Panorbital residues for elliptic summability*

Carlos E. Arreche & Matthew Babbitt
The University of Texas at Dallas

November 20, 2024

*Work in progress.

Motivation: residues for rational integrability

Each $f(x) \in \mathbb{C}(x)$ has a unique partial fraction decomposition

$$f(x) = p(x) + \sum_{k \geqslant 1} \sum_{\alpha \in \mathbb{C}} \frac{c_k(\alpha)}{(x-\alpha)^k};$$

where $p(x) \in \mathbb{C}[x]$, and $c_k(\alpha) \in \mathbb{C}$, almost all 0.

Fact: $f(x)$ is *rationally integrable*, i.e., $f(x) = g'(x)$ for some $g(x) \in \mathbb{C}(x)$, if and only if the *residues*

$$\operatorname{res}(f, \alpha, 1) := c_1(\alpha) = 0 \qquad \text{for every} \quad \alpha \in \mathbb{C}.$$

Thus we say that the residues $\operatorname{res}(f, \alpha, 1)$ form a *complete obstruction* to the rational integrability of f.

Motivation: discrete residues for rational summability

(Chen-Singer 2012) show: $f(x)$ *rationally summable*, i.e.,

$$f(x) = g(x+1) - g(x)$$

for some $g(x) \in \mathbb{C}(x)$, if and only if for every $k \geqslant 1$ and every $\mathbb{Z}$-*orbit* $\omega \in \mathbb{C}/\mathbb{Z}$ the *discrete residues*

$$\mathrm{dres}(f, \omega, k) := \sum_{\alpha \in \omega} c_k(\alpha) = 0.$$

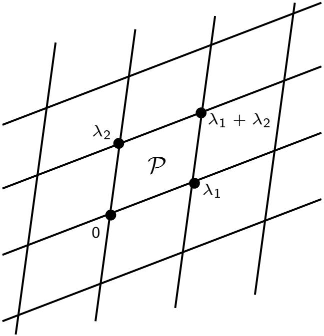
Thus we say that the discrete residues $\mathrm{dres}(f, \alpha, 1)$ form a *complete obstruction* to the rational summability of f.

The construction rests on the fundamental result (Abramov 1971): if $f(x)$ is summable, its denominator has strictly positive *dispersion*.

Similar idea is successful in q-dilation case (Chen-Singer 2012) and Mahler case (A.-Zhang 2022, A.-Zhang 2024).

Elliptic curves and functions: analytic viewpoint (1 of 2)

For a given lattice $\Lambda \subset \mathbb{C}$, the field M_Λ of Λ-*elliptic functions* consists of meromorphic functions $f(z)$ on $\mathbb{C}$ such that $f(z+\lambda) = f(z)$ for every $\lambda \in \Lambda$.



Given a $\mathbb{Z}$-basis $\{\lambda_1, \lambda_2\}$ of Λ, $f \in M_\Lambda$ is determined by its restriction to the *fundamental parallelogram* $\mathcal{P}$, the convex hull of 0, λ_1, λ_2, $\lambda_1 + \lambda_2$ in $\mathbb{C}$. The field $M_\Lambda \equiv$ meromorphic functions on the *elliptic curve* $E_\Lambda := \mathbb{C}/\Lambda$.

Facts*:

- $\#$ zeros of f in $E_\Lambda = \#$ poles of f in E_Λ.
- $\sum$ zeros of f in $E_\Lambda = \sum$ poles of f in E_Λ.
- Non-constant elliptic functions have at least two poles in E_Λ.

*Zeros and poles must be counted with multiplicity and modulo Λ.

Elliptic curves and functions: analytic viewpoint (2 of 2)

The *Weierstrass $\wp$-function* with respect to Λ is

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right].$$

Facts: $[\wp'_\Lambda(z)]^2 = 4[\wp_\Lambda(z)]^3 - g_2 \cdot \wp_\Lambda(z) - g_3$ for certain constants $g_2, g_3 \in \mathbb{C}$ that depend only on Λ; and $M_\Lambda = \mathbb{C}(\wp_\Lambda(z), \wp'_\Lambda(z))$.

The *Weierstrass ζ-function* with respect to Λ is

$$\zeta_\Lambda(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left[\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right].$$

Facts: $\zeta'_\Lambda(z) = -\wp_\Lambda(z)$; and although $\zeta_\Lambda(z)$ is not elliptic, $\zeta_\Lambda(z+s) - \zeta_\Lambda(z)$ is elliptic for any $s \in \mathbb{C}$.

Note: the Weierstrass $\zeta_\Lambda(z)$ plays a prominent role in our story!

Difference equations over elliptic curves: definitions

For a fixed $s \in \mathbb{C}$ such that $s \notin \mathbb{Q} \cdot \Lambda$, (i.e., $\bar{s} \in E_\Lambda$ is a *nontorsion point*), we define the corresponding *shift automorphism* τ of M_Λ by

$$\tau : f(z) \mapsto f(z+s).$$

An n^{th}-order linear *difference equation over E_Λ* is

$$a_n \tau^n(y) + a_{n-1}\tau^{n-1}(y) + \cdots + a_1\tau(y) + a_0 y = b,$$

where $a_0, a_1, \ldots, a_n, b \in M_\Lambda$ such that $a_n a_0 \neq 0$, and y is an "unknown function" (or a formal Laurent series).

Difference equations over elliptic curves: applications

Combinatorics [DHRS18]: given certain sets of allowed steps

$$\mathcal{D} \subseteq \left\{ \rightarrow, \nearrow, \uparrow, \nwarrow, \leftarrow, \swarrow, \downarrow, \searrow \right\},$$

the *generating function* $\sum_{i,j,n} w_{\mathcal{D}}(i,j,n) x^i y^j z^n$, where $w_{\mathcal{D}}(i,j,n)$ is the number of walks of length n from $(0,0)$ to (i,j) that only take steps from $\mathcal{D}$ and remain confined to the first quadrant.

Mathematical Physics [Spi08, ADR21]: For $p, q \in \mathbb{C}^*$ such that $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$ and $|p|, |q| < 1$, and $\underline{t} = (t_1, \ldots, t_8) \in (\mathbb{C}^*)^8$ satisfying $\prod_{j=1}^{8} t_j = p^2 q^2$, the *elliptic hypergeometric function* is

$$V(\underline{t}\ ; p, q) := \int_{S^1} \frac{\prod_{j=1}^{8} \Gamma(t_j z; p, q) \Gamma(t_j z^{-1}; p, q)}{\Gamma(z^2; p, q) \Gamma(z^{-2}; p, q)} \frac{dz}{z};$$

where the *elliptic Gamma functions* appearing in the integrand are

$$\Gamma(z; p, q) := (pq/z; p, q)_\infty / (z; p, q)_\infty; \quad \text{where}$$

$$(z; p, q)_\infty := \prod_{j,k \geqslant 0} (1 - p^j q^k z).$$

First-order difference equations over elliptic curves

The simplest kind of difference equation over E_Λ is

$$\tau(y) = ay \qquad \text{for some } 0 \neq a \in M_\Lambda. \tag{1}$$

The simplest kind of solution we could hope for is $y = b \in M_\Lambda$, which would force the special form $a = \tau(b)/b$ for some $b \in M_\Lambda$.

Is there a way to test for this?

By the chain rule, $\tau(y') = \tau(y)' = a'y + ay'$, so $y'/y =: w$ satisfies

$$\tau(w) - w = \frac{a'}{a}.$$

We say a given $f \in M_\Lambda$ is *elliptically summable* if there exists $g \in M_\Lambda$ such that $f = \tau(g) - g$.

We see that a <u>necessary condition</u> for (1) to admit an elliptic function solution is for $f = a'/a$ to be elliptically summable.

Approaching elliptic summability via residues

Goal: a complete obstruction to elliptic summability, depending $\mathbb{C}$-linearly on $f \in M_\Lambda$, analogous to the Chen-Singer discrete residues in the case of the shift on rational functions.

Obstacle: the structure of M_Λ is complicated! In particular:

- elliptic functions lack "partial fraction decompositions";
- non-constant elliptic functions must have at least two poles (this prevents us from "elliptifying" the Chen-Singer strategy).

In (Dreyfus-Hardouin-Roques-Singer 2018) the authors define a partial obstruction to elliptic summability despite these obstacles.

Our Contribution: Completion of (DHRS 2018) into a complete obstruction to elliptic summability.

Elliptic orbital residues: analytic version*

The following is an analytic reinterpretation of DHRS 2018.

For $f \in M_\Lambda$ and $\alpha \in \mathbb{C}$, there exist unique $c_k(f, \alpha) \in \mathbb{C}$ for $k \in \mathbb{N}$, almost all 0, such that

$$f(z) - \sum_{k \geqslant 1} \frac{c_k(f, \alpha)}{(z - \alpha)^k} \quad \text{is holomorphic at } \alpha.$$

Now choose one $\alpha_\omega \in \mathbb{C}$ in each *orbit* $\omega \in \mathbb{C}/(\Lambda \oplus \mathbb{Z}s) \simeq E_\Lambda/\langle \bar{s} \rangle$.

Definition (DHRS 2018)

The *analytic orbital residue* of $f \in M_\Lambda$ at the orbit ω of order k is

$$\operatorname{Ores}(f, \omega, k) := \sum_{\alpha \in \omega} c_k(f, \alpha) = \sum_{n \in \mathbb{Z}} c_k(f, \alpha_\omega + ns).$$

Theorem (DHRS 2018)

If $f \in M_\Lambda$ is elliptically summable, then $\operatorname{Ores}(f, \omega, k) = 0$ for all orbits ω and all $k \in \mathbb{N}$. On the other hand, if all these orbital residues vanish, then f is "nearly elliptically summable".

Near elliptic summability: analytic version

What it means for $f(z) \in M_\Lambda$ to be "nearly elliptically summable": there exist $g(z) \in M_\Lambda$ and $R_0, R_1 \in \mathbb{C}$ such that

$$f(z) = \underbrace{g(z+s) - g(z)}_{\text{summable part}} + \underbrace{R_0 + R_1\big[\zeta_\Lambda(z+s) - \zeta_\Lambda(z)\big]}_{\text{non-summable part}}, \qquad (2)$$

where $\zeta_\Lambda(z)$ again denotes the Weierstrass ζ-function.

Recall: $\zeta_\Lambda(z)$ is not elliptic, but $\zeta_\Lambda(z+s) - \zeta_\Lambda(z)$ is elliptic!

It is shown in [DHRS 2018] that any $f(z) \in M_\Lambda$ as in (2) is elliptically summable if and only if both $R_0 = 0$ and $R_1 = 0$.

From this analytic point of view, our goal then becomes: find an intrinsic definition of the constants R_0 and R_1 for arbitrary $f \in M_\Lambda$, i.e., not just for those with every $\mathrm{Ores}(f, \omega, k) = 0$, and not mediated by any accessory $g(z) \in M_\Lambda$.

ζ-expansions

Recall we had chosen a representative $\alpha_\omega \in \mathbb{C}$ for each orbit $\omega \in \mathbb{C}/(\Lambda \oplus \mathbb{Z}s) \simeq E_\Lambda/\langle \bar{s} \rangle =: \Omega$.

This set $\Xi := \{\alpha_\omega \mid \omega \in \Omega\}$ of choices is called an *analytic pinning*.

The following is an immediate variation on a classical result:

Proposition

There exist unique constants $c_0^\Xi(f) \in \mathbb{C}$ and $c_k(f, \alpha_\omega + ns) \in \mathbb{C}$ for $k \in \mathbb{N}$ and $n \in \mathbb{Z}$, almost all zero, such that

$$f(z) = c_0^\Xi(f) + \sum_{k \geqslant 1} \sum_{\omega \in \Omega} \sum_{n \in \mathbb{Z}} \frac{(-1)^{k-1} c_k(f, \alpha_\omega + ns)}{(k-1)!} \cdot \zeta_\Lambda^{(k-1)}(z - \alpha_\omega - ns).$$

We call this the *ζ-expansion* of $f(z) \in M_\Lambda$ relative to Ξ.

Analytic panorbital residues

Relative to the pinning $\Xi = \{a_\omega \mid \omega \in \Omega\}$, we had the ζ-expansion

$$f(z) = c_0^{\Xi}(f) + \sum_{k \geqslant 1} \sum_{\omega \in \Omega} \sum_{n \in \mathbb{Z}} \frac{(-1)^{k-1} c_k(f, \alpha_\omega + ns)}{(k-1)!} \cdot \zeta_\Lambda^{(k-1)}(z - \alpha_\omega - ns),$$

and the orbital residues $\mathrm{Ores}(f, \omega, k) := \sum_{n \in \mathbb{Z}} c_k(f, \alpha_\omega + ns)$.

Definition (A.-Babbitt 2024)

The *analytic panorbital residues* of $f \in M_\Lambda$ relative to Ξ are

$$\mathrm{PanOres}_\Xi(f, 0) := c_0^{\Xi}(f) \qquad \text{and}$$

$$\mathrm{PanOres}_\Xi(f, 1) := \sum_{\omega \in \Omega} \sum_{n \in \mathbb{Z}} n \cdot c_1(f, \alpha_\omega + ns)$$

Theorem (A.-Babbitt 2024)

$f \in M_\Lambda$ is elliptically summable if and only if all the orbital residues and both panorbital residues vanish.

Elliptic curves and functions: algebraic viewpoint (1 of 2)

In many applications, elliptic curves are described *algebraically*.

For $\mathbb{K}$ an algebraically closed field of characteristic zero, we can define an *elliptic curve* $\mathcal{E}$ over $\mathbb{K}$ as the zero locus in $\mathbf{P}^2(\mathbb{K})$ of

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3,$$

for some $g_2, g_3 \in \mathbb{K}$ such that $g_2^3 - 27 g_3^2 \neq 0$.

To reformulate the previous difference structure:

- the points of $\mathcal{E}$ form an abelian group under a group law $\oplus$, with 0-element $\mathcal{O} := [0 : 1 : 0]$;
- for a given (nontorsion) point $S \in \mathcal{E}(\mathbb{K})$ the map

  $$\tau : f(P) \mapsto f(P \oplus S)$$

  defines an (infinite order) automorphism of the field $M_{\mathcal{E}}$ of rational functions on $\mathcal{E}$.

Elliptic curves and functions: algebraic viewpoint (2 of 2)

Concretely, $M_{\mathcal{E}} \simeq \mathbb{K}(x, y)$ subject to the relation

$$y^2 = 4x^3 - g_2 x - g_3.$$

If the point $\mathcal{O} \neq S \in \mathcal{E}$ is given by

$$S = [x(S) : y(S) : 1] = [\alpha : \beta : 1] \in \mathbf{P}^2(\mathbb{K}),$$

then $\tau : M_{\mathcal{E}} \to M_{\mathcal{E}}$ is the unique $\mathbb{K}$-linear map such that

$$\tau(x) = \left(\frac{y - 2\beta}{2x - 2\alpha}\right)^2 - x - \alpha; \qquad \text{and}$$

$$\tau(y) = -\left(\frac{y - 2\beta}{x - \alpha}\right)\tau(x) + \frac{\alpha y - 2\beta x}{2x - 2\alpha}$$

In this setting, elliptic difference equations and elliptic summability are defined just like before: $f \in M_{\mathcal{E}}$ is *elliptically summable* if there exists $g \in M_{\mathcal{E}}$ such that

$$f = \tau(g) - g.$$

Compatible systems of local uniformizers

A *compatible system of local uniformizers* is a family (see [DHRS18]):

$$\mathcal{U} = \left\{ u_P \in M_{\mathcal{E}} \;\middle|\; P \in \mathcal{E} \right\}$$

such that each u_P has a zero of order 1 at P and

$$\tau\left(u_P\right) = u_{P \ominus S} \qquad \text{for every } P \in \mathcal{E}.$$

Given such a $\mathcal{U}$ as above, for $f \in M_{\mathcal{E}}$ and $P \in \mathcal{E}$ there exist unique $c_k^{\mathcal{U}}(f, P) \in \mathbb{K}$ for $k \in \mathbb{N}$, almost all 0, such that

$$f - \sum_{k \geqslant 1} \frac{c_k^{\mathcal{U}}(f, P)}{u_P^k} \qquad \text{is nonsingular at } P.$$

- In analytic setting, the usual local uniformizers $u_\alpha := (z - \alpha)$ for $\alpha \in \mathbb{C}$, which are clearly τ-compatible. Although these $u_\alpha \notin M_\Lambda$, we still used them to define the local data $c_k(f, \alpha)$.

Elliptic orbital residues: algebraic version*

The following is an algebraic summary of DHRS 2018.

For $\mathcal{U} = \{u_P\}$ as before and for $f \in M_\mathcal{E}$, there exist unique $c_k^\mathcal{U}(f, P) \in \mathbb{K}$ for $k \in \mathbb{N}$ and $P \in \mathcal{E}$, almost all 0, such that

$$f - \sum_{k \geqslant 1} \frac{c_k^\mathcal{U}(f, P)}{u_P^k} \qquad \text{is nonsingular at } P.$$

Let us further choose a representative Q_ω for each *orbit* $\omega \in \mathcal{E}/\langle S \rangle$.

Definition (DHRS 2018)

The *$\mathcal{U}$-orbital residue* of $f \in M_\Lambda$ at the orbit ω of order k is

$$\mathrm{Ores}_\mathcal{U}(f, \omega, k) := \sum_{n \in \mathbb{Z}} c_k^\mathcal{U}(f, Q_\omega \oplus nS).$$

Theorem (DHRS 2018)

If $f \in M_\mathcal{E}$ is elliptically summable, then $\mathrm{Ores}_\mathcal{U}(f, \omega, k) = 0$ for all orbits ω and all $k \in \mathbb{N}$. On the other hand, if all these orbital residues vanish, then f is "nearly elliptically summable".

Interlude: effective divisors and Riemann-Roch spaces

An *effective divisor* D on $\mathcal{E}$ is a formal sum

$$D = \sum_{P \in \mathcal{E}} n_P \cdot [P]$$

such that every $n_P \in \mathbb{Z}_{\geqslant 0}$ and almost every $n_P = 0$. The *degree*

$$\deg(D) := \sum_{P \in \mathcal{E}} n_P \in \mathbb{Z}_{\geqslant 0}.$$

For D as above, the *Riemann-Roch space* $\mathcal{L}(D)$ is the $\mathbb{K}$-vector space of elements $\varphi \in M_{\mathcal{E}}$ such that

- φ has a pole of order at most n_P at P.

Theorem (Riemann-Roch for Effective Divisors in Genus 1)

If $D \neq 0$ is an effective divisor on an elliptic curve $\mathcal{E}$ then

$$\dim_{\mathbb{K}}\big(\mathcal{L}(D)\big) = \deg(D).$$

Near elliptic summability: algebraic version

What it means for $f \in M_{\mathcal{E}}$ to be "nearly elliptically summable": there exist $g \in M_{\mathcal{E}}$ and $\varphi \in \mathcal{L}\big([\mathcal{O}] + [S]\big)$ such that

$$f = \underbrace{\tau(g) - g}_{\text{summable part}} + \underbrace{\quad \varphi \quad}_{\text{non-summable part}}, \tag{3}$$

DHRS 2018 show that any $f \in M_{\mathcal{E}}$ as in (3) is elliptically summable if and only if $\varphi = 0$.

From this algebraic point of view, our goal is now to define certain constants, intrinsically in terms of an arbitrary $f \in M_{\mathcal{E}}$, that detect whether φ in (3) is 0 in the special case when we happen to have every $\mathrm{Ores}_{\mathcal{U}}(f, \omega, k) = 0$.

Admissible algebraic pinnings

Given an elliptic curve $\mathcal{E}$ over $\mathbb{K}$ and nontorsion $S \in \mathcal{E}$, let us again denote $\Omega := \mathcal{E}/\langle S \rangle$.

An *algebraic pinning* is a choice $\Xi = \left(\mathcal{U}, \mathcal{Q}, \breve{\omega}\right)$ of:

- $\mathcal{U}$: a compatible system of local uniformizers $\{u_P \mid P \in \mathcal{E}\}$;
- $\mathcal{Q}$: a choice of representatives $\{Q_\omega \in \omega \mid \omega \in \Omega\}$;
- $\breve{\omega}$: a choice of *distinguished orbit* in Ω.

We denote the chosen representative $\check{Q} := Q_{\breve{\omega}} \in \mathcal{Q}$, for readability.

Given $f \in M_{\mathcal{E}}$, such an algebraic pinning Ξ is *f-admissible* if:

- f has no poles in $\breve{\omega}$; and
- f is nonsingular at $Q_\omega \ominus nS$ for every $\omega \in \Omega$ and $n \in \mathbb{Z}_{\geqslant 0}$.

Since each $f \in M_{\mathcal{E}}$ has only finitely many poles, it is clear there always exist pinnings that are simultaneously f_i-admissible for any finite collection $f_1, \ldots, f_N \in M_{\mathcal{E}}$.

Ancillary data obtained from algebraic pinning

For a pinning $\Xi = \big(\mathcal{U}, \mathcal{Q}, \breve{\omega}\big)$ we obtain the following auxiliary data as consequences of the Riemann-Roch Theorem.

Technical Lemma (A.-Babbitt 2024)

1. *For each $\breve{\omega} \neq \omega \in \Omega$ and $k \in \mathbb{N}$, there exist unique*

$$\varphi^{\Xi}_{\omega,k} \in \mathcal{L}\left(k[Q_\omega] + [\check{Q}]\right) \qquad \textit{and} \qquad d^{\Xi}_{k}(\omega) \in \mathbb{K} \qquad \textit{such that}$$

- $\varphi^{\Xi}_{\omega,k} - u^{-k}_{Q_\omega}$ *is nonsingular at* Q_ω*; and*
- $\varphi^{\Xi}_{\omega,k} - d^{\Xi}_{k}(\omega) \cdot u^{-1}_{\check{Q}}$ *has a zero at* $\check{Q}$*.*

2. *For each $j \in \mathbb{Z}_{\geqslant 2}$ there exists a unique*

$$\psi^{\Xi}_{j} \in \mathcal{L}\left([\check{Q} \oplus jS] + [\check{Q} \oplus S]\right) \qquad \textit{such that}$$

- $\psi^{\Xi}_{j} + u^{-1}_{\check{Q}\oplus S}$ *has a zero at* $\check{Q} \oplus S$*.*

Algebraic panorbital residues

Definition (A.-Babbitt 2024)

The *algebraic panorbital residues* of $f \in M_{\mathcal{E}}$ relative to an f-admissible pinning Ξ of orders 1 and 0 are, respectively,

$$\mathrm{PanOres}_{\Xi}(f,1) := \sum_{\substack{\omega \in \Omega \\ \omega \neq \breve{\omega}}} \sum_{n,k \geqslant 1} d_k^{\Xi}(\omega) \cdot n \cdot c_k^{\mathcal{U}}(f, Q_\omega \oplus nS); \quad \text{and}$$

$$\mathrm{PanOres}_{\Xi}(f,0) :=$$

$$f(\check{Q}) + \sum_{\substack{\omega \in \Omega \\ \omega \neq \breve{\omega}}} \sum_{n,k \geqslant 1} \Big(d_k^{\Xi}(\omega) \cdot \Psi_n^{\Xi}(\check{Q}) - \varphi_{\omega,k}^{\Xi}(\check{Q} \ominus nS) \Big) \cdot c_k^{\mathcal{U}}(f, Q_\omega \oplus nS),$$

where $\psi_1^{\Xi} := 0$ and $\Psi_n^{\Xi} := \sum_{j=1}^{n} \psi_j^{\Xi}$.

Theorem (A.-Babbitt 2024)

$f \in M_{\mathcal{E}}$ is elliptically summable if and only if all the orbital residues and both panorbital residues vanish.

References

- S.A. Abramov. *On the summation of rational functions*. USSR Computational Mathematics and Mathematical Physics **11**(4), 324–330, (1971). doi:10.1016/0041-5553(71)90028-0
- C. Arreche & Y. Zhang. *Mahler discrete residues and summability for rational functions*. Proceedings of ISSAC'22, 525–533, (2022). doi:10.1145/3476446.3536186
- C. Arreche & Y. Zhang. *Twisted Mahler discrete residues*. International Mathematics Research Notices (2024). doi:10.1093/imrn/rnae238
- C. Arreche, T. Dreyfus, & J. Roques. *Differential transcendence criteria for second-order linear difference equations and elliptic hypergeometric functions*. Journal de l'École Polytechnique — Mathématiques **8**, 147–168, (2021). doi:10.5802/jep.143
- S. Chen & M.F. Singer. *Residues and telescopers for bivariate rational functions*, Advances in Applied Mathematics **49**(2), 111–133, (2012). doi:10.1016/j.aam.2012.04.003
- T. Dreyfus, C. Hardouin, J. Roques, & M.F. Singer, *On the nature of the generating series of walks in the quarter plane*, Inventiones mathematicae **213**(1), 139–203, (2018). doi:10.1007/s00222-018-0787-z
- V.P. Spiridonov, *Essays on the theory of elliptic hypergeometric functions*, Russian Mathematical Surveys **63**(3), 405–472, (2008). doi:10.1070/RM2008v063n03ABEH004533