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PBW-pairs of varieties of linear algebras and symbolic computation in free algebras

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Abstract.

The notion of a PBW-pair of varieties of linear algebras over a field is under consideration. If (V,W) is a PBW-pair of varities and V is Schreier, then so is W. Similar results are also true for Freiheitssatz and Word problem. If V(X) and W(X) are free algebras with the set X of free generators of the varieties V and W, respectively, then V(X) is the universal enveloping algebra of W(X). In the case where V(X) is a free nonassociative algebra it gives a possibility to construct algorithms for symbolic computation in the algebra W(X) (recognizing automorphisms, primitive elements, constructing of normal forms of elements and standard bases of ideals). The talk is based on the article by A.A.Mikhalev and I.P.Shestakov.

The article is organized as follows. Section 1 contains examples of PBW-pairs of varieties of linear algebras. A variety of algebras is called Schreier if every subalgebra of a free algebra in this variety is also free. In Section 2 we prove that if $(\mathcal{V}, \mathcal{W})$ is a PBW-pair and the variety $\mathcal{V}$ is homogeneous and Schreier then so is $\mathcal{W}$. Section 3 is devoted to the Freiheitssatz for free algebras (originally the Freiheitssatz for free groups was proved by W. Magnus [24]). We prove that the results similar to the Schreier property for PBW-pairs are also true for the Freiheitssatz and Word problem. In particular, it follows that the Freiheitssatz is true for the varieties of Akivis and Sabinin algebras. We give also examples of varieties that do not satisfy the Freiheitssatz. In Section 4 we consider the problem to recognize automorphisms of free algebras. In Section 5 it is shown that an element u of a free algebra $\mathcal{W}[X]$ in a homogeneous Schreier variety of algebras $\mathcal{W}$ satisfying the Freiheitssatz is a primitive element (a coordinate polynomial) if and only if the factor algebra of $\mathcal{W}[X]$ by the ideal generated by the element u is a free algebra in $\mathcal{W}$ (originally this result for free groups was proved by J. H. C. Whitehead [48]). We consider also properties of primitive elements. Section 6 gives new examples of PBW-pairs.

Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of linear algebras over a field F and assume that there exists a functor $\mathcal{K} : \mathcal{V} \longrightarrow \mathcal{W}$ which associates to every algebra $A \in \mathcal{V}$ an algebra $\mathcal{K}(A) \in \mathcal{W}$ by changing multiplication in A.

Let $\mathcal{K} : \mathcal{V} \to \mathcal{W}$ be a multiplication changing functor, then $\mathcal{K}$ admits a left adjoined functor $\mathcal{U} : \mathcal{W} \to \mathcal{V}$ such that there exists a bijection

$$Hom_{\mathcal{W}}(A, \mathcal{K}(B)) \cong Hom_{\mathcal{V}}(\mathcal{U}(A), B),$$

for every $A \in \mathcal{W}$ and $B \in \mathcal{V}$. Moreover, there exists a canonical $\mathcal{W}$-homomorphism $i : A \to \mathcal{K}(\mathcal{U}(A))$ such that for any $B \in \mathcal{V}$ and a $\mathcal{W}$-homomorphism $\varphi : A \to \mathcal{K}(B)$ there exists a unique $\mathcal{V}$-homomorphism $\tilde{\varphi} : \mathcal{U}(A) \to B$ satisfying the equality $\tilde{\varphi} \circ i = \varphi$ (see Proposition 1). The algebra $\mathcal{U}(A)$ is called the $\mathcal{V}$-universal enveloping algebra of a $\mathcal{W}$-algebra A. It is generated by $i(A)$ and has a natural ascending filtration with the associated graded $\mathcal{W}$-algebra $gr\,\mathcal{U}(A)$.

We will call a pair of varieties $(\mathcal{V}, \mathcal{W})$ with a multiplication changing functor $\mathcal{K} : \mathcal{V} \to \mathcal{W}$ a *PBW-pair*, if it satisfies the following property:

(PBW) for every $A \in \mathcal{W}$ there is an isomorphism $gr\,\mathcal{U}(A) \cong \mathcal{U}(Ab\,A)$ where $Ab\,A$ is the $\mathcal{W}$-algebra whose underlying vector space is A and all the multiplication operations are trivial (abelian).

A classical example of a PBW-pair is the pair (As, Lie) of the varieties of associative and Lie algebras, where PBW-condition is given by the celebrate Poincaré-Birkhoff-Witt Theorem: for a Lie algebra L, $gr\,\mathcal{U}(L) \cong S(L) \cong \mathcal{U}(Ab\,L)$, where $S(L)$ is the symmetric algebra of the vector space L.

Let $\mathcal{V}[X]$ be the free algebra of a variety $\mathcal{V}$ with a set X of free generators. An element u of $\mathcal{V}[X]$ is said to be a primitive element (a coordinate polynomial) if it is an element of some set of free generators of the algebra $\mathcal{V}[X]$. A set of nonzero pairwise distinct elements of $\mathcal{V}[X]$ is said to be a primitive system of elements if it is a subset of some set of free generators of $\mathcal{V}[X]$.