GRACEFUL BASES
IN SPACES OF ANALYTIC SOLUTIONS
TO DIFFERENTIAL AND DIFFERENCE EQUATIONS

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NOTATION AND DEFINITIONS

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$\mathcal{D}_n$ = Weyl's algebra of linear differential operators with polynomial coefficients in $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$:

$$\mathcal{D}_n := \left\{ \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)} p_\alpha(x_1, \ldots, x_n) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right\}.$$

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Here $\alpha \in \mathrm{N}_0^n$, $|\alpha| := \alpha_1 + \ldots + \alpha_n$, $p_\alpha(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$.

An ideal $J \subset \mathcal{D}_n$ is called holonomic, if the complex dimension of its characteristic variety,

$$\operatorname{char}(J) := \{(x, z) \in \mathbb{C}^{2n} : \sigma(P)(x, z) = 0, \text{ for all } P \in J\},$$

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$$P = \sum_{|\alpha| \leq d} p_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

and

$$\sigma(P) := \sum_{|\alpha| = d} p_\alpha(x_1, \dots, x_n) z_1^{\alpha_1} \dots z_n^{\alpha_n}.$$

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The elements of the above ring will be referred to as Frobenius–Puiseux polynomials.

DEFINITION. The set of solutions $\{f_\tau\}$ to the ideal $J \subset \mathcal{D}_n$ is called a generating set for $\mathcal{S}_D(J)$, if the linear span of all germs of the functions f_τ in the domain D is the whole solution space $\mathcal{S}_D(J)$.

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DEFINITION. The analytic solutions to an ideal $J \subset \mathcal{D}_n$ depending on parameters $\alpha \in \mathbb{C}^m$ form a graceful basis in the space $\mathcal{S}_D(J)$, if they are linearly independent and generate the space $\mathcal{S}_D(J)$ for all $\alpha \in \mathbb{C}^m$.

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The canonical basis in the space of analytic solutions to the ODE with constant coefficients

$$\left(\prod_{j=1}^{m}\left(\frac{d}{dx}-\alpha_j\right)\right) f(x) = 0$$

has the form

$$\{e^{\alpha_j x},\, j = 1, \dots, m\}$$

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1) $f_j(x;\alpha)$ are entire functions of $x\in\mathbb{C}$ and $\alpha\in\mathbb{C}^m$;

2) for any fixed $\alpha\in\mathbb{C}^m$ the univariate functions $f_j(x;\alpha)$ are linearly independent.

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The family of m functions

$$\mathfrak{g}_i(x) := \sum_{j=1}^{m} \frac{(-1)^{m-i}\, \boldsymbol{\mathcal{S}}_{m-i}(\alpha_1, \ldots [j] \ldots, \alpha_m)}{(\alpha_j - \alpha_1) \cdot \ldots [j] \ldots \cdot (\alpha_j - \alpha_m)}\, e^{\alpha_j x}, \quad i = 1, \ldots, m$$

is a graceful basis in the space of analytic solutions to the above equation.

Denote by $\boldsymbol{V(\alpha)}$ the Vandermonde matrix defined by the roots of the characteristic polynomial:

$$\boldsymbol{V(\alpha) \equiv V(\alpha_1, \ldots, \alpha_m) := \begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_1^{m-1} \\ 1 & \alpha_2 & \ldots & \alpha_2^{m-1} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha_m & \ldots & \alpha_m^{m-1} \end{pmatrix}.}$$

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The functions

$$V(\alpha)^{-1} (e^{\alpha_1 x}, \dots, e^{\alpha_m x})^T$$

form a graceful basis in the space of analytic solutions to the ODE with constant coefficients.

For instance, a graceful basis in the solution space of the 3rd order equation

$$\left(\frac{d}{dx}-\alpha_1\right)\left(\frac{d}{dx}-\alpha_2\right)\left(\frac{d}{dx}-\alpha_3\right)f(x)=0$$

is given by the entire functions

$$\frac{\alpha_2\alpha_3\, e^{\alpha_1 x}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}-\frac{\alpha_1\alpha_3\, e^{\alpha_2 x}}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}+\frac{\alpha_1\alpha_2\, e^{\alpha_3 x}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)},$$

$$-\frac{(\alpha_2+\alpha_3)\, e^{\alpha_1 x}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}+\frac{(\alpha_1+\alpha_3)\, e^{\alpha_2 x}}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}-\frac{(\alpha_1+\alpha_2)\, e^{\alpha_3 x}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)},$$

$$\frac{e^{\alpha_1 x}}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}-\frac{e^{\alpha_2 x}}{(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}+\frac{e^{\alpha_3 x}}{(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}.$$

EXAMPLE. The standard basis in the solution space of the Cauchy–Euler equation

$$\left(\prod_{j=1}^{m}\left(x\frac{d}{dx}-\alpha_j\right)\right)f(x)=0$$

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Its last element

$$\frac{(\ln x)^{m-1}}{(m-1)!} \equiv \operatorname*{res}_{s=0}\frac{x^s}{s^m}$$

is a generating solution to the above equation and admits a representation as the residue at a multiple pole of a meromorphic (w.r.t. s) function.

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THEOREM. The family of m functions in $x \in \mathbb{C}$ depending on the parameters $\lambda_1, \ldots, \lambda_m$

$$V(\log\lambda_1, \ldots, \log\lambda_m)^{-1}(\lambda_1^x, \ldots, \lambda_m^x)^T \tag{1}$$

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is linearly independent for all $\lambda_1, \ldots, \lambda_m \in \mathbb{C}^*$.

The Wronskian of this graceful basis equals $(\lambda_1 \cdot \ldots \cdot \lambda_m)^x$.

EXAMPLE. **Consider the system of PDEs with constant coefficients and parameters $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{C}$**

$$\begin{cases} a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + cf = 0, \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f. \end{cases}$$

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If $a^2 + b^2 \neq c^2$ then a basis in the space of its analytic solutions is given by

$$f = \exp\left(-\frac{ac+b\sqrt{a^2+b^2-c^2}}{a^2+b^2}\,x - \frac{bc-a\sqrt{a^2+b^2-c^2}}{a^2+b^2}\,y\right),$$

$$g = \exp\left(-\frac{ac-b\sqrt{a^2+b^2-c^2}}{a^2+b^2}\,x - \frac{bc+a\sqrt{a^2+b^2-c^2}}{a^2+b^2}\,y\right).$$

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If $a^2 + b^2 - c^2 = 0$ then the line is tangent to the circle and the basis is degenerate.

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The limit of this graceful basis as $c \to \sqrt{a^2+b^2}$ yields the linearly independent solutions

$$\exp\left(-\frac{ax+by}{\sqrt{a^2+b^2}}\right), \quad 2\,\frac{ay-bx}{a^2+b^2}\,\exp\left(-\frac{ax+by}{\sqrt{a^2+b^2}}\right).$$

Let $h, f_1, \dots, f_n$ be holomorphic in $(x_1, \dots, x_n) \in \mathbb{C}^n$ in a neighborhood U_ζ of $\zeta \in \mathbb{C}^n$, such that the mapping $f = (f_1, \dots, f_n) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ has isolated zero at ζ: $f^{-1}(0) \cap U_\zeta = \emptyset$.

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The Grothendieck residue of the meromorphic differential form

$$\omega = \frac{h(x) dx_1 \wedge \ldots \wedge dx_n}{f_1(x) \cdot \ldots \cdot f_n(x)}$$

is defined to be

$$\operatorname*{res}_{\zeta} \omega = \frac{1}{(2\pi\sqrt{-1})^n} \int\limits_{\triangle_\zeta} \frac{h(x) dx_1 \wedge \ldots \wedge dx_n}{f_1(x) \cdot \ldots \cdot f_n(x)},$$

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where

$$\triangle_\zeta = \{x \in U_\zeta : |f_j(x)| = \varepsilon, j = 1, \dots, n\},$$

$\varepsilon > 0$ is small enough, and the orientation of $\triangle_\zeta$ is given by

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The above integral is denoted by $\operatorname*{res}_{\zeta}[f](h)$.

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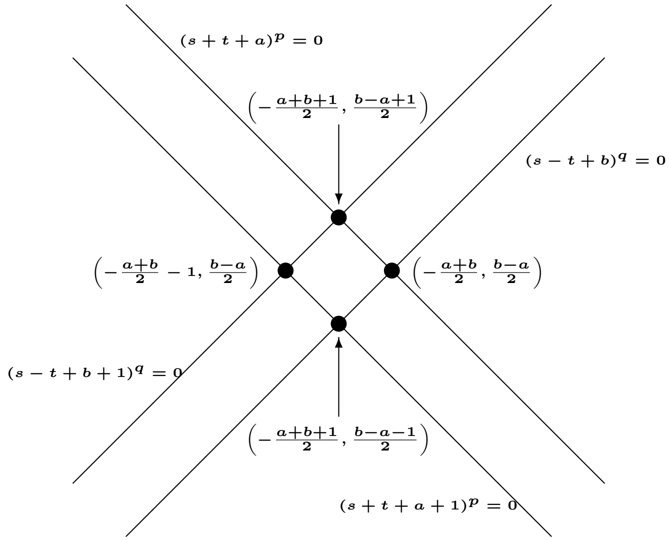

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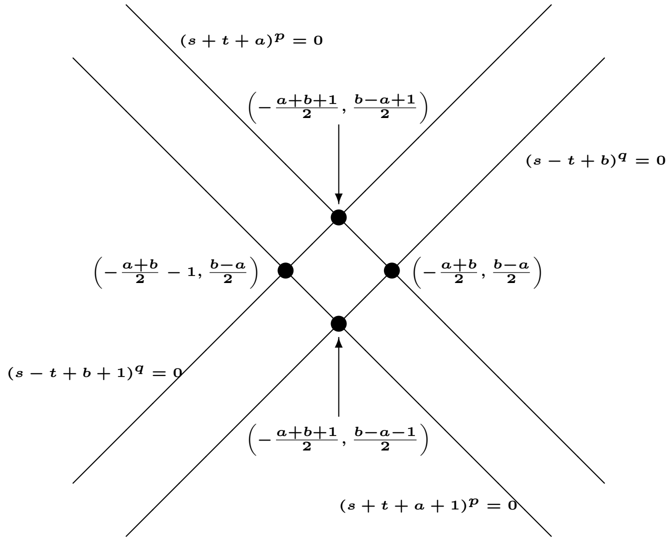

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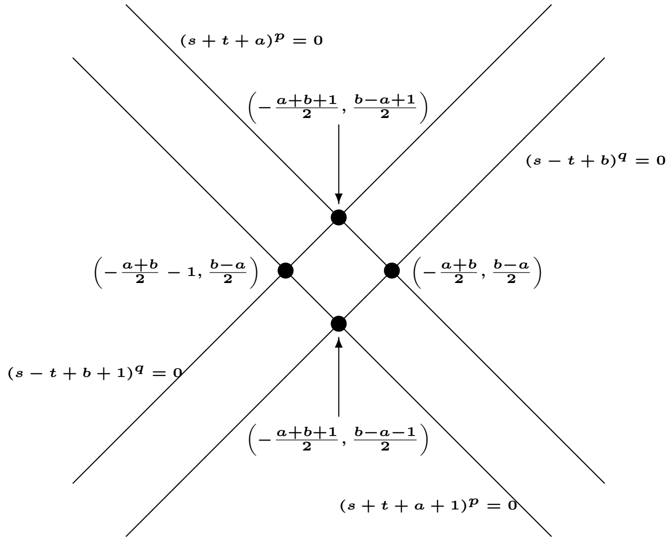

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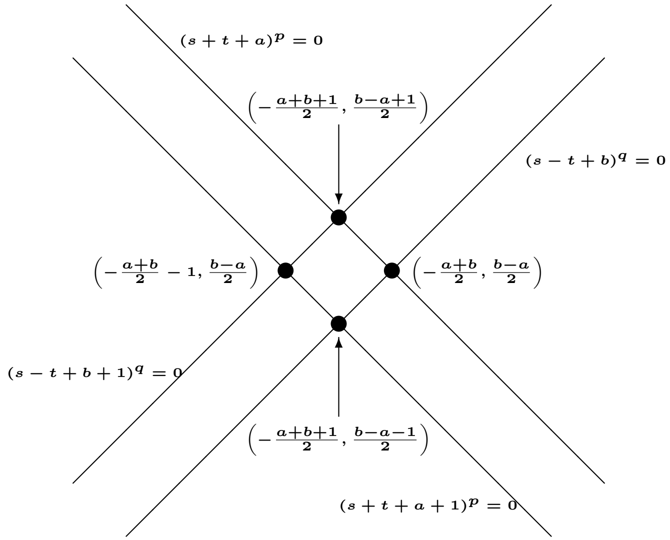

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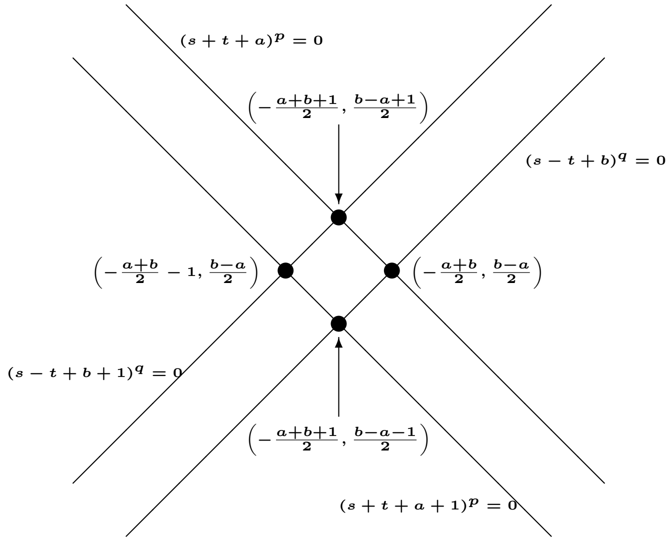

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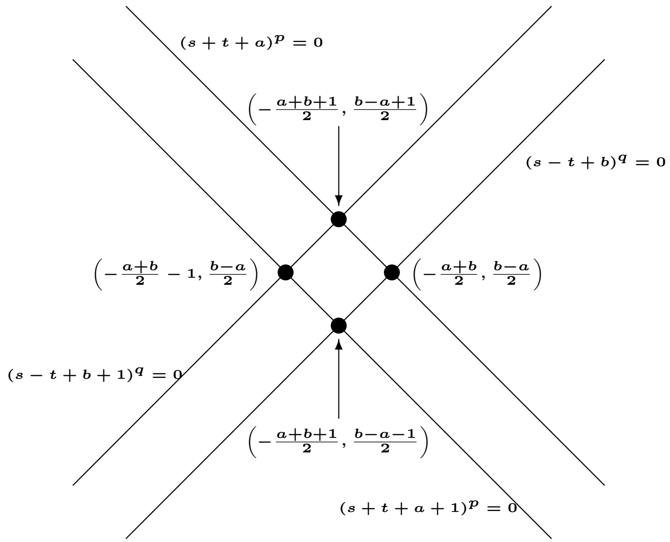

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satisfy the initial system of hypergeometric differential equations, while the linearly independent branches of these functions span the whole 8-dimensional space of its analytic solutions.

THEOREM. If the hypergeometric ideal $\mathcal{H}$ is regular and holonomic then the Grothendieck residues

$$\operatorname*{res}_{s} [\langle A_I, s\rangle + c_I](\varphi(s)\, x^s)$$

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CONJECTURE. Any regular holonomic ideal $J \subset \mathcal{D}_n$ with algebraically independent symbolic parameters, treated as the coefficients in the generators of J, admits a graceful basis in its space of analytic solutions $\mathcal{S}_D(J)$, for any domain $D \subset \mathbb{C}^n$ which does not intersect the singular locus of J. If the assumptions of Saito–Sturmfels–Takayama's theorem are satisfied by the ideal J then the elements of the graceful basis are Frobenius–Puiseux polynomials.