On Ramanujan's identities with cubic radicals

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Plan

- On simplification of expressions involving nested radicals
- Two symbolic Ramanujan's identities with cubic radicals
- Universality of Ramanujan's identities over the field of rational numbers
- Example of non-universality
- Extra results

On simplification of expressions involving nested radicals I

Using CAS, we can verify numerical equalities with radicals of the form

$$L := \sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1 =: R.$$

For this purpose, in Maple CAS there are the following functions:

- `simplify(L-R)`
- `evala(L-R)`
- `evala(Minpoly(L-R,x))`

If the right hand side R of the simplifying equality

$$L = R$$

is «not too complicated» then one can obtain it directly via the cited functions.

On simplification of expressions involving nested radicals II

A more interesting example:

$$\sqrt{14+2\sqrt{7}+8\sqrt{3+\sqrt{7}}}+\sqrt{6-2\sqrt{7}}=2\sqrt{3+\sqrt{2}}+2\sqrt{2}.$$

Here, one can find the minimal polynomial

$$x^4-40x^2-64x-16$$

for L and then find its roots explicitly (one of these roots must be R). But this does not work in the case of (as an example)

$$L=\sqrt[3]{1+\sqrt[3]{3+2\sqrt{2}}}$$

with the minimal polynomial

$$x^{18}-6x^{15}+15x^{12}-26x^9+33x^6-24x^3+8.$$

Unfortunately, this will be the case in general.

Two symbolic Ramanujan's identities with cubic radicals I

In the book

- *Berndt B.C.* Ramanujan's notebooks IV. New York: Springer, 1994.

one can find the following well-known symbolic identities:

$$\pm\sqrt{m\sqrt[3]{4(m-2n)}+n\sqrt[3]{4m+n}}= \\ =\frac{\sqrt[3]{(4m+n)^2}+\sqrt[3]{4(m-2n)(4m+n)}-\sqrt[3]{2(m-2n)^2}}{3}, \tag{1}$$

$$\sqrt[3]{(m^2+mn+n^2)\sqrt[3]{(m-n)(m+2n)(2m+n)}-m^3+3mn^2+n^3}= \\ =\sqrt[3]{\frac{(m-n)(m+2n)^2}{9}}-\sqrt[3]{\frac{(2m+n)(m-n)^2}{9}}+\sqrt[3]{\frac{(m+2n)(2m+n)^2}{9}}. \tag{2}$$

Substituting $(m,n)=(-1,0)$ in (2), we get a particular equality

$$\sqrt[3]{1-\sqrt[3]{2}}=-\sqrt[3]{\frac{1}{9}}+\sqrt[3]{\frac{2}{9}}-\sqrt[3]{\frac{4}{9}}. \tag{3}$$

Universality of Ramanujan's identities over the field of rational numbers I

As usual, we suppose the values of variables m, n to be in the field of rational numbers $\mathbb{Q}$ (in general: the values of m and n belong to some subfield P of the field of real numbers $\mathbb{R}$). In the paper

- *Zippel R.* Simplification of expressions involving radicals // J. of Symbolic Computation. 1985. Vol. 1. P. 189 – 210.

the author write the following: "...it would be desirable to show that these two identities are the only denesting formulae of their type.". A natural question must be asked here.

Question

What type of denesting formulae are we talking about?

But the answers could be different.

Universality of Ramanujan's identities over the field of rational numbers II

One can suppose that the left hand sides in (1) and (2) reduce to the form

$$\rho_1 = \sqrt{1+\omega}, \quad \rho_2 = \sqrt[3]{1+\omega},$$

where ω is a real cubic radical over $\mathbb{Q}$. What is the natural form of the simplifying right hand side? In the paper

- *Antipov M.A., Pimenov K.I.* Ramanujan denesting formulae for cubic radicals // Vestnik of Saint Petersburg University. 2020. Vol. 7. P. 187 – 196.

for the case of nested radical ρ_2 the simplifying right hand side was supposed contain only cubic real radicals over $\mathbb{Q}$. From our point of view, it is more natural to allow any real radicals (i.e., radicals of all degrees) over $\mathbb{Q}$ on the simplifying right hand side.

Universality of Ramanujan's identities over the field of rational numbers III

Let $P \subset \mathbb{R}$ be an arbitrary subfield.

Definition

A number $\omega \in \mathbb{R}$ is called real radical over P of degree $n > 1$ if $\omega^k \notin P$ for $1 \leqslant k < n$ and $\omega^n \in P$.

Denote by $R(P)$ the subfield of $\mathbb{R}$ obtained by joining to P all real radicals over P.

Theorem 1

Let ω be a real radical over P of degree n and $0 < \alpha \in P(\omega)$. If

$$\rho = \sqrt[r]{\alpha} \in R(P)$$

then $\alpha = c\omega^k\beta^r$ for some $0 \neq c \in P$, $k \in \{0, 1, \dots, n-1\}$ and $\beta \in P(\omega)$.

Universality of Ramanujan's identities over the field of rational numbers IV

The various proofs of Theorem 1 can be found in the papers

- *Blömer J.* How to Denest Ramanujan's Nested Radicals // Proc. 33rd Annual Symposium on Foundations of Computer Science. 1992. P. 447 – 456.
- *Osipov N.N., Kytmanov A.A.* Simplification of nested real radicals revisited // Computer Algebra in Scientific Computing. CASC 2021. LNCS. Vol. 12865. Springer, 2021. P. 293 – 313.

Write a real cubic radical ω in the form

$$\omega = \sqrt[3]{a}$$

where $a \in \mathbb{Q}$ does not a perfect cube. We want to give an explicit description of these a's for which $\rho_1 \in R(\mathbb{Q})$ and $\rho_2 \in R(\mathbb{Q})$.

Universality of Ramanujan's identities over the field of rational numbers V

The following two theorems can be derived from Theorem 1.

Theorem 2

Let $\omega = \sqrt[3]{a}$ where $a \in \mathbb{Q}$ does not a perfect cube. The number $\rho_1 = \sqrt{1+\omega}$ is in $R(\mathbb{Q})$ if and only if

$$a = t\left[\frac{t-8}{4(t+1)}\right]^3 \tag{4}$$

where $t \in \mathbb{Q}$ does not a perfect cube and $t > -1$.

For a's of the form (4), the simplifying equality for ρ_1 is given by

$$\sqrt{1+\frac{(t-8)\sqrt[3]{t}}{4(t+1)}} = \varepsilon\left[-\frac{1}{\sqrt{t+1}}+\frac{\sqrt[3]{t}}{\sqrt{t+1}}+\frac{\sqrt[3]{t^2}}{2\sqrt{t+1}}\right] \tag{5}$$

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where the sign $\varepsilon \in \{-1, 1\}$ is

$$\varepsilon = \begin{cases} -1 & \text{if } -1 < t < -10 + 6\sqrt{3}, \\ 1 & \text{if } t > -10 + 6\sqrt{3}. \end{cases}$$

It can be verified directly that the identity (5) and Ramanujan's identity (1) are obtained from each other by substitutions

$$(m, n) = (t - 8, 4(t + 1)), \quad t = \frac{4(m - 2n)}{4m + n}.$$

Thus, from Theorem 2 it follows that Ramanujan's identity (1) is universal over $\mathbb{Q}$.

Remark 1

The statement of Theorem 2 is true for any field $P \subset \mathbb{R}$ instead of the field $\mathbb{Q}$.

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The case of ρ_2 is more interesting.

Theorem 3

Let $\omega = \sqrt[3]{a}$ where $a \in \mathbb{Q}$ does not a perfect cube. The number $\rho_2 = \sqrt[3]{1+\omega}$ is in $R(\mathbb{Q})$ if and only if

$$a = \left[\frac{27t(t+1)(t^2+t+1)^3}{(t^3-3t^2-6t-1)^3}\right]^{\varepsilon} \tag{6}$$

where $t \in \mathbb{Q} \setminus \{-1, 0\}$ and $\varepsilon \in \{-1, 1\}$.

For a's of the form (6), the simplifying equalities can be reduced to

$$\sqrt[3]{t^3-3t^2-6t-1+3(t^2+t+1)\sqrt[3]{t(t+1)}} = \\ = \sqrt[3]{t^2} - \sqrt[3]{t+1} + \sqrt[3]{t(t+1)^2}. \tag{7}$$

Universality of Ramanujan's identities over the field of rational numbers VIII

It can be verified directly that the identity (7) and Ramanujan's identity (2) are obtained from each other by substitutions

$$(m, n) = (t + 2, t - 1), \quad t = -\frac{2m + n}{m + 2n}.$$

Thus, Theorem 3 means that Ramanujan's identity (2) is universal over $\mathbb{Q}$ (as well as Ramanujan's identity (1)).

Remark 2

Unfortunately, in Zippel's paper (1985) an homogeneous version of (7) is given incorrectly:

$$\begin{aligned} &\sqrt[3]{3\sqrt[3]{ab(a+b)} - (b^3 + 6ab^2 + 3ba^2 - a^3)} = \\ &= \sqrt[3]{a^2 b} - \sqrt[3]{b^2(a+b)} + \sqrt[3]{a(a+b)^2}. \end{aligned} \tag{8}$$

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Remark 3

The incorrect identity (8) was reproduced in Antipov&Pimenov's paper (2020) in the form

$$\sqrt[3]{3\sqrt[3]{xy(x+y)} - (y^3 + 6xy^2 - 3x^2y - x^3)} =$$
$$= \sqrt[3]{x^2y} - \sqrt[3]{y^2(x+y)} + \sqrt[3]{x(x+y)^2}$$

with an additional typo.

A correct homogeneous version of (7) is

$$\sqrt[3]{a^3 - 3a^2b - 6ab^2 - b^3 + 3(a^2 + ab + b^2)\sqrt[3]{ab(a+b)}} =$$
$$= \sqrt[3]{a^2b} - \sqrt[3]{b^2(a+b)} + \sqrt[3]{a(a+b)^2}.$$

Example of non-universality I

However, Ramanujan's identity (2) does not universal over any field $P \subset \mathbb{R}$ (in contrast to Ramanujan's identity (1) which continues to remain universal).

An example

We have

$$\sqrt[3]{1+\sqrt[3]{3+2\sqrt{2}}} = \sqrt[9]{\frac{32+16\sqrt{2}}{729}} + \sqrt[9]{\frac{239+169\sqrt{2}}{729}} + \sqrt[9]{\frac{41-29\sqrt{2}}{729}} \qquad (9)$$

where $\omega = \sqrt[3]{3+2\sqrt{2}}$ is a real cubic radical over $P = \mathbb{Q}(\sqrt{2})$. The equality (9) cannot be obtained from (7) (even by substituting any real values of the variable t, and not just values from P).

Similar examples can be given for the fields

$$P = \mathbb{Q}(\sqrt{5}),\ \mathbb{Q}(\sqrt{29}),\ \dots$$

Extra results I

- The equality (9) can be rewritten in the form

$$\left(\left(1+\sqrt{2}\right)^{1/3}-\left(1-\sqrt{2}\right)^{1/3}\right)^{1/3}=9^{-1/3}\left(\left(1+\sqrt{2}\right)^{2/3}-\left(1-\sqrt{2}\right)^{2/3}+\sqrt{2}\right)$$

which is a particular case of the symbolic identity

$$\sqrt[3]{\sqrt[3]{\frac{729w^3+27g\sqrt{f}}{2}}+\sqrt[3]{\frac{729w^3-27g\sqrt{f}}{2}}}=\sqrt[3]{\frac{h+\sqrt{f}}{2}}+\sqrt[3]{\frac{h-\sqrt{f}}{2}}-\sqrt[3]{w^2}$$

where

$$\begin{aligned} f &= w^4+12w^3+36w^2-4w, \\ g &= 2w^4+6w^3+w, \\ h &= w^2+6w. \end{aligned}$$

We need to demand $f \geqslant 0$ that is equivalent to $w \notin (0, w_+)$ where

$$w_+ = \sqrt[3]{4}\cdot(\sqrt[3]{2}-1)^2 \approx 0.10724.$$

Extra results II

- Based on (7), one can derive another symbolic identity

$$\sqrt[3]{A+B\sqrt[3]{w(w+1)}}+\sqrt[3]{C+D\sqrt[3]{w(w+1)}}=1 \qquad (10)$$

where

$$A=\frac{w(w^3+6w^2+3w-1)}{(w-1)^3(w+1)},$$
$$B=-\frac{3w(w^2+w+1)}{(w-1)^3(w+1)},$$
$$C=\frac{w^3-3w^2-6w-1}{(w-1)^3(w+1)},$$
$$D=\frac{3(w^2+w+1)}{(w-1)^3(w+1)}.$$

Substituting $w=2$ in (10), we get a particular equality

$$\sqrt[3]{\frac{74}{3}-14\sqrt[3]{6}}+\sqrt[3]{-\frac{17}{3}+7\sqrt[3]{6}}=1.$$

Extra results III

Proposition 1

There are no equalities of the form

$$\sqrt[3]{a_1 + \omega_1} + \sqrt[3]{a_2 + \omega_2} = 1$$

where $a_i \in \mathbb{Q}$ and ω_i are non-proportional cubic radicals over $\mathbb{Q}$.

This fact can be briefly explained as follows. On the elliptic curve

$$y^2 + xy + y = x^3 + x^2$$

we have no rational points except

$$(x, y) \in \{(-1, 0), (0, 0), (0, -1)\}$$

(see `https://www.lmfdb.org/EllipticCurve/Q/15/a/7`).

Extra results IV

Proposition 2

Suppose that $a \in \mathbb{Z}$ is not a perfect cube. Then there are only two values of a for which

$$\rho_2 = \sqrt[3]{1 + \sqrt[3]{a}} \in R(\mathbb{Q})$$

(i.e., ρ_2 can be simplified): $a = -2$ (see (3)) and $a = -6860$ with

$$\sqrt[3]{1 - \sqrt[3]{6860}} = \sqrt[3]{\frac{5}{9}} - \sqrt[3]{\frac{16}{9}} - \sqrt[3]{\frac{100}{9}}.$$

The proof is based on Theorem 3. Substituting $t = p/q$ with

$$\gcd(p, q) = 1$$

in (6), we have to find all pairs (p, q) for which the number a is an integer. This leads us to solving certain diophantine equations (the so-called Thue equations) that can be performed via CAS.

The end

Thank you for attention!