Computing Elementary Integrals by Additive Decomposition and Homomorphic Valuation

Ziming Li

Key Laboratory of Mathematics Mechanization
AMSS, Chinese Academy of Sciences

Seminar on Computer Algebra
CMC faculty of MSU and CCAS
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joint work with S. Chen, H. Du, Y. Gao and J. Guo

Outline

- Additive decompositions
- Elementary integrals
- On-going projects

What is an additive decomposition?

Given $f(x)$, compute $g(x)$ and $r(x)$ of the same type as f's s.t.

$$f(x) = g(x)' + r(x) \quad \leftarrow \text{remainder}$$

with the properties

(i) (minimality) $r(x)$ is minimal in some sense;

(ii) (integrability) $\exists\ h(x)$ of the same type as f's s.t.

$$f(x) = h(x)'$$

if and only if $r(x) = 0$.

Remark. $f(x)$ may be replaced by a sequence $f(n)$, and derivative $'$ by the difference operator Δ_n.

Rational case (I)

Let C be a field of characteristic zero.

Hermite-Ostrogradsky ($\approx$ 1850).

For $f \in C(x)$, one can compute $g, r \in C(x)$ s.t.

$$f = g' + r \quad \text{where } ' = d/dx,$$

with the properties:

(i) the denominator of r is of minimal degree,

- r is proper,
- r has a squarefree denominator;

(ii) $f = h'$ for some $h \in C(x)$ if and only if $r = 0$.

Rational case (II)

Abramov (1975).

For $f \in \mathbb{C}(n)$, one can construct $g, r \in \mathbb{C}(n)$ s.t.

$$f = \Delta_n(g) + r$$

with the properties:

(i) the denominator of r is of minimal degree,

- r is proper,
- r has a shiftfree denominator;

(ii) $f = \Delta_n(h)$ for some $h \in \mathbb{C}(n)$ if and only if $r = 0$.

Inspiration

S. Abramov and M. Petkovšek.

Minimal decomposition of indefinite hypergeometric sums,

Proc. ISSAC 2001, and its expanded version in *JSC*, 2002.

Theorem 11 *Let a term T be regularly described by a triple $(f_1/f_2, v_1/v_2, n_0)$, i.e.,*

$$T(n) = \frac{v_1(n)}{v_2(n)} \prod_{k=n_0}^{n-1} \frac{f_1(k)}{f_2(k)}.$$

Then there exists a term T_1 of the form

$$T_1(n) = S(n) \prod_{k=n_0}^{n-1} \frac{f_1(k)}{f_2(k)},$$

$S \in K[n]$, such that the term $T_2 = T - (E-1)T_1$ is of the form

$$\frac{P(n)}{v_2(n)} \prod_{k=n_0}^{n-1} \frac{f_1(k)}{f_2(k+1)},$$

where P is a polynomial whose degree is less than

$$\lambda = \begin{cases} \deg v_2 + \deg f_2 & \text{if } \deg(f_2 - f_1) > \deg f_1, \\ \deg v_2 + \deg f_1 & \text{if } \deg(f_2 - f_1) = \deg f_1 \\ & \text{or } \deg(f_2 - f_1) < \deg f_1 - 1, \\ \deg v_2 + \deg f_1 + \tau & \text{if } \deg(f_2 - f_1) = \deg f_1 - 1, \end{cases}$$

where in the last case τ is equal to $\mathrm{lc}\,(f_2 - f_1)/\mathrm{lc}\, f_1$ if this is a nonnegative integer, and -1 otherwise.

Some further developments

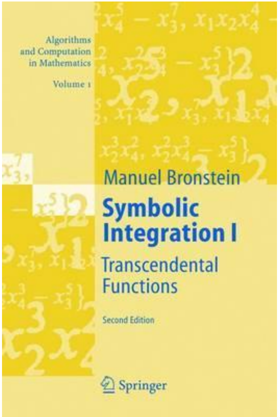
(*q*)-Hypergeometric sequences.

- Hyperexponential (Bostan, Chen, Chyzak, L and Xin 2013)
- Hypergeometric (Chen, Huang, Kauers and L 2015)
- q-Hypergeometric (Du, Huang and L 2018)

D-finite functions.

- Algebraic (Chen, Kauers, Koutschan 2016)
- Fuchsian D-finite (Chen, van Hoeij, Kauers, Koutschan 2018)
- D-finite (van der Hoeven 2017, 2018; Bostan, Chyzak, Lairez, Salvy 2018, van der Hoeven 2020)

Additive decompositions in symbolic integration



Compute elementary integrals of transcendental functions over $\mathbb{C}(x)$.

Differential fields

Let K be a field and let $' : K \to K$ satisfy

$$\forall u, v \in K, \ (u+v)' = u' + v' \quad \text{and} \quad (uv)' = u'v + uv'.$$

Call $'$ a derivation and and $(K, ')$ a differential field (D-field).

- Call $c \in K$ a constant if $c' = 0$.
- $C_K := \{c \in K \mid c' = 0\}$.
- Call $\ell \in K$ a logarithmic derivative if $\ell = a'/a$ for some $a \in K$.
- The set of generalized logarithmic derivatives

$$L_K := \text{span}_{C_K} \{\ell \mid \ell \text{ is a logarithmic derivative}\}.$$

Example. Let $(K, ') = (\mathbb{C}(x), d/dx)$. Then

$$C_K = \mathbb{C} \quad \text{and} \quad L_K = \{f \mid f \text{ is proper with squarefree denominator}\}.$$

Primitive and logarithmic towers

Set $K_0 = \mathbb{C}(x)$ and $K_i = K_0(t_1, \ldots, t_i)$, $i = 1, \ldots, n$. Then

$$K_0 \subset K_1 \subset \cdots \subset K_n = K_0(t_1, \ldots, t_n).$$

The tower K_n is primitive if

(i) $t_1, \ldots, t_n$ are algebraically independent over K_0,

(ii) $t_i' \in K_{i-1}$, $i = 1, \ldots, n$,

(iii) $C_{K_n} = \mathbb{C}$.

Such a tower is logarithmic if $t_i' \in L_{K_{i-1}}$, $i = 1, \ldots, n$.

Example.

- $K_0(\log(x), \arctan(x))$ is logarithmic,
- $K_0\left(\log(x), \int \frac{1}{\log(x)}\, dx\right)$ is primitive but not logarithmic.

Additive decomposition in symbolic integration

Let F be a D-field.

Given $f \in F$, compute $g, r \in F$ s.t.

$$f = g' + r$$

with the properties

(i) r is minimal in some sense,

(ii) $f = h'$ for some $h \in F$ if and only if $r = 0$.

Supervisor's suggestion. Develop an additive decomposition in logarithmic towers.

Students' adventure. Develop an additive decomposition in primitive towers.

Matryoshka decomposition (I)

Let $K_n = K_0(t_1, \ldots, t_n)$ and $f = a/b \in K_n$ with $\gcd(a, b) = 1$.

Call f t_i-proper if $f \in K_i$ and $\deg_{t_i}(a) < \deg_{t_i}(b)$.

Set

- $P_0 = K_0[t_1, \ldots, t_n]$,
- $P_j = \{f \in K_j[t_{j+1}, \ldots, t_n] \mid \text{coeffs are } t_j\text{-proper}\}$, $j = 1, \ldots, n-1$,
- $P_n = \{f \in K_n \mid f \text{ is } t_n\text{-proper}\}$.

Then

$$K_n = \underset{K_0[t_1, \ldots, t_n]}{\overset{P_0}{\bigcap}} \oplus \underset{K_1[t_2, \ldots t_n]}{\overset{P_1}{\bigcap}} \oplus \cdots \oplus \underset{K_{n-1}[t_n]}{\overset{P_{n-1}}{\bigcap}} \oplus P_n.$$

Matryoshka decomposition (II)

Let π_i be the projection: $K_n \to P_i$ w.r.t. the direct sum.

$$\text{For } f \in K_n, \quad f = \pi_0(f) + \pi_1(f) + \cdots + \pi_n(f)$$

is called the matryoshka decomposition of f.

Example. Let $K_3 = \mathbb{C}(x)(t_1, t_2, t_3)$ and

$$f = \frac{x(t_1 t_2 + x)(t_3^2 - t_1 t_3 + x t_2)}{t_2 t_3}.$$

Then

$$f = \underbrace{x t_1 t_3 - x t_1^2}_{\pi_0(f)} + \underbrace{0}_{\pi_1(f)} + \underbrace{(x^2/t_2) t_3 - x^2 t_1 / t_2}_{\pi_2(f)} + \underbrace{(x^2 t_1 t_2 + x^3)/t_3}_{\pi_3(f)}.$$

A partial order on a tower (I)

Monomial order. Set

$$T = \{ t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{N} \}$$

and $\prec$ to be the plex order w.r.t. $t_1 \prec t_2 \prec \cdots \prec t_n$.

Definition. Let $f \in K_n$ and $i \in \{0, 1, \ldots, n-1\}$.

$\mathrm{hm}_i(f)$:= the highest monomial in $t_{i+1}, \ldots, t_n$ in $\pi_i(f)$ if $\pi_i(f) \neq 0$

and

$\mathrm{hm}(f)$:= the highest monomial among $\mathrm{hm}_0(f), \ldots \mathrm{hm}_{n-1}(f)$.

A partial order on a tower (II)

Example. Let $f \in K_3$. Then

$$f = \underbrace{x t_1 t_3 - x t_1^2}_{\pi_0(f)} + \underbrace{0}_{\pi_1(f)} + \underbrace{(x/t_2) t_3 - x t_1/t_2}_{\pi_2(f)} + \underbrace{(x t_1 t_2 + x^2)/t_3}_{\pi_3(f)}$$

$$\Downarrow$$

$$\mathrm{hm}(f) = t_1 t_3.$$

Definition. Let $f, g \in K_n$ with $\pi_n(f) = a/b$ and $\pi_n(g) = u/v$, where

$$a, b, u, v \in K_0[t_1, \ldots, t_n] \quad \text{and} \quad \gcd(a, b) = \gcd(u, v) = 1.$$

Then $f \prec g$ if

- either $f = 0$ and $g \neq 0$, or
- $\deg_{t_n}(b) < \deg_{t_n}(v)$, or
- $\mathrm{hm}(f) \prec \mathrm{hm}(g)$.

S-Primitive towers

Definition.

- Let $f \in K_i$, $i \in \{0, 1, \ldots, n\}$, and let $t_0 = x$. Then f is t_i-simple if it is t_i-proper with squarefree denominator w.r.t. t_i.
- $f \in K_n$ is simple if $\pi_i(f)$ is t_i-simple, where $0 \leq i \leq n$.
- A primitive tower K_n is S-primitive if each t_i' is simple.

Example.

- Logarithmic towers are S-primitive.
- $K_0\left(\log(x), \int \frac{1}{\log(x)}\, dx\right)$ is S-primitive.

Additive decomposition in S-primitive towers

Theorem. Let K_n be S-primitive. For $f \in K_n$, there is an algorithm to compute $g, r \in K_n$ s.t.

$$f = g' + r$$

with the properties

(i) r is minimal w.r.t. $\prec$,

(ii) $f = h'$ for some $h \in K_n$ if and only if $r = 0$.

Idea. Using integration by parts to reduce $\mathrm{hm}(f)$.

Publ. H. Du, J. Guo, L, and E. Wong. An additive decomposition in logarithmic extensions and beyond. *Proc. ISSAC 2020.*

Example

Let $K_3 = K_0(t_1, t_2, t_3)$, where

$$t_1 = \log(x), \quad t_2 = \arctan(x), \quad t_3 = \arctan(\log(x)).$$

Let

$$f = \frac{(x^2+4)t_2 - x}{(x^2+1)t_2^2} + \frac{2t_3}{xt_1^2 + x} + (t_1+1)^2 t_3 \ \in K_3$$

$$= \underbrace{\left(\frac{x}{t_2} + t_3^2 + (xt_1^2 + x)t_3 - x\right)'}_{g} + \underbrace{\frac{3}{(x^2+1)t_2}}_{r}.$$

$$\int f\ dx \ = \ \frac{x}{\arctan(x)} + \arctan(\log(x))^2 + (x\log(x)^2 + x)\arctan(\log(x)) - x$$

$$+ \int \frac{3}{(x^2+1)\arctan(x)}\ dx.$$

$\not\in$

K_3

How to integrate a remainder

Let K_n be S-primitive. Given $f \in K_n$ with an additive decomp

$$f = (\cdots)' + r.$$

Then

$$r \text{ has an elem. int. over } K_n \begin{cases} \Longrightarrow r \text{ is simple} \\ \Longleftrightarrow r \in \text{span}_{\mathbb{C}}\{t_1', \ldots, t_n'\} + L_{K_n} \\ \Longleftrightarrow r \in L_{K_n}, \text{ when } K_n \text{ is logarithmic.} \end{cases}$$

Logarithmic parts

Problem. Given a simple element $r \in K_n$, decide whether $r \in L_{K_n}$. Assume $r \in L_{K_n}$. Compute $c_1, \ldots, c_s \in \mathbb{C}$ and $g_1, \ldots, g_s \in K_n$ s.t.

$$r = c_1 \frac{g_1'}{g_1} + \cdots + c_s \frac{g_s'}{g_s},$$

or, equivalently,

$$\int r \, dx = c_1 \log(g_1) + \cdots + c_s \log(g_s).$$

Call $\{(c_1, g_1), \ldots, (c_s, g_s)\}$ a logarithmic part of r.

From simple to t-simple

Lemma. Let r be in a primitive tower K_n. Then

$$r \in L_{K_n} \iff \pi_i(r) \in L_{K_i}, \quad i = 0, 1, \ldots, n.$$

Let K be a D-field and t primitive over K.

Problem. Given a t-simple $r \in K(t)$, compute a logarithmic part of r if there exists one.

Rothstein-Trager resultants

Theorem. For a t-simple $r = a/b \in K(t)$ with $\gcd(a, b) = 1$, let z be an indeterminate, and

$$\mathrm{RT}(r) := \underbrace{\mathrm{resultant}_t(a - zb',\ b)}_{\text{Rothstein-Trager resultant of } r} \in K[z].$$

Then r has a logarithmic part if and only if the monic associate of $\mathrm{RT}(r)$ belongs to $\mathbb{C}[z]$.

In this case, let $c_1, \ldots, c_s$ are all distinct roots of $\mathrm{RT}(r)$. Then a logarithmic part of r is

$$\left\{ \left(c_1,\, \gcd(a - c_1 b', b)\right), \ldots, \left(c_s,\, \gcd(a - c_s b', b)\right) \right\}.$$

Resultant-based algorithm

Input: a t-simple $r = a/b \in K(t)$ with $\gcd(a, b) = 1$.

Output: a logarithmic part of r if there exists one.

1. Compute the Rothstein-Trager resultant $\mathrm{RT}(r)$;
2. Compute the monic associate $M(z)$ of $\mathrm{RT}(r)$ w.r.t. z;
3. If $M(z) \notin \mathbb{C}[z]$, then return FALSE;
4. Factor $M(z) = p_1 \ldots p_s$ over its coefficient field;
5. Set $g_i(z, t) := \gcd(a - zb', b) \mod p_i(z)$, $i = 1, \ldots, s$;
6. Return $\{(\alpha_i, g_i(\alpha_i, t)) \mid p_i(\alpha_i) = 0, i = 1, \ldots, s\}$.

Example

Let $K = \mathbb{C}(x)$, $t = \log(x)$ and

$$r = \frac{8\,t^3x^2 - 2\,t^3x + 4\,t^2x - 4\,xt + t - 2}{x(8\,t^3x^3 + 12\,t^3x^2 - 10\,t^3x + 12\,t^2x^2 + t^3 + 12\,t^2x - 5\,t^2 + 6\,xt + 3\,t + 1).}$$

The numerator of $\mathrm{RT}(r) =$

$$\begin{aligned}&51200x^7z^3 - 20480x^9z^3 - 30720x^8z^3 + 2560x^9z - 320x^9 + 3840x^8z - 480x^8\\&+104960x^6z^3 - 6400x^7z - 110080x^5z^3 + 800x^7 - 13120x^6z - 1720x^5 - 5x^3\\&+18560x^4z^3 + 1640x^6 + 13760x^5z - 320x^3z^3 - 2320x^4z + 290x^4 + 40x^3z\end{aligned}$$

$$= \underbrace{(\cdots)}_{\text{content w.r.t. } z} \underbrace{\left(z^3 - \frac{1}{8}z + \frac{1}{64}\right)}_{\text{monic associate}}.$$

Logarithmic part:

$$\left(\frac{1}{4},\ t + \frac{1}{2(x-1)}\right),\ \left(\alpha, t + 8\,\frac{\alpha}{4\,x^2 + 8\,x - 1} + \frac{3 + 2\,x}{4\,x^2 + 8\,x - 1}\right)$$

with $\alpha^2 + \frac{1}{4}\alpha - \frac{1}{16} = 0$.

Lucky points

Let $K = \mathbb{C}(x, y_1, \ldots, y_\ell)$.

Definition. Let $r = a/b \in K(t)$ be t-simple. A point $\mathbf{v} \in \mathbb{C}^{\ell+1}$ is lucky for r if

$$\begin{cases} \text{the denominator of } \left(\prod_{i=1}^{\ell} y_i'\right) \cdot t' \text{ does not vanish at } \mathbf{v} & (\star) \\ \mathrm{lc}_t(b) \cdot \mathrm{lc}_t(b')(\mathbf{v}) \neq 0 & (\star\star) \\ \mathrm{resultant}_t(b, b')(\mathbf{v}) \neq 0. \end{cases}$$

- If $\mathbf{v}$ is lucky, then

$$\mathrm{RT}(r)(\mathbf{v}, z) = \underbrace{\mathrm{resultant}_t(a(\mathbf{v}, t) - zb'(\mathbf{v}, t),\ b(\mathbf{v}, t))}_{\tilde{R}}.$$

- If $\mathbf{v}$ satisfies $(\star)$ and $(\star\star)$, then

$$\mathbf{v} \text{ is lucky } \iff \deg_z(\tilde{R}) = \deg_t(b).$$

Algorithm LuckyPoints

Input: a nonzero t-simple element $r = a/b \in K(t)$,

Output: FAIL if no lucky point is chosen, otherwise

$$\{(\mathbf{v}_1, r_1), (\mathbf{v}_2, r_2)\},$$

where $\mathbf{v}_i$ is a lucky point and $r_i = \mathrm{RT}(r)(\mathbf{v}_i, z) \in \mathbb{C}[z]$ for $i = 1, 2$.

1. For k from 1 to 5 do
   - Choose two points $\mathbf{v}_1, \mathbf{v}_2$ satisfying $(\star)$ and $(\star\star)$, and compute

     $$r_i = \mathrm{resultant}_t(a(\mathbf{v}_i, t) - zb'(\mathbf{v}_i, t),\ b(\mathbf{v}_i, t)), \quad i = 1, 2.$$

   - If $\deg_z(r_1) = \deg_z(r_2) = \deg_t(b)$, return $\{(\mathbf{v}_1, r_1), (\mathbf{v}_2, r_2)\}$.

   end do.
2. Return FAIL.

Homomorphism-based algorithm

Input: a t-simple $r = a/b \in K(t)$ with $\gcd(a, b) = 1$

Output: a logarithmic part of r if there exists one.

1. $U :=$ LuckyPoints(r).
2. If $U = \textsc{Fail}$, then call the resultant-based algorithm.
3. Assume $U = \{(\mathbf{v}_1, r_1), (\mathbf{v}_2, r_2)\}$. Compute the monic associate M_1 and M_2 of r_1 and r_2, respectively.
4. If $M_1 \neq M_2$ then return FALSE.
5. Factor $M_1 = p_1 \cdots p_s$ over its coefficient field.
6. For j from 1 to s do
   - $g_j = \gcd(a - zb', b) \mod p_j$.
   - If $\deg_t(g_j) \neq$ the multiplicity of p_j in M_1, then return FALSE.

   end do.
7. Return $\{(\alpha_j,\ g_j(\alpha_j, t)), \mid p_j(\alpha_j) = 0,\ j = 1, \ldots, s\}$.

Other algorithms

- Subresultant-based algorithm (D. Lazard and R. Rioboo, 1990):
  - avoiding algebraic gcd-computation.
- Gröbner-based algorithm (G. Czichowski, 1995):
  - constructing the squarefree part of a Rothstein-Trager resultant directly and avoiding algebraic gcd-computation.

Experiments (I)

Let $K_2 = \underbrace{\mathbb{Q}(x, t_1)}_{K}(t_2)$, where $t_1 = \log(x)$ and $t_2 = \log(\log(x))$.

Let

$$f = \texttt{randpoly}([x, t_1, t_2], \texttt{dense}, \texttt{degree} = \texttt{d}),$$

$$g = \texttt{randpoly}([x, t_1, t_2], \texttt{dense}, \texttt{degree} = \texttt{d}),$$

and

$$r = \frac{\text{rem}(f, g, t_2)}{g}.$$

Compute a logarithmic part of r if there exists one.

d	5	6	7	8	9	10	11	12
RES	0.3	1.3	3.6	8.7	10.0	40.0	81.3	422.6
HOM	$*$	$*$	$*$	$*$	$*$	$*$	0.1	0.1

where $*$ means "< 0.1 sec".

Experiments (II)

Let

$$f = \texttt{randpoly}([x, t_1, t_2], \texttt{dense}, \texttt{degree} = \texttt{d}),$$

$$g = \texttt{randpoly}([x, t_1, t_2], \texttt{dense}, \texttt{degree} = \texttt{d}),$$

and

$$r = 4\frac{f'}{f} - \frac{1}{3}\frac{g'}{g}.$$

Compute a logarithmic part of r.

d	1	2	3	4	5	6	7	8
RES	$*$	$*$	$*$	0.1	0.3	3.0	7.4	18.5
HOM	$*$	$*$	$*$	$*$	$*$	0.1	0.1	0.1

where $*$ means "$<$ 0.1 sec".

Experiments (III)

Let $p = z^3 - z + 1$ and

$$f = \texttt{randpoly}([y, x, t_1, t_2], \texttt{dense}, \texttt{degree} = \texttt{d}).$$

Compute a logarithmic part of

$$r = \sum_{p(y)=0} y \frac{f(y, x, t_1, t_2)'}{f(y, x, t_1, t_2)}.$$

d	1	2	3	4	5	6	7	8
RES	$*$	0.7	15.6	139.5	> 400	> 400	> 400	> 400
HOM	$*$	$*$	0.1	0.1	0.3	0.6	0.8	1.2

where $*$ means "< 0.1 sec".

Risch's algorithm for logarithmic towers

Input: f in a logarithmic tower K_n.

Output: an elementary integral of f if there exists one.

1. Hermite-reduce w.r.t. t_n: $f = g_n' + h_n + p_n$, where h_n is t_n-simple and $p_n \in K_{n-1}[t_n]$.
   - If h_n has no log part, then return "no elem integral";
   - compute a log part $\{(c_1, g_1), \ldots, (c_s, g_s)\}$.
2. Polynomial-reduce w.r.t. t_n: $p_n = u_n' + r_{n-1}$ with $r_{n-1} \in K_{n-1}$ by solving Risch's equations in K_{n-1}. If no solution, then return "no elem integral".
3. Recursion. Integrate r_{n-1} over K_{n-1}.
   - If r_{n-1} has no elem integral, then return "no elem integral";
   - return

$$g_n + u_n + \sum_{i=1}^{s} c_i \log(g_i) + \int r_{n-1}\, dx.$$

Addition decomposition and homomorphic valuation

Input: $f \in K_n = \mathbb{C}(x)(t_1, \ldots t_n)$, a logarithmic tower.

Output: an elementary integral of f if there exists one.

1. Compute an additive decomposition

$$f = g' + r, \quad \text{where } r \text{ is a remainder.}$$

   If r is not simple, then return "no elem integral ".

2. For i from 1 to n do

   compute a logarithmic part of $\pi_i(r)$ in $K_{i-1}(t_i)$.

$$\{(c_{i,1}, g_{i,1}), \ldots, (c_{i,s_i}, g_{i,s_i})\};$$

   if such a part does not exist, then return "no elem integral ".

   end do.

3. Return

$$g + \sum_{i=1}^{n} \sum_{j=1}^{s_i} c_{i,j} \log(g_{i,j}) + \int \pi_0(r)\, dx.$$

Example

Let $K_3 = \mathbb{C}(x)(t_1, t_2, t_3)$, where

$$t_1 = \log(x), \quad t_2 = \arctan(x), \quad t_3 = \arctan(\log(x)).$$

Let

$$f = \frac{(x^2+4)t_2 - x}{(x^2+1)t_2^2} + \frac{2t_3}{xt_1^2 + x} + (t_1+1)^2 t_3 \ \in K_3$$

$$\int f \ dx \ = \ \underbrace{\frac{x}{\arctan(x)} + \arctan(\log(x))^2 + (x\log(x)^2 + x)\arctan(\log(x)) - x}_{g}$$

$$+ \int \underbrace{\frac{3}{(x^2+1)\arctan(x)}}_{\notin K_3} \ dx = g + 3\log(\arctan(x)).$$

Experiments

Let $K_1 = K_0(t)$, where $t = \log(x)$.

Let $f = \texttt{randpoly}([x, t], \texttt{degree} = \texttt{d}) + \texttt{xt}^{\texttt{d}}$,
$g = \texttt{randpoly}([x, t], \texttt{degree} = \texttt{d}) + (\texttt{x} - 1)\texttt{t}^{\texttt{d}}$,
$u = \texttt{randpoly}([x, t], \texttt{degree} = \texttt{d}) + 2\texttt{t}^{\texttt{d}}$,
$v = \texttt{randpoly}([x, t], \texttt{degree} = \texttt{d}) - \texttt{t}^{\texttt{d}}$,
$w = \texttt{randpoly}([y, x, t], \texttt{degree} = \texttt{d}) + \texttt{t}^{\texttt{d}}$ and $p = z^2 - z + 2$.

Integrate

$$\left(\frac{f}{g}\right)' + \frac{1}{2}\frac{u'}{u} - \frac{1}{3}\frac{v'}{v} + \frac{\left(\sum_{p(y)=0} yw(y, x, t)\right)'}{\sum_{p(y)=0} yw(y, x, t)}.$$

d	1	2	3	4	5	6	7	8
int	4.0	24.3	45.3	20.1	30.0	48.7	> 400	> 400
A & H	0.1	0.2	0.7	2.1	3.8	13.6	36.0	240.2

Elementary integrals over S-primitive towers

Input: $f \in K_n$, where $K_n = \mathbb{C}(x)(t_1, \ldots, t_n)$ is S-primitive.

Output: an elementary integral of f if there exists one.

1. Compute an additive decomposition

   $$f = g' + r, \quad \text{where } r \text{ is a remainder.}$$

   If r is not simple, then return "no elem integral".
2. Apply Raab's algorithm to compute $c_1, \ldots, c_n \in \mathbb{C}$ s.t.

   $$h := r - c_1 t_1' - \cdots - c_n t_n' \in L_{K_n}.$$

   If such constants do not exist, then return "no elem integral".
3. Compute a log part of h to get

   $$\{(c_1, g_1), \ldots, (c_s, g_s)\}.$$
4. Return $g + c_1 t_1 + \cdots + c_n t_n + \sum_{i=1}^{s} c_i \log(g_i)$.

Example

Let an S-primitive tower $K_3 = K_0(t_1, t_2, t_3)$, where

$$t_1 = \log(x),\ \ t_2 = \int \frac{1}{\log(x)}\, dx,\ \ t_3 = \log(\log(x)).$$

Integrate.

$$\underbrace{t_3 + \frac{t_2 - 2xt_1}{t_1^2} + \frac{1}{t_1\,t_2}}_{f} \overset{\text{A.D.}}{=} \underbrace{\left(\frac{1}{2}\,\frac{{t_2}^2 t_1 + 2\,x t_3\,t_1 - 2\,x t_2 - 2\,x^2}{t_1}\right)'}_{g} + \underbrace{\frac{1}{t_1\,t_2} - \frac{1}{t_1}}_{r}$$

$$\overset{\text{Raab}}{=} (g - t_2)' + \frac{1}{t_1\,t_2} \overset{\text{L.P.}}{=} (g - t_2 + \log(t_2))'\,.$$

Result.

$$\int f\, dx = g - t_2 + \log(t_2).$$

Remark. Both Maple and Mathematica return the integral unevaluated.

Additive decomposition in hyperexponential towers

A D-field $K_n = K_0(t_1, \ldots, t_n)$ is a hyperexponential tower if

(i) $t_1, \ldots, t_n$ are algebraically independent over K_0,

(ii) $t_i'/t_i \in K_{i-1}$ with $1 \leq i \leq n$,

(iii) $C_{K_n} = \mathbb{C}$.

Such a tower is exponential if each t_i'/t_i is a derivative in K_{i-1}.

Additive decompositions can be carried out in several cases, e.g.

- each $t_i'/t_i \in K_0$,
- K_n is exponential.

Summary

Results.

- an additive decomposition in S-primitive towers,
- an algorithm for computing elementary integrals over S-primitive towers.

Goal.

- develop an additive decomposition in $K_n = K_0(t_1, \ldots, t_n)$, where t_i is either logarithmic or exponential over K_{i-1},
- compute elementary integrals over K_n.

Thanks for your attention!