On Divergence of Puiseux Series Asymptotic Expansions of Solutions to the Third Painlevé Equation

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The talk is based on our joint works [4], [5] with Andrey V. Vasilyev.

Consider the third Painlevé equation

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w},$$

with $\gamma = 0$, $\alpha, \beta, \delta \in \mathbb{C}$, $\alpha\delta \neq 0$. By making the transform $w(z) = z^{1/3}u(x)$, $x = \frac{3}{2}z^{2/3}$, we obtain the equation:

$$u''_{xx} = \frac{(u'_x)^2}{u} - \frac{u'_x}{x} + \alpha u^2 + \frac{3}{2}\frac{\beta}{x} + \frac{\delta}{u}. \tag{1}$$

All formal power series satisfying equation (1) near infinity have the form

$$\sum_{n=0}^{\infty} a_n x^{-n}, \tag{2}$$

where coefficients a_n are defined by recurrence conditions.

Using the general theory [6] we can say that these series asymptotically approximate analytic solutions of the equation (1) in an appropriate sectors with the vertex at infinity.

The aim of our present work is to answer the questions "How precise is such an approximation?" and "What is about convergence or divergence of these series (2)?".

Definition 1 *Let* $\Omega_R(\varphi_1, \varphi_2) = \{z : |z| > R, \operatorname{Arg} z \in (\varphi_1, \varphi_2)\}$. *Let* w *be an analytic function in* $\Omega_R(\varphi_1, \varphi_2)$ *and* $\hat{f} = \sum\limits_{n=0}^{\infty} a_n z^{-n}$. *The function* w *is said to be [6] asymptotically approximated of Gevrey order* $1/k$ *by the series* $\hat{f}$ *in* $\Omega_R(\varphi_1, \varphi_2)$ *if for the points* z *of any closed subsector* Y *of* $\Omega_R(\varphi_1, \varphi_2)$ *there exist constants* $A_Y, C > 0$ *such that for any* $n \in \mathbb{N}$:

$$|z^n||w(z) - \sum_{p=0}^{n-1} a_p z^{-p}| < C(n!)^{1/k} A_Y^n.$$

Definition 2 *A formal series* $\hat{f}$ *is called a series of Gevrey order* $1/k$ [6] *if there exist constants* $k, M, C > 0$ *such that*

$$|a_n| \leqslant C(n!)^{1/k} M^n \; \forall n \in \mathbb{N}. \tag{3}$$

It is called a series of exact Gevrey order $1/k$ [6] *if it is of Gevrey order* $1/k$ *and there exists no* $k' > k$ *such that it is of Gevrey order* $1/k'$.

Definition 3 *The formal Borel transform of index* k*, applied to* $\hat{f}$*, is the formal series* [1] $\xi^{-k} \sum\limits_{n=1}^{\infty} \frac{a_n \xi^n}{\Gamma(n/k)}$.

In [5] it is shown that series (2) are series of Gevrey order one. So we have answered the first question under consideration.

As is proved in [2] series (2) represent a rational function iff $\beta = 0$ or $\beta \neq 0, \delta = -\beta^2/(4k)^2, k \in \mathbb{Z} \setminus \{0\}$, so for such values of the parameters estimates (3) are not precise. To answer the second question we should find (or show that they do not exist) the parameters of the third Painlevé equation for which series (2) are of exact Gevrey order one, hence diverge.

Consider the values of the parameters of equation (1) $\alpha = -1/32$, $\beta = -1/4$, $\delta = -1/32$ and assume that the branch of the cube root fixed so that $a_0 = -1$ (under these conditions all the coefficients $a_n \in \mathbb{R}$).

Assertion 4 *For the coefficients a_n of series (2) with the values of the parameters fixed above the following holds:*

$$4\left(\frac{32}{3}\right)^{\frac{n}{2}-1}(n-2)! \leqslant a_n \leqslant \left(\frac{32}{3}\right)^{\frac{n}{2}-1} n! \text{ with } n \geqslant 5.$$

Assertion 5 *Series (2) with the parameters $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, $\delta = -\beta^2/2$ are of exact Gevrey order one and diverge.*

We show that series (2) with the parameters $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, $\delta = -\beta^2/2$ are 1-summable. Then we obtain the formula for 1-sum of these series in the direction d applying the composition of formal Borel transform of index one and the Laplace transform of index one for all but finitely many directions d:

$$f(x^{-1}) = \int_0^{\infty(d)} \left(\xi^{-1}\sum_{n=1}^{\infty}\frac{a_n}{(n-1)!}\xi^n e^{-x\xi}\right) d\xi. \tag{4}$$

Finally, we conclude, using the theorem in [3], that functions (4) are analytic functions approximated of Gevrey order one by series under consideration in $\{x \in \mathbb{C} : \mathrm{Re}(xe^{id}) > c, |x| > \sqrt{\frac{3}{32}}\}$ for some $c > 0$. The difference of these analytic functions and solutions to equation (1) are functions which are approximated of Gevrey order one by series with zero coefficients in sectors with the vertices at infinity.

References

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