Hessian matrices of reducible third degree polynomials

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Theorem. (I.V. Latkin & S., 2015) *The next problem is NP-complete. Given a cubic form of the type*

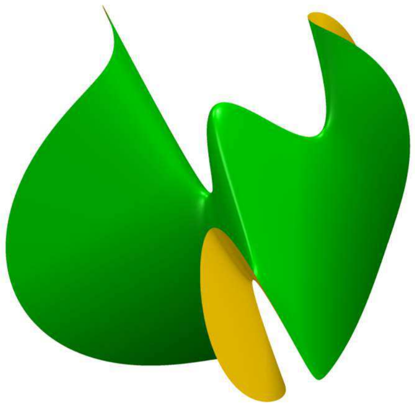
$$f = \alpha_{n+1}^2(\alpha_0 x_0^3 + \ldots + \alpha_n x_n^3) - (\alpha_0 x_0 + \ldots + \alpha_n x_n)^3$$

over $\mathbb{Z}$. Does the projective hypersurface defined by the equation $f = 0$ contain any singular point whose coordinates belong to the set $\{-1, 1\}$?

The set partition problem, which is NP-complete, can be reduced to this problem.
Thus, if $NP \neq coNP$, then there does not exist any nondeterministic polynomial time machine recognizing the existence of a real singular point in the worst case.

A cubic hypersurface in $\mathbb{CP}^n$ contains a singular point iff its discriminant vanishes. The discriminant is a polynomial of degree $(n+1)2^n$.
Over the field of complex numbers $\mathbb{C}$, to check smoothness seems as hard as to compute the discriminant. It is very hard.
There are more tools to check whether a polynomial vanishes over reals.

Let us consider the graph of a multivariate polynomial f over reals. A point of the graph is said to be elliptic if the Hessian matrix (whose entries are second partial derivatives $\frac{\partial^2 f}{\partial x_j \partial x_k}$) is definite. Roughly speaking, in a sufficiently small analytic neighborhood of an elliptic point, the surface looks like an ellipsoid.



The surface is the graph of the polynomial $(1-0.9x_1)\cdot(x_1+x_2)\cdot(x_1-x_2)$. Its vanishing locus consist of three straight lines.

Theorem. *Given a third degree multivariate polynomial. If the graph of the polynomial contains an elliptic point, then the projective closure of the vanishing locus of the polynomial does not contain any real singular point on the hyperplane at infinity.*

Let polynomials over $\mathbb{R}$ be identified with sequences of their coefficients using some monomial order. For a positive integer k, the term "almost all k-tuples" means "all k-tuples but a set covered by a vanishing locus of a nonzero polynomial in k variables with integer coefficients".

Theorem. *For almost all inhomogeneous bivariate third degree polynomials, the graph of the polynomial contains an elliptic point.*

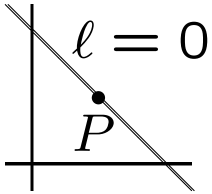
Remark. The cubic form $f = x_0^3 - x_1(x_1^2 - 3x_2^2)$ defines a smooth plane curve. But the graph of the polynomial $x_1(x_1^2 - 3x_2^2)$ is the monkey saddle; it has no elliptic point.

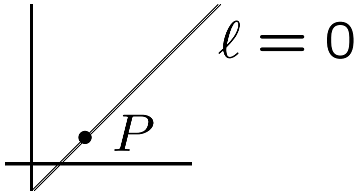
Example. Let us consider the symmetric matrix

$$\begin{pmatrix} \alpha x_1 & \ell(x_1, x_2) \\ \ell(x_1, x_2) & \beta x_2 \end{pmatrix},$$

where ℓ is a linear function, $\alpha \neq 0$, $\beta \neq 0$, and $\ell(0,0) \neq 0$.

In one of two cases, the matrix is positive definite at the point P.





In another typical case, the matrix is negatively defined at a point.

Remark. The theorem does not impose any restriction on singular points over the field of complex numbers that are not real.

So, the projective surface defined by the form $x_0^3 + x_0(x_1^2 + x_2^2) + x_3^3$ contains no real singular points. But the same form defines the complex surface containing two complex conjugate singular points $(0 : 1 : i : 0)$ and $(0 : 1 : -i : 0)$, where $i^2 = -1$. The Hessian matrix of the polynomial $1 + x_1^2 + x_2^2 + x_3^3$ is equal to the diagonal matrix $\mathrm{diag}(2, 2, 6x_3)$. It is positive definite in the affine half-space $x_3 > 0$. Thus, each point P of the half-space corresponds to the elliptic point $\check{P}$ of the graph of the polynomial.

Example. Let us consider a cuspidal cubic that is the vanishing locus of the polynomial $f = x_1^3 + x_2^2$. The curve contains a singular point at the origin. But its projective closure does not contain any singular point at infinity. The Hessian matrix

$$\begin{pmatrix} 6x_1 & 0 \\ 0 & 2 \end{pmatrix}$$

is positive definite inside the half-plane $x_1 > 0$.

On the other hand, the projective closure of the vanishing locus of the polynomial $g = x_1^3 + x_2$ contains a singular point at infinity. The Hessian matrix is degenerate

$$\begin{pmatrix} 6x_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

But these curves are projectively equivalent to each other.

Example. Let us consider the trident of Newton. An affine trident curve is defined by an equation of the type $x_1x_2 + g(x_1)$, where g denotes a univariate polynomial of degree three. The projective curve has the ordinary double point $(0 : 0 : 1)$, that is, the singular point at straight line at infinity. The Hessian matrix of the polynomial $x_1x_2 + g(x_1)$ is equal to

$$\begin{pmatrix} g''(x_1) & 1 \\ 1 & 0 \end{pmatrix}.$$

For all values of the coordinate x_1, it is neither positive nor negative definite. Thus, the graph of the polynomial $x_1x_2 + g(x_1)$ has no elliptic point.

New Results

Theorem. *For almost every reducible multivariate third degree polynomial over the field of real numbers, its Hessian matrix is semidefinite at some real point.*

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Let us consider a quadratic polynomial $q(x_1, \ldots, x_n)$ and a linear function $\ell(x_1, \ldots, x_n)$ over reals. Without loss of generality, one can assume $\ell = x_1$. So,

$$\frac{\partial^2(q\ell)}{\partial x_j \partial x_k} = \frac{\partial^2 q}{\partial x_j \partial x_k} x_1 + \frac{\partial q}{\partial x_j} \delta_{1k} + \frac{\partial q}{\partial x_k} \delta_{1j},$$

where $\delta_{kk} = 1$ and if $j \neq k$, then $\delta_{jk} = 0$. The point P is a solution to the system of linear equations

$$\begin{cases} x_1 & = & 0 \\ \frac{\partial q}{\partial x_j} & = & 0, \;\; 2 \leq j \leq n \end{cases}$$

The system consists of n equations in n variables. For almost every polynomial q, there exists a solution to the system. Moreover, if there is no solution, then there vanishes some auxiliary polynomial in coefficients of both polynomials q and ℓ. At the point P, at most one entry of the Hessian matrix is nonzero. Thus, the matrix is semidefinite.

Theorem. *For almost every reducible multivariate third degree polynomial over the field of real numbers, its Hessian matrix is definite at some real point if and only if the projective closure of the vanishing locus of the polynomial does not contain any real singular point at infinity.*

Let us consider a quadratic polynomial q and a linear function ℓ.

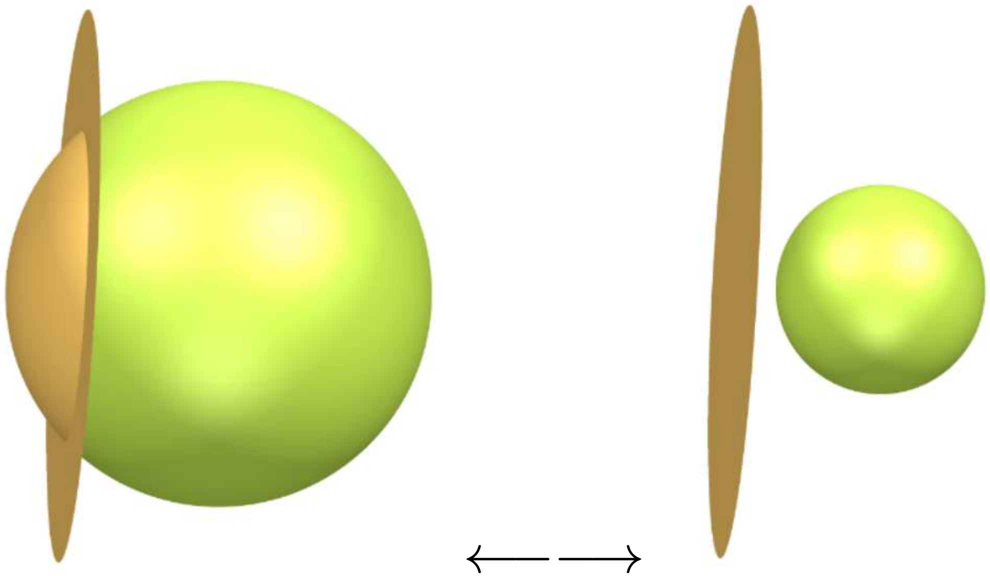
If loci of the hypersurface $q\ell = 0$ bounds a compact of full dimension, then the Hessian matrix is definite at some real point inside the compact.

For every real number α, both polynomials $q\ell$ and $(q+\alpha)\ell$ have the same Hessian matrix.

If the equation $q = 0$ defines an imaginary ellipsoid, then some equation of the type $q + \alpha = 0$ defines a real ellipsoid.

If the equation $q = 0$ defines a two-sheeted hyperboloid, then an equation of the type $q + \alpha = 0$ defines a one-sheeted hyperboloid. Moreover, another equation $q + \beta = 0$ defines a cone.

Both surfaces are vanishing loci of polynomials with the same Hessian matrix.



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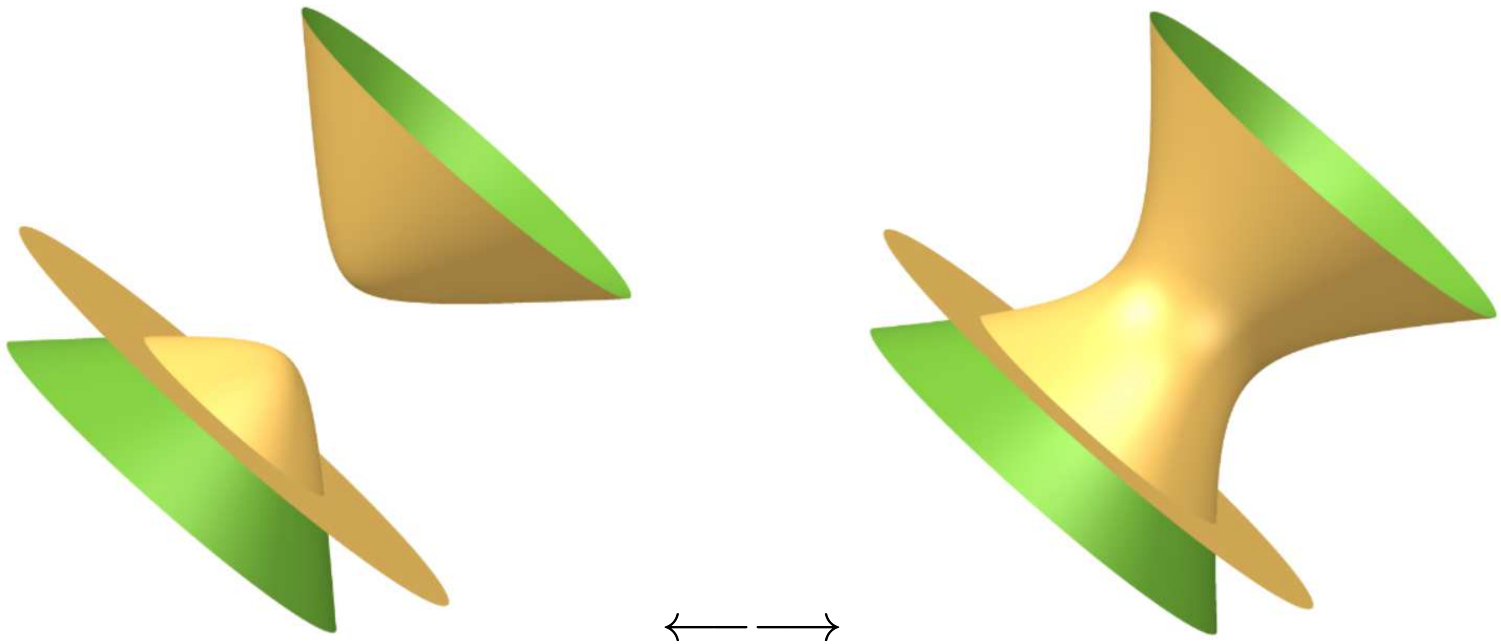

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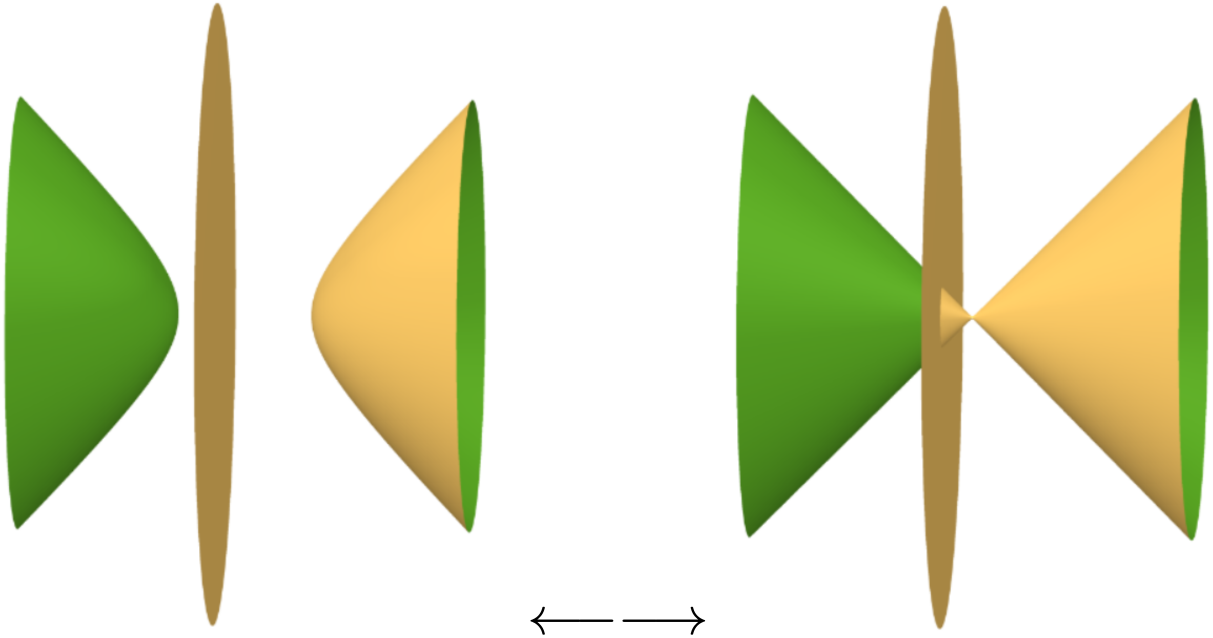

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If the set of all forms having some fixed rank contains a nonempty open subset, then the rank is called typical. The typical real rank of ternary cubic forms is equal to four according to a result by De Paolis published in 1886, but in the general case, there are several typical real ranks. Both values two and three are typical real ranks of binary cubic forms (Bernardi A., Blekherman G., Ottaviani G. 2018). For example, the real rank of the monomial $x_0 x_1^2$ is equal to three (Carlini E., Kummer M., Oneto A., Ventura E. 2017). For cubic forms in four variables, both values five and six are typical real ranks.

To show the difference between the Waring decompositions over the fields of real and complex numbers, one can consider the equality

$$4x_0^3 - (x_0 + x_1)^3 - (x_0 - x_1)^3 = (x_0 + ix_1)^3 + (x_0 - ix_1)^3,$$

where $i^2 = -1$.

The smallest typical real rank coincides with the generic rank over the field of complex numbers. For cubic forms in $n+1$ variables, the generic rank is equal to $\left\lceil \frac{1}{6}(n+2)(n+3) \right\rceil$, except for $n = 4$ when the generic rank is equal to eight (J. Alexander and A. Hirschowitz, 1995).

Theorem. *For almost every cubic form* $f(x_0, \ldots, x_n)$ *of real rank* $n+1$*, the Hessian matrix of the inhomogeneous polynomial* $f(1, x_1, \ldots, x_n)$ *is semidefinite at some real point.*

Cf. the case of bivariate polynomials.

Hypothesis. *For almost every cubic form* $f(x_0, \ldots, x_n)$ *of real rank* $n+2$*, the Hessian matrix of the inhomogeneous polynomial* $f(1, x_1, \ldots, x_n)$ *is semidefinite at some real point.*

Look from the other side

Let us consider generalized register machines over the field of reals

$$(\mathbb{R}, 0, 1, +, -, \times, <)$$

or over another real closed field. They are closely related to the machines defined by L. Blum, M. Shub, and S. Smale (1989).
Each register contains an element of $\mathbb{R}$.
There exist index registers containing nonnegative integers.
The running time is said polynomial when the total number of operations performed before the machine halts is bounded by a polynomial in the number of registers occupied by the input.
Initially, this number is placed in the zeroth index register.

One can also define a nondeterministic generalized register machine that receives a few hints over $\mathbb{R}$.

The generic computational complexity had been defined by I. Kapovich, A. G. Myasnikov, P. Schupp, and V. Shpilrain (2003) and extensively studied by A. N. Rybalov. The machine never makes mistakes, but it can warn there is no way to accept or reject some input. These rare inputs are called vague. This concept is applicable over $\mathbb{R}$.

Definition. Let us consider a generalized register machine over $\mathbb{R}$ with three halting states: ACCEPT, REJECT, and VAGUE. The machine is said to be generic when both conditions hold:
(1) the machine halts on every input and
(2) for every positive integer k and for almost all inputs that occupy exactly k registers, the machine does not halt at the VAGUE state.

A hypersurface in $\mathbb{RP}^n$ is the vanishing locus of a form, i.e., a homogeneous polynomial in $n+1$ variables.

It is hard to recognize whether a given cubic hypersurface is smooth. But a nondeterministic generalized register machine over $\mathbb{R}$ can recognize in polynomial time whether a given hypersurface contains a real singular point.

Theorem. *There exists a generic generalized register machine over* $\mathbb{R}$ *that recognizes whether a given projective cubic hypersurface defined by a form over* $\mathbb{R}$ *of the type* $x_0^3 + \ldots + x_n^3 + (\alpha_0 x_0 + \ldots + \alpha_n x_n)^3$ *is smooth at every real point of the intersection with a given projective hyperplane defined by a linear form of the type* $x_0 + \beta x_n$, *where* $\beta \in \mathbb{R}$. *The running time of the machine is polynomial in* n.

Theorem. *There exists a nondeterministic generic generalized register machine over* $\mathbb{R}$ *that recognizes whether a given reducible projective cubic hypersurface is smooth at every real point of the intersection with a given projective hyperplane over* $\mathbb{R}$. *The running time of the machine is polynomial.*

Let an affine hypersurface is defined by the equation $f = 0$ over reals. If its projective closure does not contain any real singular point at infinity, then the hint is a point, where the Hessian matrix $H_{jk}(f) = \frac{\partial^2 f}{\partial x_j \partial x_k}$ is definite.
The inputs resulted in the vague output belong to a semialgebraic set. The set can be embedded into a hypersurface whose degree is bounded by a polynomial in the number of variables. The set can also be embedded into another high degree algebraic variety of small dimension.

Example. If all entries of a matrix are non-negative and its determinant is nonzero, then its permanent is positive.
Thus, non-negative matrices with positive permanent can be accepted in generic polynomial time by a generalized register machine over $\mathbb{R}$.
But in the worst case, the permanent is hard. In accordance with Valiant's theorem, the problem of computing the permanent of a $(0,1)$-matrix is $\#P$-hard.

To read more refer to

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Thank you!