Some computational aspects of polynomials

Doru Ştefănescu
University of Bucharest

Seminar of Computer Algebra
Moscow State University
October 23, 2019

Outline

- Bounds for polynomial roots
- Dominant roots
- Bounds for positive roots of polynomials
- Jean Jacques Bret (1781–1819)
- Petre Sergescu (1893–1954)
- New bounds

Bounds for polynomial roots

- There are known many bounds for the absolute values of univariate polynomials with complex coefficients.
- A natural question is to know how far such bounds are from the true bounds. If P is such a polynomial we denote by $B(P)$ a bound for the absolute values of the roots and by $T(P)$ the true bound for the moduli of the roots.
- A first problem: What is the relation between $B(P)$ and $T(P)$?

Bounds for polynomial roots (contd.)

- We know, by the Fundamental Theorem of Algebra, that any nonconstant univariate polynomial has complex roots and that their number, counted with multiplicities, is equal to the degree.
- The computation of accurate bounds for univariate complex and real polynomial is a serious challenge.
- A key step in the numerical computation of the roots is their location, i.e. the computation of bounds for the moduli of the roots.

Dominant roots

- A root α of a univariate polynomial with complex coefficients is called *dominant* if $|\alpha| \geq |\beta|$ for any other root β.
- The dominant roots was considered by Newton (1707) in his *Arithmetica Universalis*, no. 133–137. His idea was deleloped by Daniel Bernoulli (1728), who used linear recurrent sequences for approaching the dominant roots.
- The method of Bernoulli was improved by Jacobi (1834) and another approach was proposed by Dandelin (1826), Lobatchevsky (1834) and Graeffe (1833, 1837).

The Method of Dandelin–Lobatchevsky–Graeffe

The method of Graeffe was introduced independently by Dandelin (1826), Graeffe (1833, 1837) and Lobatchevskiĭ (1834). In the restricted version there are used the polynomials $F_n(X) = \mathrm{Res}_Y\big(P(Y), Y^{2^n} - X\big)$.

For $n = 2$, we consider

$$P(x) \;=\; a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_d),$$

$$(-1)^d\, P(-x) = a_0(x+\alpha_1)(x+\alpha_2)\cdots(x+\alpha_d),$$

and

$$F_2(x) := (-1)^d F(-x)F(x) = a_0^2(x^2-\alpha_1^2)(x^2-\alpha_2^2)\cdots(x^2-\alpha_d^2)\,.$$

Graeffe (contd.)

If

$$|\alpha_1| \gg |\alpha_2| \gg \cdots \gg |\alpha_d|$$

then

$$|\alpha_j| \sim \sqrt{\frac{|a_j^{(2)}|}{|a_{j-1}^{(2)}|}}\,.$$

Graeffe (contd.)

Note that if

$$|\alpha_1| \gg |\alpha_2| \gg \cdots \gg |\alpha_d|, \qquad \text{we have}$$

$$|\alpha_1| \sim |\alpha_1| \cdot \left|1 + \frac{\alpha_2}{\alpha_1} + \cdots + \frac{\alpha_d}{\alpha_1}\right| = \left|\frac{a_1}{a_0}\right|,$$

$$|\alpha_1\alpha_2| \sim |\alpha_1\alpha_2| \cdot \left|1 + \frac{\alpha_1\alpha_3}{\alpha_1\alpha_2} + \cdots + \frac{\alpha_{d-1}\alpha_d}{\alpha_1\alpha_2}\right| = \left|\frac{a_2}{a_0}\right|$$

$$\vdots$$

so $$|\alpha_j| \sim \left|\frac{a_j}{a_{j-1}}\right|.$$

Graeffe (contd.)

If we continue the process we find

$$\begin{aligned} F_n(x) &:= a_0^{2^{n-1}}(x-\alpha_1^{2^{n-1}})(x-\alpha_2^{2^{n-1}})\cdots(x-\alpha_d^{2^{n-1}}) \\ &= \sum a_j^{(n)} x^{d-j} \end{aligned}$$

Graeffe (contd.)

Taking $F(x) = 2x^3 - 7x^2 - 13x + 16$, we have
$F_{(2)}(x) = 4x^3 + 101x^2 + 393x + 256$
$F_{(3)}(x) = 16x^3 + 7057x^2 + 102700x + 65540$
$F_{(4)}(x) = 256x^3 + 4651 \cdot 10^4\, x^2 + 9622 \cdot 10^6 x + 4295 \cdot 10^6$
and the estimates

$$
\begin{aligned}
|\alpha_1| &= \left(\frac{46510000}{256}\right)^{1/8} &\sim 4.544 \\
|\alpha_2| &= \left(\frac{9622000000}{46510000}\right)^{1/8} &\sim 1.947 \\
|\alpha_3| &= \left(\frac{4295000000}{9622000000}\right)^{1/8} &\sim 0.904
\end{aligned}
$$

which are very closed to true values of the roots.

Graeffe (contd.)

In the general case, the method of Graeffe is based on the study of the sequence $(P_n)_n$ of polynomials associated to P, where

$$P_n(X) = \mathrm{Res}_Y\big(P(Y), Y^n - X\big).$$

Let

$$P_n(X) = \sum_{i=0}^{d} a_i^{(n)} X^{d-i} \quad \text{for all} \quad n \in \mathbb{N},$$

with the convention $P_1(X) = P(X) = \sum_{i=0}^{d} a_i X^{d-i}$.

Both approaches use relations between the roots and the coefficients of the polynomial P. The simplest case is that of a unique dominant root α_1 whose absolute value is "larger enough" than the modules of the other roots. In this case $|\alpha_1| \sim |a_1/a_0|$.

Computation of complex polynomial zeros

- Since the exact computation of the zeros in function of the coeffients of the polynomial is not possible for general polynomials, for all practical purposes it is useful to handle efficient methods for their estimation.
- There exist several bounds for the absolute values of the roots of a univariate polynomial with complex coefficients.
- Bounds for complex roots were obtained, among other, by Cauchy, Kuniyeda, Fujiwara a.o.
- These bounds are expressed as functions of the degree and of the coefficients, and naturally they can be used also for the roots (real or complex) of polynomials with real coefficients.

Bounds for complex roots

If $z \in \mathbb{C}$ is a root of the polynomial $P \in \mathbb{C}[X] \setminus \mathbb{C}$ one searches for positive numbers s_0, r_0 such that

$$s_0 \leq |z| \leq r_0 .$$

The first significant result was obtained by Cauchy:

The criterion of Cauchy

Theorem (A.–L. Cauchy)

All the roots of the nonconstant polynomial

$$P(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{C}$$

are contained in the disk $|z| \leq \xi$, *where* ξ *is the unique positive solution of the equation*

$$|a_n| X^n = |a_0| + |a_1| X + \cdots + |a_{n-1}| X^{n-1} . \tag{1}$$

Kuniyeda

Among other estimates for r_0 we mention the results of Kuniyeda and Fujiwara:

Theorem (M. Kuniyeda, 1916)

If and $p, q > 0$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then all the roots of the polynomial P are contained in the disk $|z| \leq \xi$, where

$$\xi = \left(1 + \left(\sum_{j=0}^{n-1} \left|\frac{a_j}{a_n}\right|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}.$$

Fujiwara

Theorem (M. Fujiwara, 1926)

If $\lambda_1, \ldots, \lambda_d \in (0, \infty)$ *and*

$$\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_d} = 1,$$

then all the roots of the polynomial P *are contained in the disk* $|z| \leq \xi$, *where*

$$\xi = \max_{1 \leq k \leq d} \left(\lambda_k \left| \frac{a_{n-d}}{a_n} \right| \right)^{\frac{1}{k}}.$$

Fujiwara (contd.)

- One of the most efficient bound for complex roots, obtainded by Fjiwara, is the following

$$Fw(P) = 2 \cdot \max \left| \frac{a_{d-i}}{a_d} \right|^{1/i}$$

- In fact, the previous bound is a general one for complex roots.
- However it is closely related to a bound for real roots proposed by Lagrange.

Computation of real roots

Bounds for *real roots* can be derived from bounds for *omplex Roots*. Lagrange obtained two upper bounds for positive real roots. One of them is surprisingly efficient. New bounds for positive roots were obtained by Kioustelidis (1986), H. Hoon (1998) and D. Ştefănescu (2005).

Newton

Let

$$P(X) = X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_n \in \mathbb{R}[X]\, n \quad n \geq 2\,.$$

If P is hyperbolic (all the roots are real), the number

$$Nw(P) = \sqrt{a_1^2 - 2a_2}$$

is an upper bound for the roots.

(In *Arithmetica Universalis*, 1707)

Lagrange 1

Theorem (Lagrange, 1769)

Let
$P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$,
with all $a_i \geq 0$, $a_0, a_{m+1} > 0$. *Let*

$$A = \max\left\{a_i \,;\, \text{coeff}\,(X^{d-i}) < 0\right\}.$$

The number

$$1 + \left(\frac{A}{a_0}\right)^{1/(m+1)}$$

is an upper bound for the positive roots of P.

Lagrange 1 (Proof)

We remind that A is the largest absolute value of negative coeffcients. So we have, for all $x > 1$,

$$\begin{aligned} f(x) &\geq a_0x^d - A(x^{d-m-1} + \cdots + x + 1) \\ &= a_0x^d - A\frac{x^{d-m} - 1}{x-1} \\ &> a_0x^d - A\frac{x^{d-m}}{x-1} \\ &= x^d\left(1 - \frac{A}{a_0x^m(x-1)}\right) \\ &= x^d\left(1 - \frac{A}{a_0(x-1)^{m+1}}\right) . \end{aligned}$$

But the last paranthesis is positive for $(x-1)^{m+1} > A/a_0$, so $1 + A^{1/(m+1)}$ is an upper bound for the positive roots of the polynomial f .

Longchamp

Theorem (Longchamp)

Let
$P(X) = a_0X^d + \cdots + a_mX^{d-m} - a_{m+1}X^{d-m-1} \pm \cdots \pm a_d \in \mathbb{R}[X]$,
with all $a_i \geq 0$, $a_0, a_{m+1} > 0$ *and*

$$A = \max\left\{a_i \,;\, \text{coeff}\,(X^{d-i}) < 0\right\}.$$

The number

$$L_2 = 1 + \frac{A}{a_0 + a_1 + \cdots + a_m}$$

is an upper bound for the positive roots of P.

Bret (1781–1819)

Doru Ştefănescu
University of Bucharest

- Professor at the Univeristy of Grenoble
- Scientific papers in *Annales de Gergonne*
- Bounds for roots of numerical equations (1815)

First theorem of Bret

THÉORÈME I. En ajoutant successivement à l'unité une suite de fractions ayant pour numérateurs les coefficiens négatifs d'une équation proposée, pris positivement, et pour dénominateurs la somme de tous les coefficiens positifs qui les précèdent respectivement, le plus grand des nombres résultans pourra être pris pour limite supérieure des racines de cette équation.

Il est entendu au surplus que, dans la pratique, il suffira de considérer le plus grand coefficient dans chacune des séries de termes négatifs.

First theorem of Bret (contd.)

Adding scucessively to the unity a sequence of fractions having as numerators the negative coeffcients of the proposed equation, taken positively, and for denominators the sum of the positive coeeficients preceding them, the largest of the resulting numbers can be taken as an upper bound for the roots of this equation.

First theorem of Bret

$$2x^7 + 11x^4 - 10x^5 - 26x^4 + 31x^3 + 72x^2 - 230x - 348 = 0$$

The upper bound is the largest of the numbers

$$1 + \frac{26}{2+11} \quad \text{or} \quad 1 + \frac{26}{13}$$

$$1 + \frac{348}{2+11+31+72} \quad \text{or} \quad 1 + \frac{348}{116}$$

so the an upper bound is $\max\{3, 4\} = 4$.

Remark: The true upper bound 2.03.

Second theorem of Bret

THÉORÈME II. Si, après avoir divisé successivement chacun des coefficiens négatifs d'une équation par le coefficient du premier terme, on extrait de chaque quotient une racine dont le degré soit le nombre des termes positifs qui précèdent le coefficient négatif dont il s'agit, le plus grand des nombres qu'on obtiendra en augmentant chacune de ces racines d'une unité pourra être pris pour limite supérieure des racines de l'équation proposée.

Third theorem of Bret

THÉORÈME III. En ajoutant successivement à un nombre entier positif arbitraire une suite de fractions ayant successivement pour numérateurs les coefficiens négatifs d'une équation proposée, pris positivement, et pour dénominateurs la somme des produits des coefficiens positifs qui les précèdent respectivement, de droite à gauche, par les puissances successives du nombre arbitraire, à partir de sa puissance zéro ou de l'unité ; le plus grand des nombres résultans pourra être pris pour limite supérieure des racines de cette équation.

A theorem of Bret

Let

$$\begin{aligned} f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{i-1}x^{n-i+1} \\ &\quad -a_ix^{n-i} + \cdots + a_{j-1}x^{n-j+1} \\ &\quad -a_jx^{n-j} + \cdots \pm a_n \end{aligned}$$

be a polynomial with real coefficients which has only v negative coefficients $-a_i, -a_j, \ldots, -a_s, -a_t$.
J.-J. Bret (1815) obtained upper bounds for the positive roots of $f(x)$. For example, such a bound is the largest number in the sequence

$$1 + \frac{a_i}{a_0 + a_1 + \cdots + a_{i-1}}, 1 + \frac{a_j}{a_0 + a_1 + \cdots + a_{j-1}},$$

$$\ldots, 1 + \frac{a_t}{a_0 + a_1 + \cdots + a_{t-1}}.$$

Sergescu (1893–1954)

Petre Sergescu

- Professor at the University of Cluj
- Organizer of the first three Congresses of Romanian Mathematicians
- History of Mathematics
- Analytic functions
- Geometry of polynomials

Petre Sergescu (1893–1954) obtained several results on the location of polynomial roots. His main achievments were results on:

- Bounds for the absolute values of the roots of univariate polynomials with complex coefficients.
- Upper bounds for positive roots of univariate polynomials with real coefficients.

On Bret's bounds for positive roots (contd.)

- Sergescu examined the bounds of Bret, gave complete proofs for them, and obtained generalizations.
- His formulas are of the type $1 + K_{i,p}$, where $K_{i,p}$ is expressed using binomial coefficients, for example

$$\left(\frac{a_i}{\binom{i}{p-1} a_0 + \binom{i-1}{p-2} a_1 + \cdots + \binom{i-p+1}{0} a_{p-1}} \right)^{\frac{1}{i-p+1}} .$$

Petre Sergescu GM 1940

Zentralblatt MATH 1931 – 2005

Zbl. Math. 66.0061.02

Sergescu, P.
Über die Grenzen der absoluten Werte der Wurzeln algebraischer Gleichungen. (Romanian)
Gaz. mat., Bucureşti, 45, 512-515. (1940)

Ist

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{i-1}x^{n-i+1} - a_ix^{n-i} + \cdots + a_{j-1}x^{n-j+1} - a_jx^{n-j} + \cdots \pm a_n = 0$$

eine Gleichung mit den negativen Koeffizienten $-a_i, -a_j, \ldots, -a_s, -a_t$, und ist A die größte der Zahlen $a_i, a_j, \ldots, a_t$, so ist nach Mac Laurin $1 + \dfrac{A}{a_0}$ eine obere Grenze der positiven Wurzeln der Gleichung $f(x) = 0$. *J. J. Bret* (Ann. math. pures appl. 6 (1815), 112-122) betrachtet die Folge

$$1 + \frac{a_i}{a_0 + a_1 + \cdots + a_{i-1}}, \quad 1 + \frac{a_j}{a_0 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{j-1}}, \ldots$$

$$1 + \frac{a_t}{a_0 + \cdots + a_{i-1} + a_{i+1} + \cdots + a_{j-1} + a_{j+1} + \cdots + a_{t-1}},$$

in der die Zähler der Brüche die absoluten Werte der negativen Koeffizienten und die Nenner die Summen der positiven Koeffizienten sind, die dem im Zähler stehenden Koeffizienten vorangehen. Der größte Wert in dieser Folge ist eine obere Grenze der positiven Wurzeln der Gleichung $f(x) = 0$. Verf. gibt einige Anwendungen des Satzes von Mac Laurin und einen neuen Beweis des Satzes von Bret. (Data of JFM: JFM 66.0061.02; Copyright 2004 Jahrbuch Database used with permission)

Zacharias, M.; Prof. (Berlin), [ZBL]
Published: **1940**
Document Type: **J**

Jahrbuch Database – Electronic Research Archive for Mathematics

JFM 66.0062.01

Sergescu, P.
Généralisations des limites de J. J. Bret. (French)
Acad. Roumaine, Bull. Sect, Sci., 22, 460-465.

Verf. betrachtet die Teilpolynome $g_1(x), g_2(x), \ldots, g_\nu(x)$ von

$$f(x) \equiv a_0x^n + \cdots + a_{i-1}x^{n-i+1} - a_ix^{n-i} + \cdots - a_jx^{n-j} + \cdots \pm a_n,$$

die mit a_0x^n anfangen und mit $-a_ix^{n-i}, -a_jx^{n-j}, \ldots$ endigen ($a_\nu \geqq 0, \nu = 0, 1, \ldots, n$) In diesen Polynomen wird teils x durch $1+y$ ersetzt, so daß Polynome kleineren Grades in x mit von y abhängigen Koeffizienten erzeugt werden. Es wird zunächst für y eine obere Schranke gefunden, damit die Teilpolynome sukzessiv positiv werden, und durch elementare Rechnung ergibt sich der Satz: Sind $p, q, r \ldots, u$ ganze positive Zahlen, welche den Ungleichungen $p \leqq i, p+q \leqq j, \ldots, p+q+\cdots+u \leqq t$ genügen, so wird die größte der Zahlen

$$1 + \left(\frac{a_i}{C_i^{p-1}\overset{+}{a}_0 + C_{i-1}^{p-2}\overset{+}{a}_1 + \cdots + C_{i-p+1}^0\overset{+}{a}_{p-1}}\right)^{\frac{1}{i-p+1}},$$

$$1 + \left(\frac{a_j}{C_{j-p}^{q-1}(\overset{+}{a}_0 + \cdots + \overset{+}{a}_p) + C_{j-p-1}^{q-2}\overset{+}{a}_{p+1} \cdots + C_{j-p-q+1}^0\overset{+}{a}_{p+q-1}}\right)^{\frac{1}{j-p-q+1}}, \ldots$$

eine obere Schranke der positiven Wurzeln von $f(x)$. Dabei gilt $\overset{+}{a}_\sigma = a_\sigma$ für $\sigma \neq i, j, \ldots, t$; sonst $\overset{+}{a}_\sigma = 0$, wobei $-a_i, -a_j, \ldots, -a_t$ die negativen Koeffizienten von $f(x)$ sind. Durch Ersetzen von C_μ^ν durch 1 erhält Verf. erstens für $p = i, q = j-i, r = k-j, \ldots$ und zweitens für $p = q = r = \cdots = 1$ zwei Sätze von *J. J. Bret* (Ann. Math. pures appl. 6 (1815), 112-122.).

Yannopoulos, C.; Dr. (Athen)
Published: **1940**
Document Type: **J**

Petre Sergescu CR 1940

Jahrbuch Database – Electronic Research Archive for Mathematics

JFM 66.0062.02

Sergescu, P.
Sur les limites de J.-J. Bret. (French)
C. R. Acad. Sci., Paris, 210, 652-654.

Verf. beweist einige Verallgemeinerungen der Sätze von *J. J. Bret* (Ann. Math. pures appl. 6 (1815), 112-122) über Nullstellenschranken des reellen Polynoms

$$f(x) \equiv a_0x^n + a_{n-1}x^{n-1} + \cdots + a_n; \quad a_0 > 0.$$

Hat dieses Polynom genau k negative Vorzählen $a_{\nu_1}, \ldots, a_{\nu_k}$ und setzt man $A_i = a_i$ für $i \neq \nu_1, \ldots, \nu_k$ $A_i = 0$ für $i = \nu_1, \ldots, \nu_k$, so ist

$$L_1 = 1 + \max_{j=1,\ldots,k} \frac{-a_{\nu j}}{A_0 + \cdots + A_{\nu j-1}}, \quad 1 + \max_{j=1,\ldots,k} \sqrt[\nu_j - j+1]{-\frac{a_{\nu j}}{a_0}}$$

je eine obere Schranke der positiven Nullstellen von $f(x)$. Statt ihrer gibt Verf. die kleinste der Zahlen $L_1, L_2, \ldots, L_{\nu_1}$, wo

$$L_\mu = 1 + \max_{j=1,\ldots,k} \sqrt[\mu]{\frac{-a_{\nu j}}{\binom{\nu_j - 1}{\mu - 1}A_0 + \binom{\nu_j - 2}{\mu - 1}A_1 + \cdots + \binom{\mu - 1}{\mu - 1}A_{\nu_j - \mu},}}$$

als obere Schranke dieser Nullstellen an; auch die größte der Zahlen $K_1, \ldots, K_k$, wo

$$K_1 = 1 + \min_{\varrho=0,\ldots,\nu_1-1} \sqrt[\nu_1]{\frac{-a_{\nu_1}}{\sum_{\lambda=0}^{\varrho}\binom{\nu_1 - \lambda}{\varrho - \lambda}A_\lambda}},$$

$$K_j = 1 + \min_{\varrho=0,\ldots,\nu_j-\nu_{j-1}-1,} \sqrt[\nu_j - \nu_{j-1} - \varrho]{\frac{-a_{\nu j}}{\binom{\nu_j - \nu_{j-1}}{\varrho}\sum_{\lambda=0}^{\nu_{j-1}-1}A_\lambda + \sum_{\lambda=1}^{\varrho}\binom{\nu_j - \nu_{j-1} - \lambda}{\varrho - \lambda}A_{\nu_j-1+\lambda}}}$$

$$(j = 2, \ldots, k)$$

leistet dasselbe.
Vgl. auch die vorstehend besprochene Arbeit des Verf.

Geppert, H.; Prof. (Berlin)
Published: **1940**
Document Type: **J**

Montel CR 1940

MR0002815 (2,116h) 15.0X
Montel, Paul
Observations sur la communication précédente. (French)
C. R. Acad. Sci. Paris **210** (1940). 654–655

Let $f(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m (a_0>0)$ be a real polynomial in which only the coefficients a_j with $j=n_1,n_2,\cdots,n_k (0<n_1<n_2<\cdots<n_k)$ are negative. Let $b_j=a_j$ if $a_j>0$, $b_j=0$ if $a_j<0$, and let $c_j=0$ if $a_j>0$, $c_j=-a_j$ if $a_j<0$. Then, according to J. J. Bret [Ann. Math. Pures Appl. **6,** 112–122 (1815)], no positive zero of $f(x)$ exceeds $1+P$, where P is either of the limits

$$\max_{1\le r\le k}\frac{c_{n_r}}{b_0+b_1+\cdots+b_{n_r}},\quad \max_{1\le r\le k}\left(\frac{c_{n_r}}{a_0}\right)^{1/(n_r-r+1)}.$$

In his note, using the linear combinations of positive coefficients

$$S(j,k)=\sum_{i=0}^{k-j}\binom{k-1-i}{j-1}b_i,$$

$$T(j,k,l)=\binom{l-k}{j}S(1,k)+\sum_{i=1}^{j}\binom{l-k-i}{j-i}b_{k+i},$$

Sergescu asserts that Bret's methods also lead to the more general limits for P:

$$\max_{1\le r\le k}\left(\frac{c_{n_r}}{S(j,n_r)}\right)^{1/(j-1)},$$

$$\max_{1\le r\le k}\min_{n_{r-1}\le j\le n_r}\left(\frac{c_{n_r}}{T(j-n_{r-1},n_{r-1},n_r)}\right)^{1/(n_r-j)},$$

$$\max_{1\le r\le k}\left(\frac{c_{n_r}}{T(\nu_r-1,\sigma_{r-1},n_r)}\right)^{1/(n_r-\sigma_r+1)}.$$

In the above formulas, $n_0=0$, $S(1,0)=0$ and $\nu_1,\nu_2,\cdots,\nu_k$ are any positive integers such that $\sigma_r=\nu_1+\nu_2+\cdots+\nu_r\le n_r$. In his note, Montel suggests that the above limits can be derived by use of the following principle: If $P_r(x)$, a real polynomial with highest coefficient positive, has l_r as the upper limit of its positive zeros and if $P_r(x)>0$, all r, implies $f(x)>0$, then $f(x)\neq 0$ for $x>\max l_r$. For example, Sergescu's first limit may be derived by taking $r=1,2,\cdots,k$ and

$$P_r(x)=(x-1)\sum_{j=0}^{n_r-1}b_jx^{n_r-1-j}-c_{n_r}.$$

To compare with Marden's review

$$H(P) = 2 \cdot \max_{\substack{i \\ a_i<0}} \min_{\substack{j>i \\ a_j>0}} \left(\frac{|a_i|}{a_j} \right)^{\dfrac{1}{d_i - d_j}}$$

Kioustelidis

Theorem (Kioustelidis)

Let $P(X) = X^d - b_1 X^{m_1} - \cdots - b_k X^{m-k} + g(X)$, with $g(X)$ having positive coefficients and $b_1 > 0, \ldots, b_k > 0$. The number

$$K(P) \;=\; 2 \cdot \max\{b_1^{1/m_1}, \ldots, b_k^{1/m_k}\}\,.$$

is an upper bound for the positive roots of P.

Ştefănescu

Theorem (Ştefănescu)

Let $P(X) \in \mathbb{R}[X]$ be such that the number of variations of signs of its coefficients is even. If

$$P(X) = c_1 X^{d_1} - b_1 X^{m_1} + c_2 X^{d_2} - b_2 X^{m_2} + \cdots + c_k X^{d_k} - b_k X^{m_k} + g(X)$$

with $g(X) \in \mathbb{R}_+[X]$, $c_i > 0$, $b_i > 0$, $d_i > m_i > d_{i+1}$ for all i, the number

$$S(P) = \max\left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1 - m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k - m_k)} \right\}$$

is an upper bound for the positive roots of the polynomial P.

H. Hong, 1998

The bound of Fujiwara was refined by H. Hong:

Theorem

Let

$$P(X) = \sum_{i=1}^{n} a_i X^{d_i} \,,$$

with $d_1 < d_2 < \cdots < d_n = d = \deg(P)$. *The number*

$$H(P) = 2 \cdot \max_{\substack{i \\ a_i<0}} \min_{\substack{j>i \\ a_j>0}} \left(\frac{|a_i|}{a_j} \right)^{\dfrac{1}{d_i - d_j}}$$

is an upper bound for the positive roots.

Lagrange 2

Let F be a nonconstant monic polynomial of degree n over $\mathbb{R}$ and let $\{a_j\,;\, j \in J\}$ be the set of its negative coefficients. Then an upper bound for the positive real roots of F is given by the sum of the largest and the second largest numbers in the set

$$\left\{ \sqrt[j]{|a_j|}\,;\, j \in J \right\}.$$

A traditional notation for this bound is $R + \rho$, where R and ρ are the two largest numbers considered in the statement.

Improvements of the bound $R + \rho$

If we compare the bound $R + \rho$ with other bounds we note that it seems to be one of the best. So it is natural to ask if it can be improved. Such a refinement was obtained by Mignotte-Ştefănescu in 2015:

Improvements of the bound $R + \rho$

Theorem

Let $F(X) = X^d + a_1 X^{d-1} + \cdots + a_{d-1}X + a_d$ *be a polynomial with complex coefficients, with* $d > 0$ *and* $a_d \neq 0$*. Suppose that* $|a_{i_1}|^{1/i_1} \geq |a_{i_2}|^{1/i_2} \geq \cdots \geq |a_{i_d}|^{1/i_d}$ *and put* $R = |a_{i_1}|^{1/i_1}$, $\rho = |a_{i_2}|^{1/i_2}$, $i_1 = j$ *and*

Improvements of the bound $R + \rho$ (contd.)

$$C_j = \begin{cases} \dfrac{R + \rho + \sqrt{R^2 - 2R\rho + 5\rho^2}}{2} & \textit{if} \quad j = 1\,, \\ \left(R^{j-1} + R^{j-2}\rho + \cdots + \rho^{j-1}\right)^{1/(j-1)} & \textit{for} \quad j = 2, \ldots, d\,. \end{cases}$$

Then for any complex root z of F we have

$$|z| \leq \max\left\{ C_j, \ \frac{R + \rho + \sqrt{R^2 - 2R\rho + 5\rho^2}}{2} \right\}.$$

Another polynomial

Theorem

The the largest positive root of the polynomial

$$u(X) = (X - 2\rho)X^j - (R^j - \rho^j)(X - \rho)$$

is an upper bound for the absolute values of the roots of the polynomial F.

Consider a complex polynomial

$$f(X) = \sum_{i=0}^{d} a_i X^{d-i}$$

A root-bound functional V on the set of polynomials yields for every polynomial f a value

$$V(f) \geq \mu(f) = \max\{|\lambda| : f(\lambda) = 0\}.$$

If $V(\cdot)$ is a functional on the set of non-constant polynomials, we define the *maximum relative overestimation* for the set of polynomials of degree $d \in \mathbb{N}$, unequal to $\alpha x^d, \alpha \in \mathbb{C}$, by

$$\limsup_{\substack{f \in \mathbb{C}[x] \\ \deg(f)=d}} \frac{V(f)}{\mu(f)} =: \mathrm{m\text{-}r\text{-}o}(V, d).$$

Computational aspects (contd.)

By this choice, the computational effort for the bound $ML(\cdot)$ ILagrange improved) is marginally larger than the effort for $L(\cdot)$ (Lagrange 2). For any F we have $Bd(F) \leq ML(F) \leq L(F)$. The maximum relative overestimation of ML improves over m-r-o$(L, d) \sim (1 + 1/\sqrt{2})d$ as we establish the following:

Theorem

For $d \geq 3$ it is true that

$$1/(\sqrt[d]{2} - 1) \leq \mathrm{m\text{-}r\text{-}o}(Bd, d) \leq \mathrm{m\text{-}r\text{-}o}(ML, d) \leq 1.58 \cdot d.$$

Conclusions

- In the last decades there were discovered new bounds for polynomial roots.
- The bound $R + \rho$ of Lagrange gives also good results with respect to known efficient bounds for polynomials with real or complex coefficients. We proved that it can be improved.
- The methods of Kioustelidis, Hoon and Ştefănescu are used for efficient computations of the bounds.
- For superunitary roots the bounds of Bret and Sergescu should be reconsidered.

References

J. J. Bret: Analyse algébrique. Théorèmes nouveaux, sur les limites des équations numeériques, *Ann. Math. Pures Appl.*, **6**, 112–122 (1815).

P. Sergescu: Généralisations des limites de J. J. Bret, *Annales Acad. Roum.*, **22**, 460–465 (1940).

J.–L. Lagrange: *Traité de la résolution des équations numériques*, Paris (1798).
(Reprinted in *Œuvres*, t. VIII, Gauthier–Villars, Paris (1879).)

M. Mignotte, D. Ştefănescu: On an estimation of polyomial roots by Lagrange,Prépubl. IRMA, 25/2002, 17 pag., Strasbourg (2002).