Definite-Sum Solutions of Linear Recurrences With Polynomial Coefficients

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Outline

1. Introduction

Introduction

Notation:

$$\begin{array}{lcl} \mathbb{N} & = & \{0, 1, 2, \ldots\} \\ \mathbb{K} & \ldots & \text{field of characteristic } 0 \\ \mathbb{K}^{\mathbb{N}} & \ldots & \text{the set of all sequences over } \mathbb{K} \end{array}$$

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Let $E : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ be the *shift operator* w.r.t. n, acting on $a = \langle a_n \rangle_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ by

$$E a_n = a_{n+1} \quad \text{for all } n \geq 0, \quad \text{or}$$

$$(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, a_3, \ldots)$$

Introduction

Let $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in \mathbb{K}[n]$, $p_d \neq 0$. The operator $L : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ defined by

$$L \;=\; \sum_{j=0}^{d} p_j(n)\, E^j$$

is a *linear recurrence operator* with polynomial coefficients, acting on $a \in \mathbb{K}^{\mathbb{N}}$ by

$$(La)_n \;=\; \sum_{j=0}^{d} p_j(n)\, a_{n+j}.$$

Introduction

Notation:

$\mathbb{K}[n]\langle E\rangle$	...	the *algebra of linear recurrence operators* with polynomial coefficients in n
$\mathcal{P}(\mathbb{K})$	...	the set of *P-recursive sequences* over $\mathbb{K}$

Definition

A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is *P-recursive* if there is $L \in \mathbb{K}[n]\langle E\rangle$, $L \neq 0$, such that

$$(La)_n = 0$$

for all $n \geq 0$.

Creative Telescoping

GIVEN a bivariate hypergeometric sequence $F \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}} \setminus \{0\}$, i.e.,

$$\frac{F(n+1,k)}{F(n,k)}, \frac{F(n,k+1)}{F(n,k)} \in \mathbb{K}(n,k),$$

FIND $L \in \mathbb{K}[n]\langle E\rangle$ such that

$$L\left(\sum_{k=0}^{n} F(n,k)\right) = 0.$$

Solved (for many inputs) by Zeilberger's algorithm.

Inverse Creative Telescoping

GIVEN $L \in \mathbb{K}[n]\langle E \rangle$,

FIND a bivariate hypergeometric sequence $F \in \mathbb{K}^{\mathbb{N}\times\mathbb{N}} \setminus \{0\}$ s. t.

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Solved by Chen and Singer (2012) for *rational* $F(n,k)$.

A simpler version

GIVEN $L \in \mathbb{K}[n]\langle E\rangle$ and a (hypergeometric) kernel $K(n,k)$, FIND $h \in \mathbb{K}^{\mathbb{N}}$ such that

$$L\left(\sum_{k=0}^{n} h_k K(n,k)\right) = 0.$$

Example

Take $K(n,k) = \binom{n}{k} \in \mathbb{K}[n]$. Then

$$\sum_{k=0}^{n} h_k K(n,k) = \sum_{k=0}^{\infty} h_k \binom{n}{k}.$$

Idea: Interpret $f(x) := \sum_{k=0}^{\infty} h_k \binom{x}{k}$ as a *formal polynomial series*.

Introduction

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Idea: Interpret $f(x) := \sum_{k=0}^{\infty} h_k \binom{x}{k}$ as a *formal polynomial series*.

Notation:

We will also denote by $E\colon \mathbb{K}[x] \to \mathbb{K}[x]$ the shift operator w.r.t. x, acting on $p(x) \in K[x]$ by

$$Ep(x) \;=\; p(x+1).$$

So, $En = n+1$ and $Ex = x+1$.

2. Formal polynomial series

Formal polynomial series

Definition

A sequence of polynomials $\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ from $\mathbb{K}[x]$ is a factorial basis for $\mathbb{K}[x]$ if

(1) $\deg P_k = k$,

(2) $P_k \mid P_m$ for $k < m$.

Formal polynomial series

Proposition

$\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ *is a factorial basis* $\iff$

$$\exists \rho = (\rho_1, \rho_2, \rho_3, \ldots) \in \mathbb{K}^{\mathbb{N}\setminus\{0\}} \quad \exists c_0, c_1, c_2, \ldots \in \mathbb{K}^* :$$

$$P_k(x) = c_k(x - \rho_1)(x - \rho_2)\cdots(x - \rho_k) \quad \textit{for all } k \in \mathbb{N}.$$

Also known as a sequence of *polynomials with persistent roots* in *umbral calculus* (ρ is the *root sequence* of $\mathcal{B}$).

Formal polynomial series

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Notation:

$\mathcal{L}_{\mathbb{K}[x]}$ is the $\mathbb{K}$-algebra of linear operators $L \colon \mathbb{K}[x] \to \mathbb{K}[x]$.

Formal polynomial series

Example

Some factorial bases:

- $\mathcal{P} := \langle x^k \rangle_{k=0}^{\infty}$ *power basis*
- $\mathcal{F} := \langle x^{\underline{k}} \rangle_{k=0}^{\infty}$ *falling-factorial basis*
- $\mathcal{C} := \left\langle \binom{x}{k} \right\rangle_{k=0}^{\infty}$ *binomial-coefficient basis*
- $\mathcal{R} := \langle x^{\overline{k}} \rangle_{k=0}^{\infty}$ *rising-factorial basis*
- $\mathcal{A} := \left\langle \binom{x+k}{k} \right\rangle_{k=0}^{\infty}$ *Apéry basis*

Formal polynomial series

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Note:

$$\mathcal{C} = \left\langle \frac{x^{\underline{k}}}{k!} \right\rangle_{k=0}^{\infty}, \qquad \mathcal{A} = \left\langle \frac{(x+1)^{\overline{k}}}{k!} \right\rangle_{k=0}^{\infty}$$

Formal polynomial series

Definition

An operator $L \in \mathcal{L}_{\mathbb{K}[x]}$ and a factorial basis $\mathcal{B}$ are compatible iff there are $A, B \in \mathbb{N}$ such that for all $k \geq 0$

$$LP_k = \sum_{i=-A}^{B} \alpha_{i,k} P_{k+i} \tag{1}$$

for some $\alpha_{i,k} \in \mathbb{K}$. Here $P_{k+i} := 0$ if $k + i < 0$.

If (1) holds, L and $\mathcal{B}$ are (A, B)-compatible.

Formal polynomial series

Definition

Define $D, E, Q, X \in \mathcal{L}_{\mathbb{K}[x]}$ for $p \in \mathbb{K}[x]$ and $q \in \mathbb{K}^*$ by

$$\begin{array}{lcll} Dp(x) & := & p'(x) & \text{(differentiation)}, \\ Ep(x) & := & p(x+1) & \text{(shift)}, \\ Qp(x) & := & p(qx) & \text{(q-shift)}, \\ Xp(x) & := & xp(x) & \text{(multiplication by x)}. \end{array}$$

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Example

- $(x^k)' = kx^{k-1} \implies D$ is (1,0)-compatible with $\mathcal{P}$
- $(qx)^k = q^k x^k \implies Q$ is (0,0)-compatible with $\mathcal{P}$
- $\binom{x+1}{k} = \binom{x}{k-1} + \binom{x}{k} \implies E$ is (1,0)-compatible with $\mathcal{C}$

Formal polynomial series

Example

- $(x+1)^k = \sum_{j=0}^{k} \binom{k}{j} x^j \implies E$ is not compatible with $\mathcal{P}$
- $(x+1)^{\overline{k}} = \sum_{j=0}^{k} \frac{k!}{j!} x^{\overline{j}} \implies E$ is not compatible with $\mathcal{R}$
- $\binom{x+1+k}{k} = \sum_{j=0}^{k} \binom{x+j}{j} \implies E$ is not compatible with $\mathcal{A}$
- $\binom{x}{k}' = \sum_{j=0}^{k-1} \frac{(-1)^{j+k}}{j-k} \binom{x}{j} \implies D$ is not compatible with $\mathcal{C}$

Formal polynomial series

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Proposition

- $xP_k(x) = u_k P_k(x) + v_k P_{k+1}(x)$

  $\implies$ *X is (0,1)-compatible with every factorial basis*

Formal polynomial series

Proposition

E and $\mathcal{B}$ with root sequence ρ are $(A,0)$-compatible iff

$$\forall n \in \mathbb{N}\colon \{\rho_1 + 1, \rho_2 + 1, \ldots, \rho_n + 1\} \subseteq \{\rho_1, \rho_2, \ldots, \rho_{n+A}\}$$

as multisets.

Formal polynomial series

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as multisets.

Example

- $P_k(x) = x^k$: $\rho = (0, 0, 0, \ldots)$, $\rho + 1 = (1, 1, 1, \ldots)$ X
- $P_k(x) = x^{\underline{k}}$: $\rho = (0, 1, 2, \ldots)$, $\rho + 1 = (1, 2, 3, \ldots)$ $\checkmark$
- $P_k(x) = x^{\overline{k}}$: $\rho = (0, -1, -2, \ldots)$, $\rho + 1 = (1, 0, -1, \ldots)$ X

Formal polynomial series

Proposition

Let $\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ *be a factorial basis of* $\mathbb{K}[x]$. *The algebra* $\mathbb{K}[x]$ *naturally embeds into the algebra* $\mathbb{K}[[\mathcal{B}]]$ *of formal polynomial series of the form*

$$y = \sum_{k=0}^{\infty} c_k P_k(x) \qquad (c_k \in \mathbb{K}),$$

with multiplication defined by

$$\left(\sum_{i=0}^{\infty} c_i P_i(x)\right) \left(\sum_{j=0}^{\infty} d_j P_j(x)\right) = \sum_{k=0}^{\infty} e_k P_k(x),$$

$$e_k = \sum_{\max\{i,j\} \le k \le i+j} c_i d_j \, [P_k](P_i P_j).$$

Formal polynomial series

Let $L \in \mathcal{L}_{\mathbb{K}[x]}$ be (A, B)-compatible with the basis $\mathcal{B}$.

Extend $L : \mathbb{K}[x] \to \mathbb{K}[x]$ to $L : \mathbb{K}[[\mathcal{B}]] \to \mathbb{K}[[\mathcal{B}]]$ by defining

$$
\begin{aligned}
L\left(\sum_{k=0}^{\infty} c_k P_k(x)\right) &:= \sum_{k=0}^{\infty} c_k L P_k(x) \\
&= \sum_{k=0}^{\infty} c_k \sum_{i=-A}^{B} \alpha_{i,k} P_{k+i}(x) \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=-B}^{A} \alpha_{-i,k+i} c_{k+i} \right) P_k(x)
\end{aligned}
$$

where $P_k(x) = 0$ for $k < 0$ and $c_{k+i} = 0$ for $k + i < 0$.

Formal polynomial series

Theorem

For any $L \in \mathcal{L}_{\mathbb{K}[x]}$ *and* $y = \sum_{k=0}^{\infty} c_k P_k(x) \in \mathbb{K}[[\mathcal{B}]]$ *we have*

$$L\,y = 0 \quad \Longleftrightarrow \quad L'c = 0$$

where $L' = \mathcal{R}_{\mathcal{B}} L$ *is the operator induced by* L *in basis* $\mathcal{B}$*:*

$$\mathcal{R}_{\mathcal{B}} L \; := \sum_{i=-B}^{A} \alpha_{-i,k+i} E_k^i, \tag{2}$$

$$\begin{aligned} E_k^i(c)_k &= c_{k+i} \quad \text{for all } i, k \in \mathbb{Z}, \\ c_j &= 0 \qquad \text{for } j < 0. \end{aligned}$$

Formal polynomial series

Definition

Let $\mathcal{B}$ be a factorial basis of $\mathbb{K}[x]$.

- $\mathcal{L}_{\mathcal{B}} := \{L \in \mathcal{L}_{\mathbb{K}[x]};\ L \text{ compatible with } \mathcal{B}\}$,
- $\mathcal{E} :=$
$$\left\{\sum_{i=-S}^{R} a_k^{(i)} E_k^i;\ R, S \in \mathbb{N}, a^{(i)} \in \mathbb{K}^{\mathbb{Z}} \text{ for } -S \leq i \leq R\right\}.$$

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Proposition

$\mathcal{L}_{\mathcal{B}}$ and $\mathcal{E}$ are $\mathbb{K}$-algebras, and the transformation

$$\mathcal{R}_{\mathcal{B}} : \mathcal{L}_{\mathcal{B}} \to \mathcal{E}$$

is an isomorphism of $\mathbb{K}$-algebras.

Formal polynomial series

Example

Differential operators:

$$\begin{aligned} \mathcal{R}_{\mathcal{P}} D &= (k+1)E_k \\ \mathcal{R}_{\mathcal{P}} X &= E_k^{-1} \end{aligned}$$

q-Difference operators:

$$\begin{aligned} \mathcal{R}_{\mathcal{P}} Q &= q^k \\ \mathcal{R}_{\mathcal{P}} X &= E_k^{-1} \end{aligned}$$

Recurrence operators:

$$\begin{aligned} \mathcal{R}_{\mathcal{C}} E &= E_k + 1 \\ \mathcal{R}_{\mathcal{C}} X &= k(E_k^{-1} + 1) \end{aligned}$$

Formal polynomial series

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$$L \;=\; E - c \quad \text{where } c \in \mathbb{K}$$

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FIND $h_k \neq 0$ s.t. $\mathcal{R}_{\mathcal{C}} L(h_k) = 0.$

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SOLUTION: $(\mathcal{R}_{\mathcal{C}} L)\left((c-1)^k\right) = 0$

$$\Longrightarrow \quad L\left(\sum_{k=0}^{n} \binom{n}{k} (c-1)^k\right) = 0.$$

Formal polynomial series

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$$\implies \; L\left(\sum_{k=0}^{n} \binom{n}{k} (c-1)^k\right) = 0.$$

Since $c^0 = \sum_{k=0}^{0} \binom{0}{k} (c-1)^k = 1$, we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k} (c-1)^k = c^n.$$

Formal polynomial series

Example

$$\begin{aligned} L &= E^3 - (n^2 + 6n + 10)E^2 \\ &\quad + (n+2)(2n+5)E - (n+1)(n+2) \end{aligned}$$

Note: $Ly = 0$ has no nonzero Liouvillian solution.

Formal polynomial series

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Note: $Ly = 0$ has no nonzero Liouvillian solution.

$$\begin{aligned} \mathcal{R}_{\mathcal{C}} L &= E_k^3 - (k^2 + 6k + 7)E_k^2 \\ &\quad - (2k^2 + 8k + 7)E_k - (k+1)^2 \end{aligned}$$

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SOLUTION: $(\mathcal{R}_{\mathcal{C}} L)(k!^2) = 0$

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$$\mathcal{R}_{\mathcal{C}}L = E_k^3 - (k^2 + 6k + 7)E_k^2 - (2k^2 + 8k + 7)E_k - (k+1)^2$$

FIND $h_k \neq 0$ s.t. $\mathcal{R}_{\mathcal{C}}L(h_k) = 0$.

SOLUTION: $(\mathcal{R}_{\mathcal{C}}L)(k!^2) = 0 \implies L\left(\sum_{k=0}^{n} \binom{n}{k} k!^2\right) = 0.$

3. Quasi-triangular bases

Quasi-triangular bases

Question: Which formal power series solutions

$$y(x) = \sum_{k=0}^{\infty} h_k P_k(x)$$

give rise to genuine solutions in $\mathbb{K}^{\mathbb{N}}$?

Quasi-triangular bases

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Example

If $\mathcal{B} = \mathcal{C} = \left\langle \binom{x}{k} \right\rangle_{k=0}^{\infty}$ then

$$\sum_{k=0}^{\infty} h_k P_k(n) \;=\; \sum_{k=0}^{\infty} h_k \binom{n}{k} \;=\; \sum_{k=0}^{n} h_k \binom{n}{k} \;\in \mathbb{K}[[\mathcal{B}]]$$

for every $n \in \mathbb{N}$.

Quasi-triangular bases

Definition

A shift-compatible basis $\mathcal{B} = \langle P_k(n) \rangle_{k=0}^{\infty}$ is *quasi-triangular* if there is a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

1. $\forall k, n \in \mathbb{N} \colon (k > f(n) \implies P_k(n) = 0)$,
2. $\forall n \in \mathbb{N} \colon P_{f(n)}(n) \neq 0$.

Quasi-triangular bases

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Example

The basis $\mathcal{C} = \langle \binom{n}{k} \rangle_{k=0}^{\infty}$ is quasi-triangular with $f(n) = n$.

Quasi-triangular bases

Proposition

Let $\mathcal{B} = \langle P_k(n) \rangle_{k=0}^{\infty}$ be a quasi-triangular basis. Then for every $a \in \mathbb{K}^{\mathbb{N}}$ there exists $b \in \mathbb{K}^{\mathbb{N}}$ such that

$$a_n = \sum_{k=0}^{\infty} b_k P_k(n)$$

for some $b \in \mathbb{K}^{\mathbb{N}}$.

Quasi-triangular bases

Proposition

Let $\mathcal{B} = \langle P_k(n) \rangle_{k=0}^{\infty}$ *be a quasi-triangular basis. Then for every* $a \in \mathbb{K}^{\mathbb{N}}$ *there exists* $b \in \mathbb{K}^{\mathbb{N}}$ *such that*

$$a_n \;=\; \sum_{k=0}^{\infty} b_k P_k(n)$$

for some $b \in \mathbb{K}^{\mathbb{N}}$.

Proposition

A basis $\mathcal{B} = \langle P_k(n) \rangle_{k=0}^{\infty}$ *is quasi-triangular if and only if its root sequence* $\rho = \langle \rho_1, \rho_2, \rho_3, \ldots \rangle$ *satisfies*

1. $\langle 0, 1, 2, 3, \ldots \rangle$ *is a subsequence of* ρ,
2. *for every* $n \in \mathbb{N}$, *the first appearance of* n *in* ρ *precedes the first appearance of* $n+1$ *in* ρ.

4. Product bases

Product bases

Problem: We need more bases compatible with E.

Product bases

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Idea: Use *products* of compatible bases.

Definition

Let $\mathcal{B}_i = \langle P_k^{(i)}(x)\rangle_{k=0}^{\infty}$ be a basis of $\mathbb{K}[x]$ for $i = 1, 2, \ldots, m$.
For all $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, m-1\}$, let

$$P_{mk+j}^{(\pi)}(x) \;:=\; \prod_{i=1}^{j} P_{k+1}^{(i)}(x) \cdot \prod_{i=j+1}^{m} P_k^{(i)}(x).$$

Then $\prod_{i=1}^{m} \mathcal{B}_i := \langle P_n^{(\pi)}(x)\rangle_{n=0}^{\infty}$ is the *product* of $\mathcal{B}_1, \ldots, \mathcal{B}_m$.

Product bases

Example

Let $m = 2$, $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{C}$. Then $P_k^{(1)}(x) = P_k^{(2)}(x) = \binom{x}{k}$, and so

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$$P_{2k}^{(\pi)}(x) = P_k^{(1)}(x) P_k^{(2)}(x) = \binom{x}{k}^2,$$

Product bases

Example

Let $m = 2$, $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{C}$. Then $P_k^{(1)}(x) = P_k^{(2)}(x) = \binom{x}{k}$, and so

$$P_{2k}^{(\pi)}(x) = P_k^{(1)}(x)P_k^{(2)}(x) = \binom{x}{k}^2,$$

$$P_{2k+1}^{(\pi)}(x) = P_{k+1}^{(1)}(x)P_k^{(2)}(x) = \binom{x}{k+1}\binom{x}{k}.$$

Product bases

Theorem

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ be factorial bases of $\mathbb{K}[x]$, and $L \in \mathcal{L}_{\mathbb{K}[x]}$.

1. $\prod_{i=1}^{m} \mathcal{B}_i$ *is a factorial basis of* $\mathbb{K}[x]$.
2. *Let L be a ring endomorphism of $\mathbb{K}[x]$, and let each $\mathcal{B}_i$ be (A_i, B_i)-compatible with L.*
   Denote $A = \max_{1 \le i \le m} A_i$ and $B = \min_{1 \le i \le m} B_i$.
   Then $\prod_{i=1}^{m} \mathcal{B}_i$ is (mA, B)-compatible with L.

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Proposition

Let $a \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{K}$, and $\mathcal{C}_{a,b} := \left\langle \binom{ax+b}{k} \right\rangle_{k=0}^{\infty}$.
Then E is $(a, 0)$-compatible with $\mathcal{C}_{a,b}$.

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Then E is $(a, 0)$-compatible with $\mathcal{C}_{a,b}$.

Proof: By the Chu-Vandermonde identity,

$$\binom{a(x+1)+b}{k} = \sum_{i=0}^{a} \binom{a}{i} \binom{ax+b}{k-i} = \sum_{i=-a}^{0} \binom{a}{-i} \binom{ax+b}{k+i}.$$

Product bases

Corollary

The product basis $\prod_{i=1}^{m} \mathcal{C}_{a_i,b_i}$ *is a factorial basis of* $\mathbb{K}[x]$*, corresponding to the kernel*

$$K(n,k) = \prod_{i=1}^{m} \binom{a_i n + b_i}{k}.$$

It is $(mA, 0)$*-compatible with* E *where* $A = max_{1 \leq i \leq m} a_i$*.*

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$\mathcal{B} = \mathcal{C}^2$: $m = 2$, $A = 1 \implies E$ is (2,0)-compatible with $\mathcal{B}$

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Example

$\mathcal{B} = \mathcal{C}^2$: $m = 2$, $A = 1 \implies E$ is (2,0)-compatible with $\mathcal{B}$

$$\begin{aligned} P_{2k}(x+1) &= P_{2k}(x) + 2P_{2k-1}(x) + P_{2k-2}(x) \\ P_{2k+1}(x+1) &= P_{2k+1}(x) + \frac{2k+1}{k+1} P_{2k}(x) + \frac{k}{k+1} P_{2k-1}(x) \end{aligned}$$

Product bases

Example (continued)

X is $(0,1)-$compatible with $\mathcal{B}$:

$$\begin{aligned} x \cdot P_{2k}(x) &= (k+1)P_{2k+1}(x) + kP_{2k}(x), \\ x \cdot P_{2k+1}(x) &= (k+1)P_{2k+2}(x) + kP_{2k+1}(x). \end{aligned}$$

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Problems:

1. The coefficients of $\mathcal{R}_{\mathcal{B}}L$ need not belong to $\mathbb{K}(k)$.
2. $\operatorname{ord} \mathcal{R}_{\mathcal{B}}L$ may exceed $\operatorname{ord} L$ by a factor of mA.
3. We need only those $h \in \ker \mathcal{R}_{\mathcal{B}}L$ satisfying

$$k \not\equiv 0 \pmod{m} \implies h_k = 0.$$

5. Sieved polynomial bases

Sieved polynomial bases

Definition

Call the sequence $b \in \mathbb{K}^{\mathbb{N}}$ defined by

$$b_k = a_{mk+j} \text{ for all } k \in \mathbb{N}$$

the *j-th m-section* of $a \in \mathbb{K}^{\mathbb{N}}$, and denote it by $s_j^m a$.

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Notation

For

$$y(x) = \sum_{k=0}^{\infty} c_k P_k(x) \in \mathbb{K}[[\mathcal{B}]]$$

let $\beta y = c' \in \mathbb{K}^{\mathbb{Z}}$ *be the doubly-infinite sequence where*

$$c'_k = \begin{cases} c_k, & \textit{if } k \geq 0, \\ 0, & \textit{if } k < 0. \end{cases}$$

Sieved polynomial bases

Theorem

For $L \in \mathcal{L}_{\mathbb{K}[x]}$, $k \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{0\}$, $r, j \in \{0, \dots, m-1\}$, *let*

$$L_{r,j} \;:=\; \sum_{\substack{-A \le i \le B \\ i+j \equiv r \pmod m}} \alpha_{k+\frac{r-i-j}{m},j,i} E_k^{\frac{r-i-j}{m}} \;\in\; \mathcal{E}$$

where

$$LP_{mk+j}(x) \;=\; \sum_{i=-A}^{B} \alpha_{k,j,i} P_{mk+j+i}(x).$$

Then for every $y \in \mathbb{K}[[\mathcal{B}]]$ *and* $r \in \{0, 1, \dots, m-1\}$,

$$s_r^m \beta(Ly) \;=\; \sum_{j=0}^{m-1} L_{r,j} \left(s_j^m \beta y \right).$$

Sieved polynomial bases

Corollary

$Ly = 0 \iff \forall r \in \{0, 1, \ldots, m-1\}$:

$$\sum_{j=0}^{m-1} L_{r,j}\left(s_j^m \beta y\right) = 0.$$

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Notation

$$[\mathcal{R}_{\mathcal{B}} L] \;:=\; [L_{r,j}]_{r,j=0}^{m-1}$$

Proposition

$$\left[\mathcal{R}_{\mathcal{B}}\left(L^{(1)} L^{(2)}\right)\right] \;=\; \left[\mathcal{R}_{\mathcal{B}} L^{(1)}\right]\left[\mathcal{R}_{\mathcal{B}} L^{(2)}\right]$$

Sieved polynomial bases

- To construct $[\mathcal{R}_{\mathcal{B}} L]$ for some $L \in \mathbb{K}[x]\langle E\rangle$:
  1. compute $[\mathcal{R}_{\mathcal{B}} E]$ and $[\mathcal{R}_{\mathcal{B}} X]$;
  2. everywhere in L substitute
     - $E \mapsto [\mathcal{R}_{\mathcal{B}} E]$,
     - $x \mapsto [\mathcal{R}_{\mathcal{B}} X]$,
     - $c \in \mathbb{K}^* \mapsto c\, I_m$.

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     - $E \mapsto [\mathcal{R}_{\mathcal{B}} E]$,
     - $x \mapsto [\mathcal{R}_{\mathcal{B}} X]$,
     - $c \in \mathbb{K}^* \mapsto c\, I_m$.

- We are looking for $y \in \ker L$ of the form

$$y(x) = \sum_{k=0}^{\infty} h_k P_{mk}(x),$$

so we have $s_0^m \beta y = h$ and $s_j^m \beta y = 0$ for all $j \neq 0$.

Sieved polynomial bases

- For such y, the last Corollary implies

$$Ly = 0 \iff \forall r \in \{0, 1, \ldots, m-1\} \colon L_{r,0}\, h = 0$$
$$\iff \mathrm{gcrd}(L_{0,0}, L_{1,0}, \ldots, L_{m-1,0})\, h = 0.$$

- So we only need column 0 of $[\mathcal{R}_{\mathcal{B}} L]$ to construct the desired annihilator

$$L' = \mathrm{gcrd}(L_{0,0}, L_{1,0}, \ldots, L_{m-1,0})$$

of the unknown h.

Sieved polynomial bases

Example

Find a solution $y \neq 0$ of $L\,y = 0$ where

$$\begin{aligned} L &= 4(2n+3)^2(4n+3)E^2 \\ &\quad - 2(4n+5)\left(20n^2+50n+27\right)E + 9(4n+7)(n+1)^2. \end{aligned}$$

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Find a solution $y \neq 0$ *of* $L\,y = 0$ *where*

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For $\mathcal{B} = \mathcal{C}^2$ we compute

$$[\mathcal{R}_{\mathcal{B}}E] = \begin{bmatrix} E_k + 1 & \frac{2k+1}{k+1} \\ 2E_k & \frac{k+1}{k+2}E_k + 1 \end{bmatrix},$$

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Sieved polynomial bases

Example (continued)

This yields

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$$
\begin{aligned}
L_{0,0} &= 4(2k+3)^2(4k+3)E_k^2 \\
&+ \frac{2\left(592k^4+1388k^3+1254k^2+519k+81\right)}{k+1}E_k \\
&+ 676k^3-889k^2-466k-99-(244k+41)k^2E_k^{-1}, \\
L_{1,0} &= \frac{8(2k+3)\left(28k^3+108k^2+132k+51\right)}{k+2}E_k^2 \\
&+ 4\left(360k^3+720k^2+451k+82\right)E_k \\
&- 2(k+1)\left(74k^2+377k+133\right)-60(k+1)k^2E_k^{-1},
\end{aligned}
$$

Sieved polynomial bases

Example (continued)

with $L' = \operatorname{gcrd}\left(L_{0,0}, L_{1,0}\right) = 1 - \dfrac{k}{2(2k-1)} E_k^{-1}$.

Sieved polynomial bases

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with $L' = \operatorname{gcrd}(L_{0,0}, L_{1,0}) = 1 - \dfrac{k}{2(2k-1)} E_k^{-1}$.

Looking for hypergeometric solutions h of $L'h = 0$, we obtain $h_k = \frac{1}{\binom{2k}{k}}$, hence

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$$y_n = \sum_{k=0}^{\infty} \frac{\binom{n}{k}^2}{\binom{2k}{k}}$$

is a definite-sum solution of $L\,y = 0$.

Sieved polynomial bases

Example

Find a definite-sum solution $y \neq 0$ *of* $L\,y = 0$ *where*

$$L = (n+2)^2 E^2 - (11n^2 + 33n + 25)E - (n+1)^2$$

using the kernel $K(n,k) = \binom{n}{k}\binom{n+k}{2k}$.

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Find a definite-sum solution $y \neq 0$ of $L\,y = 0$ where

$$L = (n+2)^2 E^2 - (11n^2 + 33n + 25)E - (n+1)^2$$

using the kernel $K(n,k) = \binom{n}{k}\binom{n+k}{2k}$.

Define the basis $\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ by

$$\begin{aligned} P_{3k}(x) &= \binom{x}{k}\binom{x+k}{2k}, \\ P_{3k+1}(x) &= \binom{x}{k}\binom{x+k}{2k+1}, \\ P_{3k+2}(x) &= \binom{x}{k+1}\binom{x+k}{2k+1}. \end{aligned}$$

Example (continued)

It is not difficult to see that basis $\mathcal{B}$ is factorial, hence the operator X is $(0, 1)$-compatible with $\mathcal{B}$:

$$
\begin{aligned}
xP_{3k}(x) &= (2k+1)P_{3k+1}(x) + kP_{3k}(x) \\
xP_{3k+1}(x) &= (k+1)P_{3k+2}(x) + kP_{3k+1}(x) \\
xP_{3k+2}(x) &= 2(k+1)P_{3k+3}(x) - (k+1)P_{3k+2}(x)
\end{aligned}
$$

Sieved polynomial bases

Example (continued)

The shift operator E is $(3,0)$-compatible with $\mathcal{B}$:

$$\begin{aligned} P_{3k}(x+1) &= P_{3k}(x) \\ &\quad + \frac{3}{2}P_{3k-1}(x) + \frac{8k-3}{2k}P_{3k-2}(x) + P_{3k-3}(x) \\ P_{3k+1}(x+1) &= P_{3k+1}(x) \\ &\quad + \frac{3k+1}{2k+1}P_{3k}(x) + \frac{k}{2k+1}P_{3k-1}(x) + \frac{2k-1}{2k+1}P_{3k-2}(x) \\ P_{3k+2}(x+1) &= P_{3k+2}(x) + \frac{3k+2}{k+1}P_{3k+1}(x) + P_{3k}(x) \end{aligned}$$

Sieved polynomial bases

Example (continued)

The associated operator matrices are:

$$[\mathcal{R}_{\mathcal{B}}E] = \begin{bmatrix} E_k + 1 & \frac{3k+1}{2k+1} & 1 \\ \frac{8k+5}{2(k+1)}E_k & \frac{2k+1}{2k+3}E_k + 1 & \frac{3k+2}{k+1} \\ \frac{3}{2}E_k & \frac{k+1}{2k+3}E_k & 1 \end{bmatrix},$$

$$[\mathcal{R}_{\mathcal{B}}X] = \begin{bmatrix} k & 0 & 2k\,E_k^{-1} \\ 2k+1 & k & 0 \\ 0 & k+1 & -(k+1) \end{bmatrix}$$

Sieved polynomial bases

Example (continued)

The associated operator matrices are:

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For $L = (n+2)^2 E^2 - (11n^2 + 33n + 25)E - (n+1)^2$ we find:

Sieved polynomial bases

Example (continued)

$$
\begin{aligned}
L_{0,0} &= (k+2)^2 E_k^2 + \frac{29k^3+46k^2+14k-1}{2k+1} E_k \\
&\quad - 2(37k^2+41k+11), \\
L_{1,0} &= \frac{(k+2)(4k+5)(12k^2+26k+11)}{2(k+1)(2k+3)} E_k^2 \\
&\quad - \frac{79+237k+199k^2+47k^3}{2(1+k)} E_k - (2k+1)(49k+31), \\
L_{2,0} &= \frac{(k+2)(22k^2+62k+43)}{2(2k+3)} E_k^2 \\
&\quad - \frac{3}{2}(11k^2+34k+25) E_k - 11(k+1)(2k+1),
\end{aligned}
$$

Example (continued)

and

$$\mathrm{gcrd}(L_{0,0}, L_{1,0}, L_{2,0}) = E_k - 2\,\frac{2k+1}{k+1}.$$

Sieved polynomial bases

Example (continued)

and

$$\mathrm{gcrd}(L_{0,0}, L_{1,0}, L_{2,0}) = E_k - 2\,\frac{2k+1}{k+1}.$$

So $h_k = \binom{2k}{k}$ satisfies $L_{0,0}h_k = L_{1,0}h_k = L_{2,0}h_k = 0$.

Sieved polynomial bases

Example (continued)

and

$$\operatorname{gcrd}(L_{0,0}, L_{1,0}, L_{2,0}) = E_k - 2\,\frac{2k+1}{k+1}.$$

So $h_k = \binom{2k}{k}$ satisfies $L_{0,0}h_k = L_{1,0}h_k = L_{2,0}h_k = 0$. Since

$$h_k P_{3k}(n) \;=\; \binom{2k}{k}\binom{n}{k}\binom{n+k}{2k} \;=\; \binom{n}{k}^2\binom{n+k}{k},$$

Sieved polynomial bases

Example (continued)

and

$$\operatorname{gcrd}(L_{0,0}, L_{1,0}, L_{2,0}) = E_k - 2\,\frac{2k+1}{k+1}.$$

So $h_k = \binom{2k}{k}$ satisfies $L_{0,0}h_k = L_{1,0}h_k = L_{2,0}h_k = 0$. Since

$$h_k P_{3k}(n) = \binom{2k}{k}\binom{n}{k}\binom{n+k}{2k} = \binom{n}{k}^2\binom{n+k}{k},$$

we have found that Apéry's $\zeta(2)$-sequence

$$y_n = \sum_{k=0}^{\infty} h_k P_{3k}(n) = \sum_{k=0}^{\infty}\binom{n}{k}^2\binom{n+k}{k}$$

Example (continued)

is a solution of

$$(n+2)^2 y_{n+2} - (11n^2 + 33n + 25) y_{n+1} - (n+1)^2 y_n = 0.$$

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$$(n+2)^2 y_{n+2} - (11n^2 + 33n + 25) y_{n+1} - (n+1)^2 y_n = 0.$$

Similarly, using the kernel $K(n,k) = \binom{n+k}{2k}^2$, we can show that Apéry's $\zeta(3)$-sequence

Example (continued)

is a solution of

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Similarly, using the kernel $K(n,k) = \binom{n+k}{2k}^2$, we can show that Apéry's $\zeta(3)$-sequence

$$y_n = \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k}$$

is a solution of

$$(n+2)^3 y_{n+2} - (2n+3)(17n^2 + 51n + 39) y_{n+1} + (n+1)^3 y_n = 0.$$

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Thank you for your attention!