Computer algebra methods and algorithms for solving the Cauchy problem for difference equations

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Problem №1. Statement of the problem and known results

Denote by $\mathbb{Z}$ the set of integers, and let $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ be the 2-dimensional integer lattice and $\mathbb{Z}^2_{\geqslant}$ be the subset of this lattice consisting of the points with integer nonnegative coordinates.

Let δ_1 the shift operator along the axis x_1, i.e.

$$\delta_1 f(x_1, x_2) = f(x_1 + 1, x_2),$$

and δ_2 be the shift operator along the axis x_2, i.e.,

$$\delta_2 f(x_1, x_2) = f(x_1, x_2 + 1).$$

Consider the polynomial difference operator

$$P(\delta) = \sum_{|\alpha| \leqslant m} c_\alpha \delta^\alpha,$$

where $\alpha = (\alpha_1, \alpha_2)$ in the multi-index, $|\alpha| = \alpha_1 + \alpha_2$, $\delta^\alpha = \delta_1^{\alpha_1} \cdot \delta_2^{\alpha_2}$, c_α are constant coefficients, and m is the order of the operator $P(\delta)$.

Statement of the problem and known results

We consider the difference equations

$$P(\delta_1,\delta_2)f(x_1,x_2)=0, \quad (x_1,x_2)\in\mathbb{Z}^2_{\geqslant}, \tag{1}$$

where $f(x_1,x_2)$ is an unknown function.

Let us fix $\beta=(\beta_1,\beta_2)$ such that $\beta\neq(0,m)$, $\beta\neq(m,0)$, $|\beta|=m$, and $c_\beta\neq0$. Define

$$X_{0,\beta}=\{(x_1,x_2)\in\mathbb{Z}^2_{\geqslant}:(x_1,x_2)\not\geqslant\beta\}$$

and formulate the problem:

find a function $f(x_1,x_2)$ that coincides with the given function $\varphi(x_1,x_2)$ for all $(x_1,x_2)\in X_{0,\beta}$, i.e., that satisfies the condition

$$f(x_1,x_2)=\varphi(x_1,x_2), \quad (x_1,x_2)\in X_{0,\beta}. \tag{2}$$

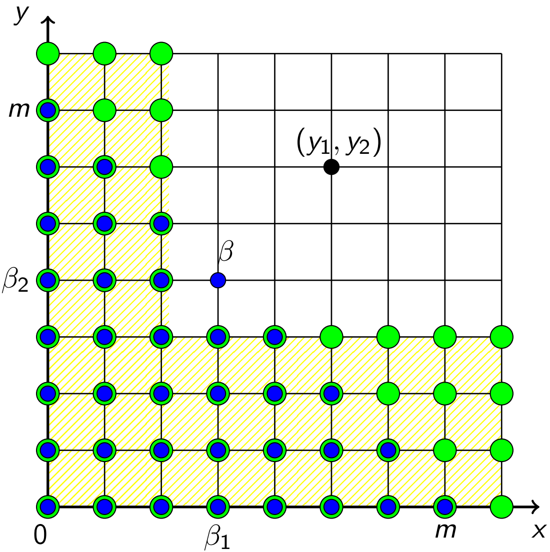
It is known (see [1,2]) that problem (1), (2) is uniquely solvable if

$$|c_\beta| > \sum_{\substack{|\alpha| \leqslant m, \\ \alpha \neq \beta}} |c_\alpha|. \tag{3}$$

The problem is to find the value of the function $f(x_1, x_2)$ at the point (y_1, y_2).

1. Rogozina (Apanovich), M. S. On the solvability of the Cauchy Problem for the polynomial difference operator, Vest. Novosibirsk. Gos. Univ., 2014, vol. 14, no. 3, pp. 83—94.

2. Leinartas, E.K. and Rogozina (Apanovich), M. S. Solvability of the Cauchy Problem for a polynomial difference operator and monomial bases of factor in the ring of polynomials. Sib. Mat. Zh., vol. 2015, no 1, pp. 111–121.



$\beta = (\beta_1, \beta_2)$
$\beta_1 + \beta_2 = m$
The set $X_{0,\beta}$ is hatched yellow.

Description of the initial data

1. The point $\beta = (\beta_1, \beta_2)$, which determines the size of the matrix of coefficients C.
2. The lower triangular matrix $C = (c_{\alpha_1,\alpha_2})$, $\alpha_1 = 0, \ldots, m$, $\alpha_2 = 0, \ldots, m$ of size $(m+1) \times (m+1)$ consisting of the coefficients c_{α_1,α_2} of the two-dimensional difference equation.
3. The point f with the coordinates (y_1, y_2), which determines the position of the unknown element $f(x_1, x_2)$ in the matrix F the size of this matrix.
4. The initial data matrix $F = (\varphi(x_1, x_2))$ of size $(y_1 + y_2 + 1) \times (y_1 + y_2 + 1)$, where $(x_1, x_2) \in X_{(0,\beta)}$; for all other (x_1, x_2), $\varphi(x_1, x_2) = 0$.

Description of the initial data

Since the coordinates of the coefficient matrix of the difference operator and the initial data matrix in the Cartesian system of coordinates (X, Y) are different from their coordinates in the matrix (row and column), it is reasonable for the technical implementation of the algorithm to change the Cartesian system of coordinate $D(d_1, d_2)$ for the "matrix" system of coordinates $(M(m_1, m_2))$ using the rule $D(d_1, d_2) \to M(m_1, m_2)$, where $m_1 = p - d_2$, $m_2 = d_1 + 1$, and p is the dimension of the coefficient matrix or the initial data matrix.

For example, the element c_{00} of the matrix $C(3 \times 3)$, the coordinates of which in the Cartesian system of coordinates are $(d_1, d_2) = (0, 0)$, has the coordinates $(m_1, m_2) = (3, 1)$ in the matrix system of coordinates, and the element $\varphi(1, 0)$ of the initial data matrix $F(4 \times 4)$ has the coordinates $(4, 2)$ in the matrix system of coordinates.

Next, we must verify condition (3) for the coefficients of the operator $P(\delta)$ to be sure that the Cauchy problem (1), (2) is uniquely solvable.

Example

We will consider the polynomial difference operator

$$P(\delta_1, \delta_2) = c_{02}\delta_2^2 + c_{01}\delta_2 + c_{11}\delta_1\delta_2 + c_{10}\delta_1 + c_{20}\delta_1^2 + c_{00}.$$

Fix $\beta = (1, 1)$.
The set of initial data is

$$X_{0,(1,1)} = \{(x_1, x_2) \in \mathbb{Z}^2_{\geqslant} : (x_1, x_2) \ngeqslant (1, 1)\}$$

Let the coefficient matrix of the polynomial difference operator $P(\delta_1, \delta_2)$ be

$$C = \begin{pmatrix} c_{02} & 0 & 0 \\ c_{01} & c_{11} & 0 \\ c_{00} & c_{10} & c_{20} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 10 & 0 \\ 4 & 2 & 4 \end{pmatrix}$$

We want to find the value of the function f at the point $A(2, 3)$, i.e. $f(2, 3)$

Example

Let the initial data matrix be

$$X_{0,(1,1)} = F =$$

$$= \begin{pmatrix} \varphi(0,5) & 0 & 0 & 0 & 0 & 0 \\ \varphi(0,4) & 0 & 0 & 0 & 0 & 0 \\ \varphi(0,3) & 0 & 0 & 0 & 0 & 0 \\ \varphi(0,2) & 0 & 0 & 0 & 0 & 0 \\ \varphi(0,1) & 0 & 0 & 0 & 0 & 0 \\ \varphi(0,0) & \varphi(1,0) & \varphi(2,0) & \varphi(3,0) & \varphi(4,0) & \varphi(5,0) \end{pmatrix} =$$

$$= \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

Example

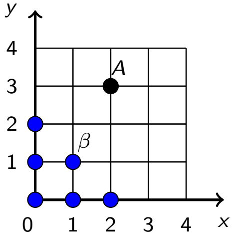
1. Transition from the Cartesian system of coordinates to the matrix system of coordinates.

$\beta_1 = [2, 2]$ – are the coordinates of the point β in the matrix system of coordinates.

The matrix system of coordinates for the coefficients of the difference operator $P(\delta)$ is

	1	2	3
1	●		
2	●	● β	
3	●	●	●

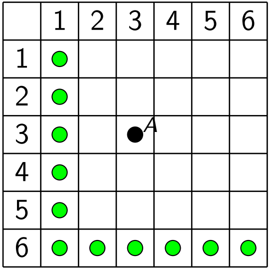
Location of the elements of the matrix C in the Cartesian system of coordinates.



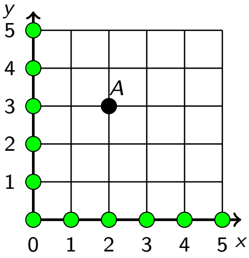
Here ● are the coefficients c_α.

Example

The matrix system of coordinates for the initial data is



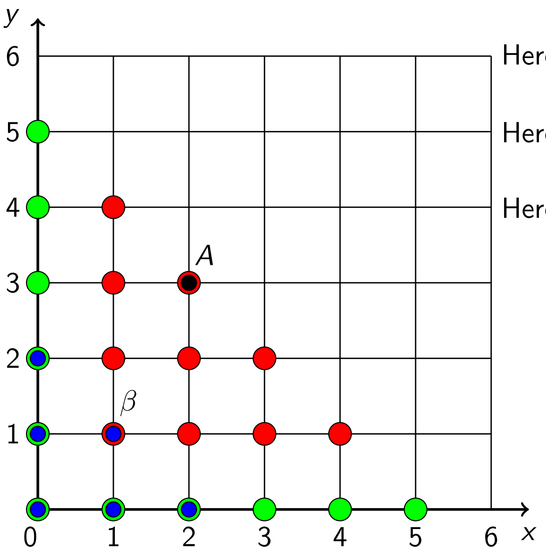
Location of the elements of the matrix F in the Cartesian system of coordinates



Here • are the initial data.

$F_1 = [3, 3]$ are the coordinates of the point A in the matrix system of coordinates.

Example



Here $\bullet$ are the coefficients c_α.

Here $\bullet$ are the initial data.

Here $\bullet$ is the unknown value.

Example

2. Verification of the matrix C to ensure that the Cauchy problem (1)–(2) is uniquely solvable: $|10| = |c_{1,1}| > |c_{0,2}| + |c_{2,0}| = 5$; therefore, condition (3) is satisfied and the Cauchy problem is uniquely solvable.

3. The diagonal of the coefficient matrix C is $a = \begin{pmatrix} 1 & 10 & 4 \end{pmatrix}^T$

4. Computation of the Toeplitz matrix [3], the size of which depends on the value $f(y_1, y_2)$ to be found:

$$column_c = \begin{pmatrix} 10 & 1 & 0 & 0 \end{pmatrix}^T, row_c = \begin{pmatrix} 10 & 4 & 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 10 & 4 & 0 & 0 \\ 1 & 10 & 4 & 0 \\ 0 & 1 & 10 & 4 \\ 0 & 0 & 1 & 10 \end{pmatrix}.$$

3. Iokhvidov, I.S., Hankel and Toeplitz Matrices, Moscow: Nauka, 1974 [in Russian].

Example

5. Solution of the linear system of equations to find the values of f that can be needed for computing the desired value.
At each step, the size of the Toeplitz matrix increases together with the increasing number of unknowns, and the found values of the function F are written into the initial data matrix F:

$$F = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 \\ 1 & -0.02 & 0 & 0 & 0 & 0 \\ 2 & -4.90 & -1.98 & 0 & 0 & 0 \\ 3 & 0.72 & 7.14 & 4.33 & 0 & 0 \\ 4 & -7.50 & -4.82 & -7.26 & -7 & 0 \\ 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}.$$

6. Transition to the Cartesian system of coordinates
$f(2,3) = F(3,3) = -1.98$

Problem №2. Statement of the problem and known results

Let us define the "strip" $\Pi = \{(x, y) \in \mathbb{Z}^2, 0 \leqslant x \leqslant B, y \geqslant 0\}$ in the positive octant of the integer lattice, with number $B+1$ being the width of Π. We consider the difference polynomial operator with constant coefficients

$$P(\delta_1, \delta_2) = \sum_{j=0}^{m} \sum_{i=0}^{b} c_{ij} \delta_1^i \delta_2^j = \sum_{j=0}^{m} P_j(\delta_1) \delta_2^j, \tag{4}$$

where m and b determine the size of the scheme, and $P_j(\delta_1) = \sum\limits_{i=0}^{b} c_{ij} \delta_1^i$, $j = 0, 1, ..., m$.

Polynomial $P(z, w) = \sum\limits_{j=0}^{m} \sum\limits_{i=0}^{b} c_{ij} z^i w^j$ is called a characteristic polynomial.

Degree m of polynomial $P(z, w)$ with respect to variable w is referred to as the order of difference operator $P(\delta_1, \delta_2)$, and it is assume that $b < B$.

Statement of the problem and known results

We fix $\beta = (x_\beta, m)$ such that $c_\beta \neq 0$, and consider the set $\Pi_\beta = \{(x, y) \in \mathbb{Z}_+^2 : 0 \leqslant x - x_\beta \leqslant B - b, y > m - 1\}$. Then, we denote $L_\beta = \Pi \setminus \Pi_\beta$ and formulate the following problem:
Find a solution of the difference equation

$$P(\delta_1, \delta_2) f(x, y) = g(x, y), (x, y) \in \Pi, \tag{5}$$

that satisfies the condition

$$f(x, y) = \varphi(x, y), (x, y) \in L_\beta, \tag{6}$$

where $g(x, y)$ and $\varphi(x, y)$ – are functions of integer arguments.
Problem (5)–(6) is known [4] to be unambiguously solvable if the following condition holds:

$$|c_\beta| \geqslant \sum_{|\alpha|=0, \alpha\neq\beta}^{b} |c_\alpha|. \tag{7}$$

Let us pose the problem of evaluating function $f(x, y)$ at a point A with coordinates (x_A, y_A), i.e. $f\,(x_A, y_A)$.

Description of input data

The input data are finite and are represented as follows:

1. point $\beta = (x_\beta, m)$;
2. $(m+1) \times (b+1)$ -dimensional rectangular matrix of coefficients $C = (c_\alpha)$, $(\alpha = (\alpha_1, \alpha_2)$, $\alpha_1 = 0, \dots, b$, $\alpha_2 = 0, \dots, m)$, which consists of the coefficients c_α of polynomial difference operator (4);
3. point A with coordinates (x_A, y_A), which determines the coordinates of the desired value of $f(x, y)$ and the dimension of initial data matrix F;
4. $(y_A + 1) \times (B + 1)$-dimensional matrix of initial data $F = (\varphi(x, y))$, where $(x, y) \in L_\beta$; for all other values (x, y), $\varphi(x, y) = 0$.

Then, it is required to check the Cauchy problem (5)–(6) for solvability, i.e., check whether condition (7) holds for the coefficients of difference operator (4).

Example

Let us fix $\beta = (x_\beta, m)$, $x_\beta = 2$, $m = 1$, $b = 3$, $B = 5$.
For the polynomial difference operator

$$\begin{gathered} P(\delta_1, \delta_2) = c_{00} + c_{10}\delta_1 + \\ + c_{20}\delta_1^2 + c_{30}\delta_1^3 + c_{01}\delta_2 + \\ + c_{11}\delta_1\delta_2 + c_{21}\delta_1^2\delta_2 + c_{31}\delta_1^3\delta_2 \end{gathered} \tag{8}$$

the Cauchy problem is represented as follows:

$$\begin{gathered} c_{00}f(x,y) + c_{10}f(x+1,y) + \\ + c_{20}f(x+2,y) + c_{30}f(x+3,y) + \\ + c_{01}f(x,y+1) + c_{11}f(x+1,y+1) + \\ + c_{21}f(x+2,y+1) + \\ c_{31}f(x+3,y+1) = 0, (x,y) \in \Pi, \\ f(x,y) = \varphi(x,y), (x,y) \in L_{(2,1)}, \end{gathered} \tag{9}$$

where $\Pi = \left\{(x,y) \in \mathbb{Z}^2,\ 0 \leqslant x \leqslant 5, y \geqslant 0\right\}$,
$\Pi_{(2,1)} = \{(x,y) \in \mathbb{Z}_+^2 : 2 \leqslant x \leqslant 4, y \geqslant 1\}$, and $L_{(2,1)} = \Pi \setminus \Pi_{(2,1)}$.

Example

The coefficient matrix of polynomial difference operator (8) is defined as:

$$C = \begin{pmatrix} c_{01} & c_{11} & c_{21} & c_{31} \\ c_{00} & c_{10} & c_{20} & c_{30} \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 3 & 10 & 1 \\ 2 & 4 & 2 & 0 \end{pmatrix}.$$

The problem is formulated as follows: evaluate function $f(x, y)$ at point A with coordinates $(4, 4)$, т.е. $f(4, 4)$.
The initial data matrix is

$$F = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 3 \\ 2 & 3 & 0 & 0 & 0 & 4 \\ 3 & 4 & 0 & 0 & 0 & 5 \\ 4 & 5 & 0 & 0 & 0 & 6 \\ 5 & 6 & 6 & 8 & 1 & 7 \end{pmatrix},$$

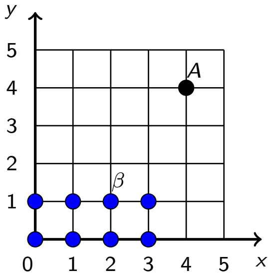
Example

1. The transition from the Cartesian coordinate system to the matrix coordinate system is carried out by mirroring the initial data matrix and coefficient matrix with respect to the horizontal axis, which is why point β with coordinates $(2, 1)$ passes into a point with coordinates $(2, 3)$ in the matrix coordinate system.

The matrix system of coordinates for the coefficients of the difference operator $P(\delta)$ is

	1	2	3	4
1	●	●	●	●
2	●	●	● β	●

Location of the elements of the matrix C in the Cartesian system of coordinates.



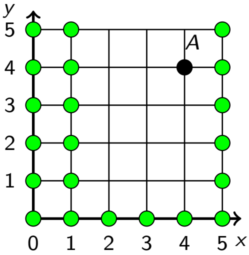
Here ● are the coefficients c_α.

Example

The matrix system of coordinates for the initial data is

	1	2	3	4	5	6
1	●	●	●	●	●	●
2	●	●				●
3	●	●				●
4	●	●				●
5	●	●			A ●	●

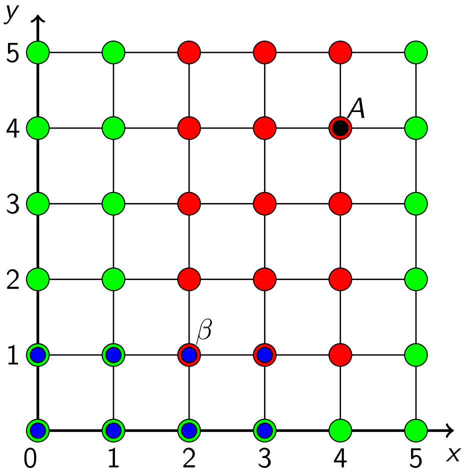
Location of the elements of the matrix F in the Cartesian system of coordinates



Here ● are the initial data.

Point A are has coordinates $[5, 5]$ in the matrix system of coordinates.

Example



Here • are the coefficients c_α.

Here • are the initial data.

Here • is the unknown value.

Example

2. The solvability of the Cauchy problem (9), is checked; i.e., it is checked whether condition (7) holds for the coefficients of polynomial difference operator (8). Since

$$|10| = |c_{21}| \geqslant |c_{01}| + |c_{11}| + |c_{31}| = 5,$$

condition (7) holds and the Cauchy problem is solvable.
3. Matrices C and F are mirrored with respect to the horizontal axis:

$$C_{work} = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 1 & 3 & 10 & 1 \end{pmatrix},$$

$$F_{work} = \begin{pmatrix} 5 & 6 & 6 & 8 & 1 & 7 \\ 4 & 5 & 0 & 0 & 0 & 6 \\ 3 & 4 & 0 & 0 & 0 & 5 \\ 2 & 3 & 0 & 0 & 0 & 4 \\ 1 & 2 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

Example

4. The last row in matrix C_{work} is

$$a = \begin{pmatrix} c_{01} & c_{11} & c_{21} & c_{31} \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 10 & 1 \end{pmatrix}^T$$

5. A general Toeplitz matrix the size of which depends on the desired value is obtained.

To construct the Toeplitz matrix, we need two vectors $column_c$ and row_c.

$$column_c = \begin{pmatrix} 10 & 3 & 1 \end{pmatrix}^T, row_c = \begin{pmatrix} 10 & 1 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} 10 & 1 & 0 \\ 3 & 10 & 1 \\ 1 & 3 & 10 \end{pmatrix}.$$

Example

6. The system of linear equations is solved to find the values of $f(x, y)$ that may be required to compute the desired value at point $A(x_A, y_A)$. The number of unknowns for each m, which indicates the number of a "layer," is the same at each step of the algorithm. That is why the size of the Toeplitz matrix remains constant. At each step of the algorithm, the found values of $f(x, y)$ are written into the initial data matrix:

$$F = \begin{pmatrix} 5 & 6 & 6 & 8 & 1 & 7 \\ 4 & 5 & -6.15 & -3.50 & -3.53 & 6 \\ 3 & 4 & -3.32 & 2.51 & 2.42 & 5 \\ 2 & 3 & -2.70 & 0.35 & -1.15 & 4 \\ 1 & 2 & -1.83 & 0.68 & 0.19 & 3 \end{pmatrix}.$$

7. The transition to the Cartesian coordinate system $f(4, 4) = F[5, 5]$ is performed.

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