdot (n-p-k+1)_j \cdot a_{k+i} \cdot a_{n-p-k+j}\right). \tag{25}$$

QDE to QRE

Some computations in Maxima

Normal forms for δ_2-finite power series

The last step of our computations consists of using the obtained QRE to write the highest order indexed variable in terms of the others. Together with the necessary initial values this defines a normal form for the power series considered.

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Theorem

Given a δ_2-finite function $f(z)$, the following steps

1. *Compute a δ_2-finite equation (QDE) satisfied by $f(z)$;*
2. *Expand the left-hand side of the obtained differential equation and convert it into a QRE using the rewrite rules (22) and (25);*
3. *Use the obtained QRE to write its highest-order indexed variable in terms of the preceding ones with the required initial values;*

define a normal form of the power series representation of $f(z)$.

Power series of $z/(\exp(z)-1)$

The obtained QRE leads to the recursive formula

$$a_{n+3} = -\frac{a_{n+2} + \sum_{k=1}^{n+2} a_k \cdot a_{n+3-k}}{n+4},\; n \geqslant 0,\; a_0 = 1, a_1 = -1/2, a_2 = -1/12, \tag{26}$$

Power series of $z/(\exp(z)-1)$

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$$a_{n+3} = -\frac{a_{n+2} + \sum_{k=1}^{n+2} a_k \cdot a_{n+3-k}}{n+4},\ n \geqslant 0,\ a_0 = 1, a_1 = -1/2, a_2 = -1/12, \tag{26}$$

Using the fact that $a_n = B_n/n!$ and some basic facts about the Bernoulli numbers leads to the formula

$$B_{2n+2} = -\frac{1}{2n+3}\left(2 \cdot \sum_{k=1}^{\lceil n/2 \rceil} \binom{2n+2}{2k} B_{2k} \cdot B_{2(n+1-k)} - \binom{2n+2}{n+1} B_{n+1}^2\right)\ (n \geqslant 1), \tag{27}$$

which is a well-known convolution identity for manipulating Bernoulli numbers.

Normal forms for δ_2-finite power series

Some Maple and Maxima computations

5. Guessing

The “What comes next?” exercise

$$1,\ 2,\ 4,\ 8,\ 16,\ 31,\ 57,\ 99,\ 163,\ ? \tag{28}$$

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$$1,\ 2,\ 4,\ 8,\ 16,\ 31,\ 57,\ 99,\ 163,\ ? \quad (28)$$

Globally, there is no correct answer. It is always possible to find a rule or a justification that allows continuing a sequence by any number. It generally only depends on the complexity of the rule (Wittgenstein). Therefore Grasping a rule is not enough to get the correct interpretation.

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Mathematicians look at this problem differently by defining the principle of **guess and prove**, first propagated by George Polya in 1978. The idea is to make sure after guessing that the inferred rule is correct, by proving that the latter correctly captures the scientific context.

Guess and prove paradigm

The Guess and prove paradigm mainly consists in:

1. generating lots of necessary conditions on the shape of a general *hypothetical solution*;
2. find the general solution instance that satisfies these conditions;
3. verify that this instance meets all the solution requirements of the initial problem.

Applications: Combinatorics, Number Theory, Algebraic Geometry, and many research problems where deducing a general rule from the outcomes becomes important.

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The maximum number of regions, say $R(n)$, a circle is cut into when $n+1$ points on the circumference are connected, is the solution to our introductory sequence.

$$R(0)=1,\ R(1)=2,\ R(2)=4,\ R(3)=8,\ R(4)=16,\ R(5)=31,\ R(6)=57,$$
$$R(7)=99,\ R(8)=163,\ R(9)=256,\ R(10)=386,\ldots \tag{29}$$

Guessing with δ_2-finite functions

By its nice properties, the class of *holonomic functions* is the most used *hypothetical solution* to perform the guess and prove paradigm: the Maple **gfun** package of Salvy and Zimmermann; the Mathematica **GeneratingFunctions** package of Mallinger; or the Sage **ore_algebra** package of Kauers, Jaroschek, and Johansson (see [Kauers et al., 2015]).

However, no specific class is defined when linearity fails the guessing step.

Open question: “Are there more efficient algorithms for intermediate classes, say between *D*-finite and ADE?”– Manuel Kauers, September, 2021.

Maple implementation

```
> convert(z/(exp(z)-1),FormalPowerSeries);# Maple 2022
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\, z^n, A(n+3) = -\frac{\left(\sum_{\_k=1}^{n+2} A(\_k)\, A(n+3-\_k)\right) + A(n+2)}{n+4}\right], \{A(n)\}, \left\{A(0)=1, A(1)=-\frac{1}{2}, A(2)=\frac{1}{12}\right\}, INFO\right) \quad (30)$$

Thank You!

```
> restart
> convert(tan(z),FormalPowerSeries)
```
$$\tan(z) \quad (1)$$

```
> convert(sec(z),FormalPowerSeries)
```
$$\sec(z) \quad (2)$$

```
> convert(csc(z),FormalPowerSeries)
```
$$\csc(z) \quad (3)$$

```
> convert(sech(z),FormalPowerSeries)
```
$$\operatorname{sech}(z) \quad (4)$$

```
> convert(z/(exp(z)-1),FormalPowerSeries)
```
$$\frac{z}{e^z - 1} \quad (5)$$

```
> convert(1/log(1+z),FormalPowerSeries)
```
$$\frac{1}{\ln(z+1)} \quad (6)$$

Link for download: http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm

Computing QDEs

```
> DE1:=FPS:-QDE(tan(z),y(z))
```
$$DE1 := -2 \left(\frac{d}{dz} y(z)\right) y(z) + \frac{d^2}{dz^2} y(z) = 0 \quad (7)$$

```
> dsolve(DE1,y(z))
```
$$y(z) = \frac{\tan\left(\frac{\_C2 + z}{\_C1}\right)}{\_C1} \quad (8)$$

```
> DE2:=FPS:-QDE(sec(z),y(z))
```
$$DE2 := -y(z)^2 - 2 \left(\frac{d}{dz} y(z)\right)^2 + \left(\frac{d^2}{dz^2} y(z)\right) y(z) = 0 \quad (9)$$

```
> dsolve(DE2,y(z))
```
$$y(z) = -\frac{1}{\cos(z)\ \_C2 - \sin(z)\ \_C1} \quad (10)$$

```
> DE3:=FPS:-QDE(tan(z)^k,y(z))
```
$$DE3 := (20 k^2 - 24) \left(\frac{d}{dz} y(z)\right)^2 + 4 k^2 \left(\frac{d^2}{dz^2} y(z)\right) y(z) + 3 (k-2) (k+2) \left(\frac{d^2}{dz^2} y(z)\right)^2 + (-4 k^2 + 6) \left(\frac{d^3}{dz^3} y(z)\right) \left(\frac{d}{dz} y(z)\right) + k^2 \left(\frac{d^4}{dz^4} y(z)\right) y(z) = 0 \quad (11)$$

```
> dsolve(DE3,y(z))
```

$$y(z) = \left(e^{\int \frac{\_f}{\_g(\_f)}\, d\_f + \_C2}\right) \textbf{ where } \left[\left\{\left(\frac{d^2}{d\_f^2}\_g(\_f) + \frac{6k^2-12}{k^2}\right)\_g(\_f)^2 + \frac{\left(\left(\frac{d}{d\_f}\_g(\_f)\right)^2 k^2 + 6\_f\left(\frac{d}{d\_f}\_g(\_f)\right) - 6\_f^2 + 4k^2\right)\_g(\_f)}{k^2} + \frac{-6\_f^4 + 24\_f^2k^2 - 24\_f^2}{k^2} = 0\right\}, \left\{\_f = \frac{\frac{d}{dz}y(z)}{y(z)}, \_g(\_f) = \frac{\frac{d^2}{dz^2}y(z)}{y(z)} - \frac{\left(\frac{d}{dz}y(z)\right)^2}{y(z)^2}\right\}, \left\{z = \int \frac{1}{\_g(\_f)}\, d\_f + \_C1, y(z) = e^{\int \frac{\_f}{\_g(\_f)}\, d\_f + \_C2}\right\}\right] \quad \textbf{(12)}$$

```
> simplify(eval(subs(y(z)=tan(z)^k,lhs(DE3))))
```

$$0 \quad \textbf{(13)}$$

Non-holonomic power series

```
> FPS(tan(z),z,n)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\, z^n, A(n+3) = -\frac{-2A(n+1) + \sum_{\_k=1}^{n}\left(-2(\_k+1)A(\_k+1)A(n+1-\_k)\right)}{(n+2)(n+3)}\right], \{A(n)\}, \{A(0) = 0, A(1) = 1, A(2) = 0\}, INFO\right) \quad \textbf{(14)}$$

```
> FPS(1/(1+sin(z)),z,n,fpstype=quadratic)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\, z^n, A(n+2) = -\frac{-5A(n) + \sum_{\_k=1}^{n-1}\left(-3A(\_k)A(n-\_k)\right)}{(n+1)(n+2)}\right], \{A(n)\}, \{A(0) = 1, A(1) = -1\}, INFO\right) \quad \textbf{(15)}$$

```
> FPS(z/(exp(z)-1),z,n,fpstype=quadratic)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\, z^n, A(n+3) = -\frac{\left(\sum_{\_k=1}^{n+2} A(\_k)\, A(n+3-\_k)\right) + A(n+2)}{n+4}\right], \{A(n)\}, \left\{A(0) = 1, A(1) = -\frac{1}{2}, A(2) = \frac{1}{12}\right\}, INFO\right) \quad \textbf{(16)}$$

```
> print(op(4,%))
```

$$\text{table}\left(\left[input = \frac{z}{e^z - 1}, linear = false, variable = z\right]\right) \quad \textbf{(17)}$$

Simple code for Bernoulli numbers

```
> B:= proc(n) local k; B(n):= if type(n, odd) then 0 else
      -((2*add(binomial(n,2*k)*B(2*k)*B(n-2*k),k=1..ceil((n-2)/4)))
  -binomial(n,n/2)*B(n/2)^2)/(n+1)
      end if end proc:
      B(0):=1: B(1):=-1/2: B(2):=1/6:
> seq(B(k),k=0..20)
```

$$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \frac{43867}{798}, 0, -\frac{174611}{330} \quad \textbf{(18)}$$

```
> seq(bernoulli(k),k=0..20)
```

$$1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, -\frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \frac{43867}{798}, 0, -\frac{174611}{330} \quad \textbf{(19)}$$

```
> B(500)
```

-16596380640568557229852123088077134206658664302806671892352650993155331641220960\ **(20)**
  00840149560881357709214650253239428092078518579928602134637832527454090964 20\
  9325099531654667356754859790348176199837272098442910819081455978296749801598\
  8997624424063374660112070330069832902971048260006971786691722911374979763293\
  0033559794717838407415772796504419464932337498642714226081743688706971990010\
  7342620768812383228675592757482195884044880230345282960230516388584671851732\
  0248388879434272083741373764441076556321322004347739688781289124295233630134\
  4808165757942109887803692579439427973561487863524556256869403384306433922049\
  0783007204803617576807141980442305220157754752870753156688862999789581507566\
  774171800043629814543966136466123270197841417404998354 61 / 8365830

```
> bernoulli(500)
```

-1659638064056855722985212308807713420665866430280667189235265099315533164122096\ **(21)**

0084014956088135770921465025323942809207851857992860213463783252745409096420\
9325099531654667356754859790348176199837272098442910819081455978296749801598\
8997624424063374660112070330069832902971048260006971786691722911374979763293\
0033559794717838407415772796504419464932337498642714226081743688706971990010\
7342620768812383228675592757482195884044880230345282960230516388584671851732\
0248388879434272083741373764441076556321322004347739688781289124295233630134\
4808165757942109887803692579439427973561487863524556256869403384306433922049\
0783007204803617576807141980442305220157754752870753156688862999789581507566\
7741718000436298145439661364661232701978414174049983546 1 / 8365830

Proving non-holonomic identities

```
> f:=log((1+tan(z))/(1-tan(z))); g:=2*arctanh(sin(2*z)/(1+cos(2*z))
  );
```

$$f := \ln\left(\frac{1+\tan(z)}{1-\tan(z)}\right)$$

$$g := 2\,\mathrm{arctanh}\left(\frac{\sin(2\,z)}{\cos(2\,z)+1}\right) \tag{22}$$

```
> FPS(f,z,n,fpstype=quadratic)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\,z^n,\ A(n+4) = -\frac{1}{2\,(n+2)\,(n+3)\,(n+4)}\left(-8\,(n+2)\,A(n+2) + \left(\sum_{\_k=1}^{n} (\_k+1)\,(\_k+2)\,(\_k+3)\,A(\_k+3)\,(n-\_k+2)\,A(n-\_k+2)\right) + \sum_{\_k=1}^{n} \left(-4\,(\_k+1)\,A(\_k+1)\,(n-\_k+2)\,A(n-\_k+2)\right) + \sum_{\_k=1}^{n} \left(-2\,(\_k+1)\,(\_k+2)\,A(\_k+2)\,(n-\_k+2)\,(n+3-\_k)\,A(n+3-\_k)\right)\right)\right],\ \{A(n)\},\ \left\{A(0)=0,\ A(1)=2,\ A(2)=0,\ A(3)=\frac{4}{3}\right\},\ INFO\right) \tag{23}$$

```
> FPS(g,z,n,fpstype=quadratic)
```

$$FPSsol\left(\left[\sum_{n=0}^{\infty} A(n)\,z^n,\ A(n+4) = -\frac{1}{2\,(n+2)\,(n+3)\,(n+4)}\left(-8\,(n+2)\,A(n+2) + \left(\sum_{\_k=1}^{n} (\_k+1)\,(\_k+2)\,(\_k+3)\,A(\_k+3)\,(n-\_k+2)\,A(n-\_k+2)\right) + \sum_{\_k=1}^{n} \left(-4\,(\_k+1)\,A(\_k+1)\,(n-\_k+2)\,A(n-\_k+2)\right) + \sum_{\_k=1}^{n} \left(-2\,(\_k+1)\,(\_k+2)\,A(\_k \right.\right.\right. \tag{24}$$

$$+ 2) (n - \_k + 2) (n + 3 - \_k) A(n + 3 - \_k)) \Big) \Big], \{A(n)\}, \Big\{A(0) = 0, A(1) = 2, A(2) = 0, A(3) = \frac{4}{3}\Big\}, INFO \Big)$$

```
> FPS(f-g,z,n,fpstype=quadratic)
```

$$0 \tag{25}$$

Guessing with δ_2-finite functions (guessing "not nice" sequences)

1- Exponential generating function of Bernoulli numbers

```
> L:=[seq(bernoulli(i)/i!,i=0..12)]
```

$$L := \left[1, -\frac{1}{2}, \frac{1}{12}, 0, -\frac{1}{720}, 0, \frac{1}{30240}, 0, -\frac{1}{1209600}, 0, \frac{1}{47900160}, 0, -\frac{691}{1307674368000}\right] \tag{26}$$

```
> G1:=FPS:-delta2guess(L)
```

$$G1 := \left[\_C \left(\sum_{k=0}^{n} a(k)\, a(n-k)\right) + \_C0 \left(\sum_{k=0}^{n-1} a(k)\, a(n-1-k)\right) + (\_C\, n - \_C)\, a(n) + \_C0\, a(n-2) + (\_C0\, n + \_C - 2\, \_C0)\, a(n-1) = 0, (\_C0\, z + \_C)\, y(z)^2 + \big(-\_C + (-\_C0 + \_C)\, z + \_C0\, z^2\big)\, y(z) + \big(\_C0\, z^2 + \_C\, z\big) \left(\frac{d}{dz}\, y(z)\right) = 0\right] \tag{27}$$

```
> subs([_C=1,_C0=0],G1)
```

$$\left[\left(\sum_{k=0}^{n} a(k)\, a(n-k)\right) + (n-1)\, a(n) + a(n-1) = 0, y(z)^2 + (-1+z)\, y(z) + z \left(\frac{d}{dz}\, y(z)\right) = 0\right] \tag{28}$$

```
> FPS:-FindQRE(z/(exp(z)-1),z,a(n))
```

$$\left(\sum_{\_k=0}^{n} a(\_k)\, a(n-\_k)\right) + (n-1)\, a(n) + a(n-1) = 0 \tag{29}$$

```
> FPS:-QDE(z/(exp(z)-1),y(z))
```

$$y(z)^2 + (-1+z)\, y(z) + z \left(\frac{d}{dz}\, y(z)\right) = 0 \tag{30}$$

2- Exponential generating function of Bell numbers

```
> L:=[seq(combinat:-bell(i)/i!,i=0..21)]
```

$$L := \left[1, 1, 1, \frac{5}{6}, \frac{5}{8}, \frac{13}{30}, \frac{203}{720}, \frac{877}{5040}, \frac{23}{224}, \frac{1007}{17280}, \frac{4639}{145152}, \frac{22619}{1330560}, \frac{4213597}{479001600}, \frac{27644437}{6227020800}, \frac{95449661}{43589145600}, \frac{276591709}{261534873600}, \frac{10480142147}{20922789888000}, \frac{255755771}{1097800704000}, \frac{97439543737}{914624815104000}, \frac{5832742205057}{121645100408832000}, \frac{263898766507}{12412765347840000}, \frac{158289938718917}{1703031405723648000}\right] \quad (31)$$

```
> G2:=FPS:-delta2guess(L)
```

$$G2 := \left[\_C0 \left(\sum_{k=0}^{n-1} (k+1)(k+2) a(k+2) a(n-1-k)\right) - \_C \left(\sum_{k=0}^{n-2} (k+1)(k+2) a(k+2) a(n-2-k)\right) - \_C0 \left(\sum_{k=0}^{n-1} (k+1) a(k+1)(n-k) a(n-k)\right) + \_C \left(\sum_{k=0}^{n-2} (k+1) a(k+1)(n-1-k) a(n-1-k)\right) - \_C0 \left(\sum_{k=0}^{n-1} (k+1) a(k+1) a(n-1-k)\right) + \_C \left(\sum_{k=0}^{n-2} (k+1) a(k+1) a(n-2-k)\right) = 0, \left(-\_C z^2 + \_C0\, z\right) \left(\frac{d^2}{dz^2} y(z)\right) y(z) + \left(\_C z^2 - \_C0\, z\right) \left(\frac{d}{dz} y(z)\right) y(z) + \left(\_C z^2 - \_C0\, z\right) \left(\frac{d}{dz} y(z)\right)^2 = 0\right] \quad (32)$$

```
> simplify(subs([_C=1,_C0=0],G2[2]))
```

$$z^2 \left(-\left(\frac{d^2}{dz^2} y(z)\right) y(z) + \left(\frac{d}{dz} y(z)\right)\left(y(z) + \frac{d}{dz} y(z)\right)\right) = 0 \quad (33)$$

```
> FPS:-QDE(exp(exp(z)-1),y(z))
```

$$\left(\frac{d^2}{dz^2} y(z)\right) y(z) - \left(\frac{d}{dz} y(z)\right) y(z) - \left(\frac{d}{dz} y(z)\right)^2 = 0 \quad (34)$$

3- $\dfrac{\mathrm{sqrt}(1+z)}{1-\log(1+z)}$ series coefficients (from recent slides of Manuel Kauers (September 2021))

$$\frac{\sqrt{z+1}}{1-\ln(z+1)} \quad (35)$$

```
> T:=series(sqrt(1+z)/(1-log(1+z)),z,81):
> L:=[seq(coeff(T,z,i),i=0..80)]:
```

```
> G3:=FPS:-delta2guess(L,degpoly=3)
```

$$G3 := \Bigg[(4\_C0-2\_C)\left(\sum_{k=0}^{n}(k+1)\,a(k+1)\,(n-k+1)\,a(n-k+1)\right)+(6\_C0 \tag{36}$$

$$-4\_C)\left(\sum_{k=0}^{n-1}(k+1)\,a(k+1)\,(n-k)\,a(n-k)\right)-2\_C\left(\sum_{k=0}^{n-2}(k+1)\,a(k+1)\,(n-1$$

$$-k)\,a(n-1-k)\right)-2\_C0\left(\sum_{k=0}^{n-3}(k+1)\,a(k+1)\,(n-2-k)\,a(n-2-k)\right)$$

$$+\_C0\left(\sum_{k=0}^{n-3}(k+1)\,(k+2)\,a(k+2)\,a(n-3-k)\right)+(-2\_C0+\_C)\left(\sum_{k=0}^{n}(k+1)\,(k$$

$$+2)\,a(k+2)\,a(n-k)\right)+(-3\_C0+2\_C)\left(\sum_{k=0}^{n-1}(k+1)\,(k+2)\,a(k+2)\,a(n-1$$

$$-k)\right)+\_C\left(\sum_{k=0}^{n-2}(k+1)\,(k+2)\,a(k+2)\,a(n-2-k)\right)+(-4\_C0+2\_C)\left(\sum_{k=0}^{n}(k$$

$$+1)\,a(k+1)\,a(n-k)\right)+(-2\_C0+2\_C)\left(\sum_{k=0}^{n-1}(k+1)\,a(k+1)\,a(n-1-k)\right)$$

$$+2\_C0\left(\sum_{k=0}^{n-2}(k+1)\,a(k+1)\,a(n-2-k)\right)+\left(\frac{\_C0}{2}-\frac{\_C}{4}\right)\left(\sum_{k=0}^{n}a(k)\,a(n-k)\right)$$

$$-\frac{\_C0\left(\sum_{k=0}^{n-1}a(k)\,a(n-1-k)\right)}{4}=0,\ \left(\frac{1}{2}\_C0-\frac{1}{4}\_C-\frac{1}{4}\_C0\,z\right)y(z)^2+\big(4\_C0$$

$$-2\_C+(6\_C0-4\_C)\,z-2\_C\,z^2-2\_C0\,z^3\big)\left(\frac{d}{dz}y(z)\right)^2+\big(-2\_C0+\_C+($$

$$-3\_C0+2\_C)\,z+\_C\,z^2+\_C0\,z^3\big)\left(\frac{d^2}{dz^2}y(z)\right)y(z)+\big(-4\_C0+2\_C+(-2\_C0$$

$$+ 2\_C) z + 2\_C0 z^2) \left(\frac{d}{dz} y(z)\right) y(z) = 0$$

```
> subs([_C=1,_C0=0],G3[2])
```

$$-\frac{y(z)^2}{4} + \left(-2 z^2 - 4 z - 2\right) \left(\frac{d}{dz} y(z)\right)^2 + \left(z^2 + 2 z + 1\right) \left(\frac{d^2}{dz^2} y(z)\right) y(z) + (2 + 2 z) \left(\frac{d}{dz} y(z)\right) y(z) = 0 \quad \textbf{(37)}$$

```
> FPS:-QDE(sqrt(1+z)/(1-log(1+z)),y(z))
```

$$-y(z)^2 + (8 z + 8) \left(\frac{d}{dz} y(z)\right) y(z) - 8 (z + 1)^2 \left(\frac{d}{dz} y(z)\right)^2 + 4 (z + 1)^2 \left(\frac{d^2}{dz^2} y(z)\right) y(z) = 0 \quad \textbf{(38)}$$

(%i1) `powerseries(tan(z),z,0);`

(%o1) $$\sum_{i1=0}^{inf}\left(\frac{(-1)^{i1-1}\left(2^{2\,i1}-1\right)2^{2\,i1}\,bern(2\,i1)\,z^{2\,i1-1}}{(2\,i1)!}\right)$$

(%i2) `powerseries(sec(z),z,0);`

(%o2) $$\sum_{i2=0}^{inf}\left(\frac{(-1)^{i2}\,euler(2\,i2)\,z^{2\,i2}}{(2\,i2)!}\right)$$

(%i3) `powerseries(csc(z),z,0);`

(%o3) $$2\sum_{i3=0}^{inf}\left(\frac{(-1)^{i3-1}\left(2^{2\,i3-1}-1\right)bern(2\,i3)\,z^{2\,i3-1}}{(2\,i3)!}\right)$$

(%i4) `powerseries(sech(z),z,0);`

(%o4) $$\sum_{i4=0}^{inf}\left(\frac{euler(2\,i4)\,z^{2\,i4}}{(2\,i4)!}\right)$$

(%i5) `powerseries(z/(exp(z)-1),z,0);`

(%o5) $$powerseries\left(\frac{z}{\%e^{z}-1},z,0\right)$$

(%i6) `powerseries(1/(log(1+z),z,0));`

expt: undefined: 0 to a negative exponent.
-- an error. To debug this try: debugmode(true);

link for download:

http://www.mathematik.uni-kassel.de

(%i7) `batchload(FPS);`

Download at http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/F

(%o7) *C:/Users/bertr/maxima/FPS.mac*

1 The delta_{2,z} derivative operator

(%i8) `delta2diff(f(z),z,5);`

(%o8) $$f(z)\left(\frac{d}{d\,z}f(z)\right)$$

```
(%i9) delta2diff(f(z),z,6);
```

(%o9) $\left(\frac{d}{d\,z}\,f(z)\right)^2$

```
(%i10) delta2diff(f(z),z,14);
```

(%o10) $\left(\frac{d^2}{d\,z^2}\,f(z)\right)\left(\frac{d^3}{d\,z^3}\,f(z)\right)$

2 QDE to QRE

```
(%i11) FindQRE(tan(z),z,a[n]);
```

(%o11) $(n+1)\,(n+2)\,a_{n+2}-2\sum_{k=0}^{n}\left((k+1)\,a_{k+1}\,a_{n-k}\right)=0$

```
(%i12) FindQRE(z/(exp(z)-1),z,a[n]);
```

(%o12) $\left(\sum_{k=0}^{n}(a_k\,a_{n-k})\right)+(n-1)\,a_n+a_{n-1}=0$

```
(%i13) FindQRE(log(1+sin(z)),z,a[n]);
```

(%o13) $\left(\sum_{k=0}^{n}\left((k+1)\,(k+2)\,a_{k+2}\,(n-k+1)\,a_{n-k+1}\right)\right)+(n+1)\,(n+2)\,(n+3)\,a_{n+3}=0$

3 Computing truncated series

```
(%i14) taylor(sec(z),z,0,7);
```

(%o14)/T/ $1+\frac{z^2}{2}+\frac{5\,z^4}{24}+\frac{61\,z^6}{720}+...$

```
(%i15) QTaylor(sec(z),z,0,7);
```

(%o15) $\frac{61\,z^6}{720}+\frac{5\,z^4}{24}+\frac{z^2}{2}+1$

```
(%i16) taylor(tan(z),z,0,7);
```

(%o16)/T/ $z+\frac{z^3}{3}+\frac{2\,z^5}{15}+\frac{17\,z^7}{315}+...$

(%i17) QTaylor(tan(z),z,0,7);

(%o17) $$\frac{17\,z^{7}}{315}+\frac{2\,z^{5}}{15}+\frac{z^{3}}{3}+z$$