Using computer algebra to characterize spacetimes with isotropy

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Introducing the problem

The question whose solution I want to describe arose in studying models of the universe in general relativity. So to introduce the problem, without assuming the audience is familiar with relativity, I have to introduce its basic ideas, which depend on concepts from differential geometry.

The more I thought about this talk the more background I realised I needed, meaning I have very little time for the actual computer algebra calculations, which may be the most interesting part for this audience. I shall also skip over technical details like conditions on topology and differentiability.

I shall start with special relativity, then motivate general relativity and introduce the geometric ideas needed to formulate it. Then I have to discuss spacetime symmetries and so introduce the problem I tackled. A theorem on (local) characterization of Riemannian manifolds, a computational technique for the spacetime case, and a specialized suite of software then enabled the problem to be solved.

The need for special relativity

At the turn of the 20th century it was known that Newtonian ideas of space, time and motion did not agree with what was observed for light.

Maxwell's equations for electromagnetism (in vacuum) imply that light has a specific speed c. If light behaved we might expect, the speed would be different in a moving frame: if we are in a car going at 30 kph, a car going at 100 kph has a speed relative to us of 70 kph. But light's speed, relative or not, is always c.

19th century explanations in terms of an "aether" which supported light waves ran into problems with contradictory experiments. These can be accounted for by suitable equations saying how space and time measurements by relatively moving observers are related. Various people (Lorentz, Larmor, Poincaré) deduced these. They could be considered to be real changes in moving objects.....

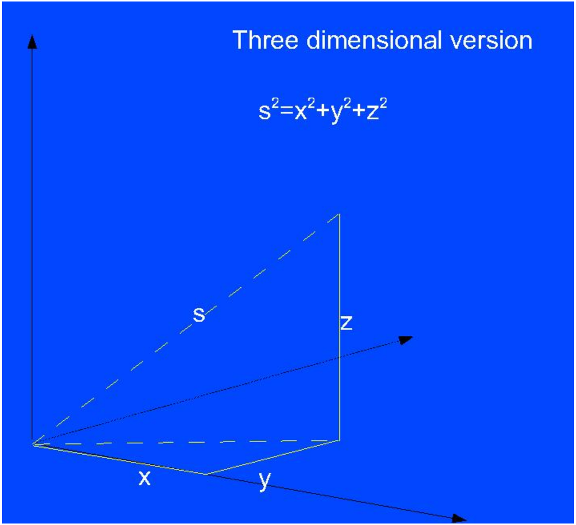
Question: are these real changes of objects or not?

Special relativity

Einstein’s radical idea was that they were apparent changes arising from the nature of space-time and how we observe it.

He was able to show that there was a certain quantity which did not change under the transformations Lorentz and others had found, and associated it with how we measure.

Pythagoras' theorem in 3D



Spacetime geometry in special relativity

So what about 4D, with time as the fourth dimension?

One might expect

$$s^2 = x^2 + y^2 + z^2 + c^2 t^2$$

where the c^2 (or some velocity squared) is needed to turn time units into length units.

Einstein realised that one actually needs

$$s^2 = x^2 + y^2 + z^2 - c^2 t^2.$$

now called the Minkowski metric (after 1908 work of Minkowski). This ensures everybody agrees on the value of c because they agree that $s = 0$ along a light ray. Of course, it means s can no longer be simply thought of as distance.

A negative total for s^2 means we are measuring a time.

Why was General Relativity needed?

Special relativity gave a new understanding of space and time. It agreed with the known properties of electromagnetic phenomena and with a suitably modified version of Newton's laws of motion. But it is not compatible with Newton's law of gravitation.

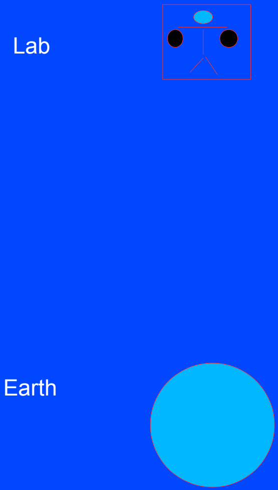
Moreover, Einstein knew that Newton's gravity theory could not explain the observed precession of Mercury's orbit round the Sun.

So from 1907 onwards Einstein looked to generalise special relativity so as to agree with both special relativity and Newton's gravity theory in appropriate limits, the corrections to either being small in everyday observations.

In 1915 he announced general relativity which meets that need. The main holdup was that he did not know the differential geometry needed in formulating the new theory. (He was taught it in particular by his friends Michele Besso and Marcel Grossmann.)

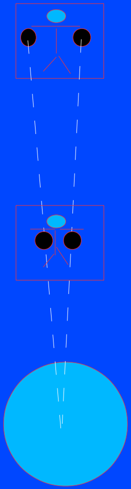
Einstein’s “equivalence principle”

Consider Einstein's lift example.



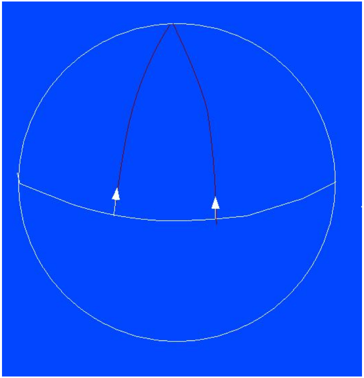
Experimenter releases two balls. From his/her point of view they are moving in time and not in space, so their motions are parallel in space-time. What happens?

Einstein's "equivalence principle"



Balls move towards one another. So parallel lines in space-time meet, when there is a gravity field.

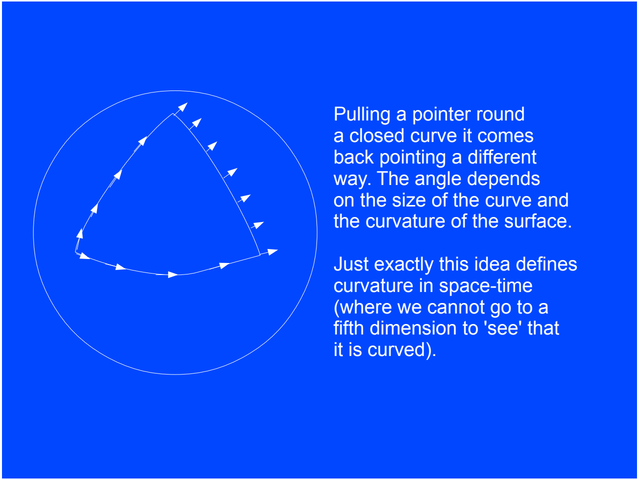
Curvature on the earth



Parallel lines meet on the curved surface of the earth. Einstein's thought experiment therefore suggests we need a curved spacetime. How can we define curvature without having an extra dimension to "see" it in?

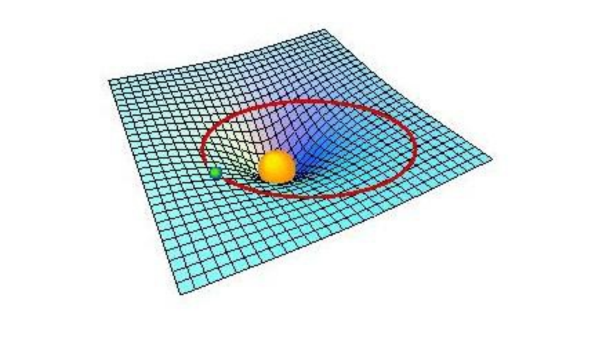
How to define curvature

How to infer a sphere (e.g. the Earth’s surface) is curved and define its curvature.



An analogy for the gravitational effects of curvature

The idea of gravity in general relativity is that it can be described as a curvature of spacetime. For a body like the Sun this gives a picture like:



The amount of curvature depends on the amount of matter.

More on general relativity and differential geometry

General relativity has by now undergone many tests and is clearly the 'best buy' theory of space, time and gravity. For example it is used in GPS systems and in working out satellite trajectories.

For today's purpose, I need to introduce more detail on the differential and Riemannian geometry which underlie general relativity. A manifold is just a topological space on which one can (in neighbourhoods) define coordinates, and a differential manifold is one where the coordinates in overlapping coordinate neighbourhoods are related by differentiable functions.

Tangent spaces, vectors and tensors

At each point p of a (real) differential manifold there is a tangent space T_p, a space of vectors V over the field $\mathbb{R}$ at that point. (There are at least 3 equivalent definitions of such vectors.)

From T_p one can define the dual "covector" space V^*, and then spaces of tensor products of V and V^* at p, in the usual algebraic way. On the manifold these enable one to define vector and tensor fields, meaning a specification of a vector or tensor at each point (in an appropriately differentiable way). It is usual to presume some basis of the tangent space, say $\{\boldsymbol{e}_a\}$ $(a \in (1 \cdots n))$ is being used, and to write vectors and tensors as indexed values in that basis; For example one writes $V = v^a \boldsymbol{e}_a$.

Riemannian manifolds

A Riemannian manifold is further equipped with a metric, a symmetric tensor field g which maps $V \otimes V \to \mathbb{R}$, so at each point, for any pair of vectors v_1 and v_2, it gives a value $g(v_1, v_2) \in \mathbb{R}$. The metric and its inverse map vectors to covectors and vice versa. Vectors v such that $g(v,\ v) > 0$ are called spacelike, those such that $g(v,\ v) < 0$ are timelike and those such that $g(v,\ v) = 0$ are lightlike or "null".

In general relativity, spacetime is assumed to be a four-dimensional real Riemannian manifold with a metric that at each point has the same signature as the metric of special relativity, so that by choice of a basis of the tangent space one gets back special relativity locally.

More geometry

To define curvature in a differential manifold one needs to define when vectors in a manifold are to be considered parallel, so that one can implement the definition implied by my earlier example of curvature of the Earth's surface.

This is done by defining a connection, which for a vector v_1 at p defines which vector v_2 at a neighbouring point p' is to be considered parallel to v_1. This is said to define parallel transport, and it can readily be extended to covectors and tensors.

The usual partial derivative of a vector field is not a tensor. To define a tensorial derivative, the "covariant derivative", one compares the value of a vector field at p' with the vector parallelly transported from p. This also extends to covector and tensor fields.

The curvature tensor

In a Riemannian manifold the metric defines a symmetric connection by the requirement that the metric be covariantly constant.

In such a manifold, applying the idea of parallel transport round a closed curve in a 2-dimensional surface with coordinates (x^k, x^ℓ) leads to the formula

$$v^m - \overset{0}{v}{}^m = \frac{1}{2} \overset{0}{R}{}^m{}_{ik\ell} \overset{0}{v}{}^i A^{k\ell} \tag{1}$$

where the left side is the difference of the final and initial vectors, $A^{kl} \equiv \oint x^\ell dx^k$ is the area within the curve, and $R^m{}_{ik\ell}$ is the Riemann curvature tensor. (It satisfies various algebraic and differential identities, notably those named after Bianchi and Ricci, which I will not give here but do need to refer to later.)

I'll skip the formulae for going from metric to connection to curvature as you do not need to know them to follow this talk.

The Weyl and Ricci tensors

For later use we will need the decomposition of the Riemannian curvature given by

$$R^{ab}{}_{cd} = C^{ab}{}_{cd} - \tfrac{1}{3}R\delta^a_{[c}\delta^b_{d]} + 2\delta^{[a}_{[c}R^{b]}{}_{d]}. \qquad (2)$$

where $\delta_a{}^b$ is the usual Kronecker delta, $R_{ab} \equiv R^a{}_{bad}$ and $R \equiv g^{ab}R_{ab}$ (using Einstein's index convention that repeated indices re summed over), and the square brackets mean we take the skew part. R_{ab} is called the Ricci tensor and C_{abcd} is the Weyl tensor or conformal curvature tensor.

Einstein's equations relating curvature to the matter content take the form

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = \kappa_0 T_{ab}. \qquad (3)$$

where T_{ab} is the energy-momentum tensor of the matter that is present and κ_0 is a constant determinable from the Newtonian limit.

(Local) characterization of Riemannian manifolds

Because the same geometry may be obtained in various coordinates, it is very useful to have a way to give a unique (local) characterization of the geometry.

The theorem on which this is based can be expressed as saying that for two spacetimes $\mathcal{M}$ and $\overline{\mathcal{M}}$, if x is a regular point of $\mathcal{M}$ and E a basis of the tangent space at x (a "frame") and similarly $(\bar{x}, \bar{E})$ in $\overline{\mathcal{M}}$, then $\mathcal{M}$ and $\overline{\mathcal{M}}$ are (locally) identical ("isometric": see next slide) around x and $\bar{x}$ if and only if the components of the curvature and its covariant derivatives in frames E and $\bar{E}$ chosen in a coordinate-independent way can be equated. [For a more formal statement see Theorem 9.1 in Stephani et al. (2003).]

The algorithm to compute the required quantities follows ideas of É. Cartan. It was initiated by Brans (1965, 1977) and Karlhede (1980) and first implemented by Karlhede and Åman (1979). It has been improved by MacCallum and Åman (1986), and further by myself and others.

Spacetime symmetries

To explain how the characterization works I need some symmetry concepts.

An **isometry** in a Riemannian manifold M is a map $M \to M$ such that the metric is mapped to itself. Isometries form a group.

If an isometry has a fixed point, say p, it is an **isotropy** at p. A transformation in the tangent space at p, T_p, that preserves the Riemannian structures there is called a **linear isotropy**. An isotropy always induces a linear isotropy at p. For both, the adjective '**local**' will mean existence in some neighbourhood of p. I will not put the word 'local' into all statements to which it would apply, but just remind you of it sometimes.

Obtaining a (local) characterization

One usually starts with a given metric. The idea is to compute the Riemann tensor and its derivatives up to the q-th ($q = 0, 1, \cdots$): we call that set of quantities $\mathcal{R}^q$. At each step we choose a frame in which $\mathcal{R}^q$ takes a canonical form: this implies that at each step we are reducing the freedom in the choice of frame.

We then find the linear isotropy group $\hat{I}_q$ preserving $\mathcal{R}^q$, denoting its dimension by s_q, and the number t_q of independent functions of position in $\mathcal{R}^q$. We stop when increasing q gives no new information. It has been proved that for spacetimes the maximum q required (in a very special case) is 7 (Milson and Pelavas (2009)).

Two spacetimes will be locally isometric (i.e. really the same) if and only if they have the same values s_q, t_q and the remaining independent components can be equated (which would specify the isometry). In principle that step is formally undecideable but in practice it is doable in cases encountered so far.

More on spacetime symmetries

If the isometry group includes maps such that for any pair of points (p, q) there is a isometry mapping p to q the manifold is called **homogeneous**.

Among spacetimes an important subset (because they give interesting cosmological models) are those which are **spatially homogeneous**, i.e. contain spacelike hypersurfaces which are homogeneous.

Robertson (1929, 1935, 1936) and (independently) Walker (1936) proved that spacetimes which are spherically symmetric about each point must be spatially homogeneous. Such Robertson-Walker metrics, with appropriate dynamics for their time evolution, are the standard models of modern cosmology.

The work I want to talk about involves a generalization of that result using weaker isotropy assumptions.

Local invariance

A local linear isotropy preserves the curvature tensor and all its derivatives at p. I consider a slightly weaker idea which I call **local invariance**.

This is that "at each point p in an open neighbourhood U of a point p_0, the same non-trivial subgroup g of the Lorentz group acts in the tangent space T_p and leaves invariant the curvature tensor and its covariant derivatives up to the m-th".

Following Ellis (1967), if g is a continuous group this assumption will be denoted (A_m).

The problem

I have now given enough background to be able to introduce the problem whose solution I wanted to describe.

In my PhD thesis (1970) I studied spatially homogeneous spacetimes, without assuming any isotropy. While considering the observational properties of such models of the universe, I noted that there were several cases in which the observations had reflection symmetries at each point (MacCallum and Ellis 1970). Schmidt (1969) then showed these cases necessarily had a discrete isotropy group at every point.

This prompted the question of whether assuming local invariance under those discrete isotropy groups would imply that there was a continuous group of isometries (not necessarily giving as much symmetry as spatial homogeneity).

Schmidt (1969) proved that if all reversals of spatial axes were isotropies, spatial homogeneity was implied.

The problem (cont)

With my then graduate student Filipe Mena I set out to study the remaining groups of discrete isotropies that had appeared in spatially homogeneous spacetimes. We used a more algebraic method than Schmidt, and were able to make progress (MacCallum and Mena 2002, and Mena's PhD thesis) but we still have not completed the work in a fully satisfactory manner.

I realised that part of our difficulties was that one will have reflection symmetries if there is a continuous group of isotropies which includes (e.g.) rotation through π radians, since that reverses both axes in a plane. It made our proofs easier if we could avoid there being such a continuous isotropy group. So during the pandemic I set out to characterize all cases with local invariance under a continuous isotropy group. That is the problem I shall talk about today.

Previous results and conjectures

In the first paper I ever studied line by line, Ellis (1967) proved that (A_3) with a group of spatial rotations implied that spacetimes with "dust" matter content were (locally) spatially homogeneous. Such spacetimes were called "locally rotationally symmetric" (LRS). Those results were generalized to other Ricci tensors by Ellis and Stewart (1968). Cahen and Defrise (1968) and Goode and Wainwright (1986) respectively showed (with different assumptions) that only (A_2) and (A_1) were needed. (Goode and Wainwright did not express the result in those terms, but I identified their assumptions as (A_1).)

In the PhD thesis (1981) of my former postdoc and co-author, the late Stephen Siklos, he wrote that spacetimes could be completely characterized by the curvature and its first covariant derivative. This is not true in general but remained a conjecture for many cases.

From these results I anticipated that something between 1 and 3 covariant derivatives would be required in the various cases to be studied, and that turned out to be true.

The strategy

I followed an extension of the strategy of Goode and Wainwright:

1. Pick a frame in which the curvature has a canonical form invariant under the linear isotropy assumed.
2. Let $m = 1$
3. Find the consequences of (A_m).
4. Use the Bianchi identities to obtain further restrictions. Also use the Ricci equations or commutator equations where possible.
5. Stop if the characterization algorithm has terminated
6. Having fully utilized (A_m), increment m and go to step 3.

The dimension of the full group of isometries can then be calculated, using the information on the number of independent quantities remaining in the equations. Note that in a fixed frame connection components are invariants although they are not tensors.

Tackling the continuous isotropy problem

Assuming a continuous isotropy implies strong restrictions on the curvature. Possible algebraic structures of the Weyl tensor were first studied by Petrov (1954). To allow a continuous isotropy either the Weyl tensor must be zero, in which case the spacetime is called conformally flat, or it must be of one of the special types now called Petrov types D or N. The Ricci tensor also must have one of the structures admitting a continuous isotropy: these are denoted by Segre types.

To solve the continuous isotropy problem I had to compute covariant derivatives for each allowable pair of Petrov and Ricci tensor types.

The final results are statements of the number of derivatives on which the isotropy has to be imposed, and the dimension of the isometry groups arising.

The software used

For the calculations I used a system called CLASSI which was developed to tackle the problem of classification of spacetimes. It is based on a system called SHEEP for use in relativity.

The starting point for SHEEP was Ray d'Inverno's system LAM (Lisp Algebraic Manipulator) originally implemented in machine-specific code and designed to study spacetimes in general relativity (see his PhD thesis, 1970, and subsequent papers cited in my review of computer algebra systems for work in gravity theory).

Using Ray's ideas as a basis Inge Frick (1977, 1982) developed SHEEP ("LAM(B) grown up") which is based on the same Lisp that underlies Reduce, so-called Standard Lisp. SHEEP can be built on any machine for which Reduce will run, and it is possible to build joint Reduce-SHEEP binaries enabling SHEEP values to be handled using Reduce's facilities.

More on CLASSI

CLASSI is a specialized set of packages built on SHEEP, primarily by Jan Åman, but also by myself, Jim Skea, Gordon Joly, and others, to compute and study the curvature and its derivatives as required for classification of spacetimes.

The calculations are usually done entirely in spinors, which I now introduce.

Reduce, SHEEP and CLASSI are all available as free software.

The Lorentz group and spinors

The set of transformations that preserves

$$s^2 = x^2 + y^2 + z^2 - c^2 t^2.$$

is called the Lorentz group. It plays an important role in physics, especially in quantum theory and particle physics. The part of the group connected with the identity is the proper (no change of parity) orthochronous (no reversal of the time direction) part $L_+^\uparrow$.

There is a 2-1 map of the group $SL(2, \mathbb{C})$ to $L_+^\uparrow$. the fundamental representation of $SL(2, \mathbb{C})$, which is a 2-dimensional complex vector space S with elements V_A ($A \in (0,\ 1)$) called (2-)spinor space. One can map vectors and tensors on spacetime to those on S. All quantities needed have concise forms because the two-dimensionality forces all tensors built on S to be expressible as combinations of terms with skew symmetry and completely symmetric terms (e.g. $S_{AB} = S_{(AB)} + S\epsilon_{AB}$ where $S_{(AB)} = (S_{AB} + S_{BA})/2$, S is the trace of S_{AB}, and ϵ_{AB} is skew and has the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Results

I succeeded in solving the continuous isotropy problem completely, in 3 papers in 2021. Knowing the relevant conditions, one can then study cases with discrete isotropy which do not satisfy those conditions, and so complete the study of the discrete isotropies. That remains to be done.

There are three main cases depending on whether the isotropy acts in spacelike, timelike or null 2-dimensional spaces. The most difficult subcases turned out to be the conformally flat (zero Weyl tensor) spacetimes, because no useful information came from the Weyl tensor and its derivatives.

I shall give the outcome for spatial rotation isotropy. (In the talk, time allowed for a short live demonstration of some of the calculations.)

Results for spatial rotations

Theorem

If in a neighbourhood in a spacetime of Petrov type D, (A_1) is true for a g containing a spatial rotation, then the spacetime is LRS.

This confirms the Goode and Wainwright result. I was able to show why Cahen and Defrise had overlooked the further reduction from (A_2) to (A_1).

For conformally flat accelerated perfect fluid type Ricci tensors (A_3) is required to ensure LRS, showing that Ellis's original condition is necessary as well as sufficient.

All other conformally flat spacetimes in which (A_1) holds with a g containing a spatial rotation must be LRS, agreeing with Siklos' conjecture.

References

A. Z. Petrov. “The classification of spaces defining gravitational fields” Scientific Proceedings of Kazan State University (named after V.I. Ulyanov-Lenin), Jubilee (1804-1954) Collection [Uch. Zapiski Kazan Gos. Univ.] 114(8), 55-69 (1954) An English translation by J. Jezierski, M.A.H. MacCallum and A.G. Polnarev appeared as *Gen. Rel. Grav.* **32(8)**, 1665-1685 (2000).

G. F. R. Ellis, “Dynamics of pressure-free matter in general relativity”, J. Math. Phys. 8, 1171 (1967)

M. Cahen and L. Defrise, “Lorentzian 4-dimensional manifolds with ‘local isotropy’“ Commun. Math. Phys. 11, 56 (1968)

S. W. Goode and J. Wainwright, “Characterization of locally rotationally symmetric space-times” Gen. Rel. Grav. 18, 315-31 (1986)

B. G. Schmidt, “Discrete isotropies in a class of cosmological models” Comm. Math. Phys. 15, 329-336 (1969)

My own related work

The book of which I am a co-author

H. Stephani, D. Kramer, M. A. H. MacCallum, C. A. Hoenselaers, and E. Herlt Exact solutions of Einstein's field equations, 2nd edition Cambridge University Press, Cambridge (2003) Corrected paperback edition, 2009.

gives a concise introduction to the mathematics of spacetimes followed by a review of the known solutions of Einstein's equations.

The papers on the problem discussed here appeared in the journal General Relativity and Gravitation as vol. 53, articles 57, 61 and 96 (2021). The example above is in the first of these.

My review of available software for gravity research and the applications thereof up to 2018 appeared as Living Reviews in Relativity 21, 6 (2018).