# Computational Group Theory and Quantum Physics

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# Preliminary Remarks

1. Quantum behavior is manifestation of universal mathematical properties of systems with indistinguishable objects — any violation of identity of particles destroys interferences
2. For systems with symmetries only invariant — independent of relabeling of "homogeneous" elements — relations and statements are objective

   E.g., no objective meaning can be attached to electric potentials $\varphi$ and $\psi$ or to space points $\mathbf{a}$ and $\mathbf{b}$, but invariants $\psi - \varphi$ and $\mathbf{b} - \mathbf{a}$ (in more general group notation $\varphi^{-1}\psi$ and $\mathbf{a}^{-1}\mathbf{b}$) are meaningful
3. Question "whether the real world is discrete or continuous" or even "finite or infinite" is metaphysical — neither empirical observations nor logical arguments can validate one of the alternatives

   The choice is a matter of belief or taste

Since no empirical consequences of choice between finite and infinite descriptions are possible — "physics is independent of metaphysics" —

we can consider quantum concepts in constructive finite background without any risk to destroy physical content of the problem

# Poincaré

1. "The sole natural object of mathematical thought is the whole number. It is the external world which has imposed the continuum upon us, which we doubtless have invented, but which it has forced us to invent. Without it there would be no infinitesimal analysis; all mathematical science would reduce itself to arithmetic or to the theory of substitutions. . . . On the contrary, we have devoted to the study of the continuum almost all our time and all our strength.

   . . . Let us not be such purists and let us be grateful to the continuum, which, if all springs from the whole number, was alone capable of making so much proceed there from." (1904)
2. "Now we can no longer maintain that «nature does not make jumps» (*Natura non facit saltus*); in fact, it behaves in quite the opposite way. And not only matter possibly reduces to atoms, but even the world history, I dare say, and even time itself. . . " (1912)
3. "However, we should not hurry too much, since at the moment it is clear only that we are quite far from completing the struggle between two styles of thinking: that of atomists, believing in the existence of primary elements, a very large but finite number of combinations of which suffices to explain the whole diversity of the Universe, and the other one, common to the adherents of continuity and infinity concepts." (1912)

# Discrete Mathematics Outperforms Continuous By Content

## Comparative overview of simple continuous and finite groups

| Lie groups | Finite groups |
|---|---|
| 4 infinite families | $16+1+1$ infinite families |
| $A_n, B_n, C_n, D_n$ | $A_n(q), B_n(q), C_n(q), D_n(q), E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$ - Chevalley; |
| 5 exceptional groups | ${}^2A_n\left(q^2\right), {}^2D_n\left(q^2\right), {}^2E_6\left(q^2\right), {}^3D_4\left(q^3\right)$ - Steinberg; ${}^2B_n\left(2^{2n+1}\right)$ - Suzuki; |
| $E_6, E_7, E_8, F_4, G_2$ | ${}^2F_4\left(2^{2n+1}\right)$ - Ree, Tits; ${}^2G_2\left(3^{2n+1}\right)$ - Ree |
| | $\mathbb{Z}_p$ - prime order cyclic groups; $\mathrm{A}_n$ - alternating groups |
| | 26 sporadic groups |
| | $M_{11}, M_{12}, M_{22}, M_{23}, M_{23}$ - Mathieu, only nontrivial 4- and 5-transitive |
| | $J_1, J_2, J_3, J_4$ - Janko; $Co_1, Co_2, Co_3$ - Conway; $Fi_{22}, Fi_{23}, Fi_{24}$ - Fischer; $HS$ - Higman-Sims; $McL$ - McLaughlin; $He$ - Held; $Ru$ - Rudvalis; $Suz$ - Suzuki; $O'N$ - O'Nan; $HN$ - Harada-Norton; $Ly$ - Lyons; $Th$ - Thompson; $B$ - Baby Monster; |
| | $M$ - Monster; largest sporadic, contains all other sporadics (excepting 6 called pariahs: $J_1, J_3, J_4, Ru, O'N, Ly$) |
| | John McKay discovered famous "monstrous moonshine" Richard Borcherds won Fields medal for proving "monstrous moonshine" using string theory methods |

# State Mixing in Flavor Physics

Fermions in Standard Model form 3 generations of quarks and leptons

| Fermions \ Generations | 1 | 2 | 3 |
|---|---|---|---|
| Up-quarks | $u$ | $c$ | $t$ |
| Down-quarks | $d$ | $s$ | $b$ |
| Charged leptons | $e^-$ | $\mu^-$ | $\tau^-$ |
| Neutrinos | $\nu_e$ | $\nu_\mu$ | $\nu_\tau$ |

Transitions between up- and down- quarks in quark sector and flavor and mass neutrino states in lepton sector are described by Cabibbo–Kobayashi–Maskawa

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

and Pontecorvo–Maki–Nakagawa–Sakata

$$U_{\mathrm{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

mixing matrices

# Observational Evidences of Fundamental Finite Symmetries

Most sharp picture comes from numerous neutrino oscillation data

Phenomenological pattern

- $\nu_\mu$ and $\nu_\tau$ flavors are presented with equal weights in all 3 mass eigenstates $\nu_1, \nu_2, \nu_3$ (called "bi-maximal mixing")
- all three flavors are presented equally in $\nu_2$ ("trimaximal mixing")
- $\nu_e$ is absent in $\nu_3$

implies probabilities $\left(|U_{\alpha\beta}|^2\right) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

$\longrightarrow$ unitary matrix (Harrison, Perkins, Scott) $U_{\text{HPS}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$U_{\text{HPS}}$ (also "tribimaximal mixing matrix") coincides with a matrix decomposing permutation representation of $S_3$ into irreducible components

This caused a burst of activity in building models based on finite symmetry groups

In the quark sector the picture is not so clear, but there are some encouraging empirical observations, e.g., quark-lepton complementarity (QLC)

— observation that sum of quark and lepton mixing angles $\approx \pi/4$

# Popular Groups for Constructing Models in Flavor Physics

- $\mathrm{T} = \mathrm{A}_4$ — the tetrahedral group;
- $\mathrm{T}'$ — the double covering of $\mathrm{A}_4$;
- $\mathrm{O} = \mathrm{S}_4$ — the octahedral group;
- $\mathrm{I} = \mathrm{A}_5$ — the icosahedral group;
- $\mathrm{D}_N$ — the dihedral groups ($N$ even);
- $\mathrm{Q}_N$ — the quaternionic groups (4 divides $N$);
- $\Sigma\left(2N^2\right)$ — the groups in this series have the structure $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_2$;
- $\Delta\left(3N^2\right)$ — the structure $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_3$;
- $\Sigma\left(3N^3\right)$ — the structure $(\mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathbb{Z}_3$;
- $\Delta\left(6N^2\right)$ — the structure $(\mathbb{Z}_N \times \mathbb{Z}_N) \rtimes \mathrm{S}_3$.

# Permutations

- Any set $\Omega = \{\omega_1, \ldots, \omega_n\}$ with transitive symmetries $G = \{g_1, g_2, \ldots, g_M\}$ is in 1-to-1 correspondence with set of right (or left) cosets of some subgroup $H \leq G$
  $\Omega \cong H \backslash G$ (or $G/H$) is called homogeneous space ($G$-space)

  Action of $G$ on $\Omega$ is faithful if $H$ does not contain normal subgroups of $G$

  Action by permutations $\pi(g) = \begin{pmatrix} \omega_i \\ \omega_i g \end{pmatrix} \cong \begin{pmatrix} Ha \\ Hag \end{pmatrix}$ $\quad g, a \in G \quad i = 1, \ldots, n$

- Maximum transitive set $\Omega \cong \{\mathbf{1}\} \backslash G \cong G$ corresponds to right regular action

$$\Pi(g) = \begin{pmatrix} g_i \\ g_i g \end{pmatrix} \qquad i = 1, \ldots, M$$

- For "quantitative" ("statistical") description elements of $\Omega$ are equipped with numerical "weights" from suitable number system $\mathcal{N}$ containing 0 and 1
  — permutations can be rewritten as matrices

  $\pi(g) \to \rho(g) \quad \rho(g)_{ij} = \delta_{\omega_i g, \omega_j} \quad i, j = 1, \ldots, n$ $\qquad$ permutation representation
  $\Pi(g) \to \mathrm{P}(g) \quad \mathrm{P}(g)_{ij} = \delta_{e_i g, e_j} \quad i, j = 1, \ldots, M$ $\qquad$ regular representation

- For the sake of freedom of algebraic manipulations, one assumes usually that $\mathcal{N}$ is algebraically closed field — ordinarily complex numbers $\mathbb{C}$.
  If $\mathcal{N}$ is a field, then the set $\Omega$ can be treated as basis of linear vector space $\mathcal{H} = \mathrm{Span}(\omega_1, \cdots, \omega_n)$.

# Linear Representations of Finite Group

1. Any linear representation of G is unitary — there is always unique invariant inner product $\langle \cdot \mid \cdot \rangle$ making $\mathcal{H}$ into Hilbert space
2. All possible irreducible unitary representations of G are contained in regular representation

$$\mathrm{T}^{-1}\mathrm{P}(g)\mathrm{T} = \begin{pmatrix} D_1(g) & & & & & & \\ & d_2\left\{\begin{matrix} D_2(g) & & \\ & \ddots & \\ & & D_2(g) \end{matrix}\right. & & & \\ & & \ddots & & \\ & & & d_\mathrm{m}\left\{\begin{matrix} D_\mathrm{m}(g) & & \\ & \ddots & \\ & & D_\mathrm{m}(g) \end{matrix}\right. \end{pmatrix}$$

- $\mathrm{m} =$ number of $\begin{cases} \text{different irreducible representations } D_j \text{ of G} \\ \text{conjugacy classes in G} \end{cases}$
- $d_j = \dim D_j$ = multiplicity of $D_j$ in regular representation
- obviously $d_1^2 + d_2^2 + \cdots + d_\mathrm{m}^2 = \mathrm{M} \equiv |\mathrm{G}|$ besides: $d_j$ divides M

# Character Table Describes All Irreducible Representations

Example: character table of icosahedral group $A_5$

| | $K_1$ | $K_{15}$ | $K_{20}$ | $K_{12}$ | $K_{12'}$ |
|---|---|---|---|---|---|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_3$ | 3 | $-1$ | 0 | $\phi$ | $1-\phi$ |
| $\chi_{3'}$ | 3 | $-1$ | 0 | $1-\phi$ | $\phi$ |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | 1 | $-1$ | 0 | 0 |

$\phi = \frac{1+\sqrt{5}}{2}$ — "golden ratio"
$\phi$ and $1-\phi$ are cyclotomic integers (even "cyclotomic naturals"):
$\phi = -r^2 - r^3 \ \equiv\ 1 + r + r^4$ and $1 - \phi = -r - r^4 \ \equiv\ 1 + r^2 + r^3$
$r$ is primitive $5$th root of unity

Some general properties of characters

- Characters determine representations uniquely
- Isoclinism. Character table determines group almost entirely: nonisomorphic groups with identical character tables have identical derived groups (commutator subgroups)
  Example. Dihedral and quaternionic groups of order $8$ are isoclinic:
  $D_8 = \{\text{symmetries of square}\}$ and $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

# Computational Group Theory

## Some computer implementations

- **GAP** (Groups, Algorithms, Programming)
  http://www.gap-system.org/
  sufficiently comprehensive, efficient free system for working
  with groups and various other structures of discrete mathematics
  some shortcomings are total ignoring unitarity issues
  and unhandy command line oriented interface
- **Magma**
  http://magma.maths.usyd.edu.au/magma/
  quality enough (by all accounts) but expensive system
- **Nauty** (No automorphisms, yes?)
  http://cs.anu.edu.au/∼bdm/nauty/
  author Brendan D. McKay
  program for determining automorphism groups of graphs
  regarded as most efficient at present
  (apparently ideas of the algorithm can be easily adapted
  to computing symmetries of other combinatorial structures),
  it is written in **C**, free available

# Classical and Quantum Evolution of Dynamical System

**Classical evolution is a sequence of states evolving in time**

$\cdots \to s_{t-1} \to s_t \to s_{t+1} \to \cdots \qquad t \in \mathcal{T} \subseteq \mathbb{Z} \qquad \Omega = \{\omega_1, \ldots, \omega_{\mathrm{N}}\}$

**Quantum evolution is a sequence of permutations of states**

$\cdots \to p_{t-1} \to p_t \to p_{t+1} \to \cdots \qquad p_t \in \mathrm{G} = \{\mathrm{g}_1, \ldots, \mathrm{g}_{\mathrm{M}}\} \leq \mathrm{Sym}(\Omega)$

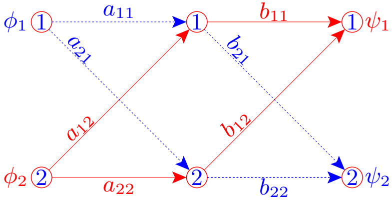
**In physics systems with space $\mathrm{X} = \left\{\mathrm{x}_1, \ldots, \mathrm{x}_{|\mathrm{X}|}\right\}$ are usual**

- Set of states takes special structure of functions on space $\Omega = \Sigma^{\mathrm{X}}$
- $\Sigma = \left\{\sigma_1, \ldots, \sigma_{|\Sigma|}\right\}$ is set of local states
- Space symmetry group $\mathrm{F} = \left\{\mathrm{f}_1, \ldots, \mathrm{f}_{|\mathrm{F}|}\right\} \leq \mathrm{Sym}(\mathrm{X})$
- Internal symmetry group $\Gamma = \left\{\gamma_1, \ldots, \gamma_{|\Gamma|}\right\} \leq \mathrm{Sym}(\Sigma)$
- Whole symmetry group G can be expressed as split extension $\mathbf{1} \to \Gamma^{\mathrm{X}} \to \mathrm{G} \to \mathrm{F} \to \mathbf{1}$ determined by an antihomomorphism $\mu : \mathrm{F} \to \mathrm{F}$
  if $\mu(f) = f^{-1}$ (natural antihomomorphism)
  then $\mathrm{G} \cong \Gamma \wr_{\mathrm{X}} \mathrm{F}$ is wreath product

# Unifying Feynman’s and Matrix Formulations of Quantum Mechanics

Feynman’s rules “multiply subsequent events” and “sum up alternative histories” is nothing else than rephrasing of matrix multiplication rules

Quantum evolution $\quad |\psi\rangle = U\,|\phi\rangle \qquad |\phi\rangle = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

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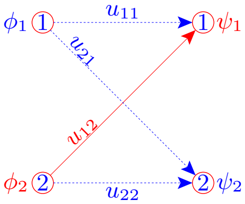



$\Updownarrow \qquad\qquad \Updownarrow$

$$BA = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} \sim \; U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$BA = U$$

All this works also for generalized amplitude with non $\mathrm{U}(1)$-valued connection

One should only take into account non-commutativity of matrix entries

# Standard and Finite Quantum Mechanics

Our aim is to reproduce main features of quantum mechanics in finite background

Our strategy is Occam's razor — not to introduce entities unless we really need them

| Standard QM | Finite QM |
|---|---|
| **State vectors $\|\psi\rangle$ form** | |
| Hilbert space $\mathcal{H}$ over $\mathbb{C}$ | K-dimensional Hilbert space $\mathcal{H}_K$ over abelian number field $\mathcal{F}$ — extension of rationals $\mathbb{Q}$ with abelian Galois group ("*cyclotomics*") |
| **Unitary operators $U$ belong to** | |
| general unitary group $\mathrm{Aut}(\mathcal{H})$ acting in $\mathcal{H}$ | unitary representation $\mathrm{U}$ in space $\mathcal{H}_K$ of finite group $\mathrm{G} = \{\mathrm{g}_1, \ldots, \mathrm{g}_M\}$. Field $\mathcal{F}$ depends on structure of $\mathrm{G}$ |
| **Quantum evolution is unitary transformation $\|\psi_{out}\rangle = U\|\psi_{in}\rangle$** | |
| Elementary step of evolution is described by Schrödinger equation $i\dfrac{\mathrm{d}}{\mathrm{d}t}\|\psi\rangle = H\|\psi\rangle$ | Only finite number of possible evolutions: $U_j \in \{\mathrm{U}(\mathrm{g}_1)\ldots, \mathrm{U}(\mathrm{g}_j), \ldots, \mathrm{U}(\mathrm{g}_M)\}$ No need for any kind of Schrödinger equation. Formally Hamiltonians can be introduced: $H_j = i\ln U_j \equiv \sum_{k=0}^{p-1}\lambda_k U_j^k,$ $p$ is period of $U_j$ |

# Standard and Finite Quantum Mechanics. Continuation

- More general Hermitian operators describing observables in quantum formalism can be expressed in terms of group algebra representation:

$$A = \sum_{k=1}^{M} \alpha_k \mathrm{U}\left(\mathrm{g}_k\right)$$

- The Born rule: probability to register particle described by $\left|\psi\right\rangle$ by apparatus tuned to $\left|\phi\right\rangle$ is

$$\mathbf{P}(\phi, \psi) = \frac{\left|\left\langle\phi \mid \psi\right\rangle\right|^2}{\left\langle\phi \mid \phi\right\rangle\left\langle\psi \mid \psi\right\rangle} \qquad \text{(BR)}$$

  Conceptual refinement is needed — the only reasonable meaning of probability for finite sets is frequency interpretation: probability is ratio of number of "favorable" combinations to total number of all combinations

  Our guiding principle: formula (BR) must give rational numbers if all things are arranged correctly

Other elements of quantum theory are obtained in standard way. E.g., Heisenberg principle follows from Cauchy-Bunyakovsky-Schwarz inequality

$$\left\langle A\psi \mid A\psi\right\rangle\left\langle B\psi \mid B\psi\right\rangle \geq \left|\left\langle A\psi \mid B\psi\right\rangle\right|^2$$

equivalent to standard property of any probability $\mathbf{P}(A\psi, B\psi) \leq 1$

# Embedding Quantum System into Permutations

- Any (always unitary) representation $\mathsf{U}$ of group $\mathsf{G}$ in $\mathsf{K}$-dimensional Hilbert space $\mathcal{H}_\mathsf{K}$ can be embedded into permutation representation $\mathrm{P}$ of faithful realization of $\mathsf{G}$ by permutations of $\mathsf{N} \geq \mathsf{K}$ things:
$$\Omega = \{\omega_1, \ldots, \omega_\mathsf{N}\}$$
- If $\mathsf{N} > \mathsf{K}$ then representation $\mathrm{P}$ in $\mathsf{N}$-dimensional Hilbert space $\mathcal{H}_\mathsf{N}$ has the structure
$$\mathrm{T}^{-1}\mathrm{P}\mathrm{T} = \begin{pmatrix} \mathbf{1} & & \\ & \mathsf{U} & \\ & & \mathrm{V} \end{pmatrix} \equiv \mathbf{1} \oplus \mathsf{U} \oplus \mathrm{V}$$
Here $\mathbf{1}$ is trivial one-dimensional representation — obligatory component of any permutation representation, $\mathrm{V}$ may be empty
- Additional “hidden parameters” — appearing due to increase of Hilbert space dimension from $\mathsf{K}$ to $\mathsf{N}$ — in no way can effect on data relating to space $\mathcal{H}_\mathsf{K}$ since both $\mathcal{H}_\mathsf{K}$ and its complement in $\mathcal{H}_\mathsf{N}$ are invariant subspaces of extended space $\mathcal{H}_\mathsf{N}$

With trivial assumption that components of state vectors are arbitrary elements of $\mathcal{F}$ we can set arbitrary (e.g., zero) data in subspace complementary to $\mathcal{H}_\mathsf{K}$

Dropping this assumption leads to more natural meaning of quantum amplitudes

# Natural Quantum Amplitudes

- Permutation representation $\mathrm{P}$ makes sense over any number system with 0 and 1
- Very natural number system is semi-ring of natural numbers

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

- With this semi-ring we can attach counters to elements of set $\Omega$ interpreted as "multiplicities of occurrences" or "population numbers" of elements $\omega_i$ in state of system involving elements from $\Omega$
- Such state can be represented by vector with natural components

$$|n\rangle = \begin{pmatrix} n_1 \\ \vdots \\ n_{\mathrm{N}} \end{pmatrix}$$

Thus, we come to representation of G in N-dimensional module $\mathrm{H_N}$ over semiring $\mathbb{N}$. Representation $\mathrm{P}$ simply permutes components of vector $|n\rangle$
For further development we turn module $\mathrm{H_N}$ into N-dimensional Hilbert space $\mathcal{H}_{\mathrm{N}}$ by extending $\mathbb{N}$ to field $\mathcal{F}$

# Why Cyclotomics?

cyclic subgroups are most important constituents of groups

Field $\mathbb{C}$ consists almost entirely of useless non-constructive elements
What is needed actually are combinations of basics:

- Natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ — counters of states and dimensions
- Irrationalities:
  - Roots of unity — all eigenvalues of linear representations
  - Square roots of dimensions — coefficients to provide unitarity

  Irrationalities of both types have common nature — they are cyclotomic integers
  e.g., $i$ is simultaneously square root of integer $\sqrt{-1}$ and primitive $4^{\text{th}}$ root of unity

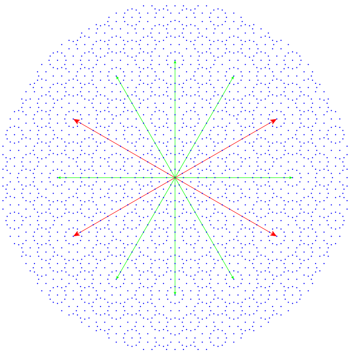
Purely mathematical derivation leads to minimal abelian number field $\mathcal{F}$ containing these basics

- Kronecker-Weber theorem:
  Any abelian number field is subfield of some cyclotomic field $\mathbb{Q}_{\mathcal{P}}$:
  $\mathcal{F} \leq \mathbb{Q}_{\mathcal{P}} = \mathbb{Q}[\mathrm{r}] / \langle \Phi_{\mathcal{P}}(\mathrm{r}) \rangle$, $\Phi_{\mathcal{P}}(\mathrm{r})$ is $\mathcal{P}^{\text{th}}$ cyclotomic polynomial
- Period $\mathcal{P}$ — called conductor — is determined by structure of $\mathsf{G}$
- $\mathbb{Q}_{\mathcal{P}}$ can be embedded into $\mathbb{C}$, but we do not need this possibility.
  Purely algebraic properties of $\mathbb{Q}_{\mathcal{P}}$ are sufficient for all purposes

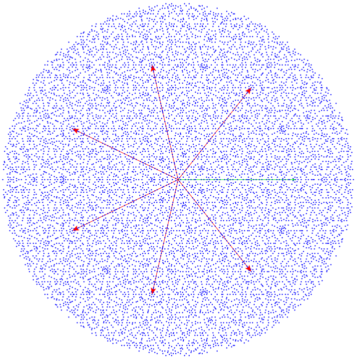
All irrationalities are intermediate elements of quantum description whereas final values are rational — this is refinement of interrelation between complex and real numbers in standard quantum mechanics

# Embedding Cyclotomic Integers $\mathcal{N}_\mathcal{P}$ into Complex Plane $\mathbb{C}$

$\mathcal{P} = 12$



$\mathcal{P} = 7$



Red (green) arrows — primitive (nonprimitive) roots

Complex conjugation in $\mathcal{N}_\mathcal{P}$ is defined via rule $\overline{\mathrm{r}^k} = \mathrm{r}^{\mathcal{P}-k}$

# Cyclotomics and Eigenvalues of Representations

Roots of unity and abelian number fields

- Cyclotomic equation $\mathrm{r}^n = 1$ describes all roots of unity
- Cyclotomic polynomial $\Phi_n(\mathrm{r})$ describes all primitive $n$th roots of unity (and only them)
- $\Phi_n(\mathrm{r})$ is irreducible over $\mathbb{Q}$ divisor of $\mathrm{r}^n - 1$
- Natural combinations of roots of unity are sufficient for constructing cyclotomic integers.

  Negative integers can be introduced via identity $(-1) = \sum_{k=1}^{p-1} \mathrm{r}^{\frac{\mathcal{P}}{p}k}$, $p$ is any divisor of $\mathcal{P}$
- Conductor $\mathcal{P}$ determining ring of integers $\mathcal{N}_{\mathcal{P}}$ and field $\mathbb{Q}_{\mathcal{P}}$ may be proper divisor of $n$
  To compute basis of lattice $\mathcal{N}_{\mathcal{P}}$ algorithms like *LLL* are used
- Abelian number field $\mathcal{F} \leq \mathbb{Q}_{\mathcal{P}}$ is fixed in $\mathbb{Q}_{\mathcal{P}}$ by additional symmetries called Galois automorphisms

All eigenvalues of linear representations are roots of unity

- any linear representation is subrepresentation of some permutation representation
- characteristic polynomial of matrix $\mathrm{P}$ of permutation of $\mathrm{N}$ elements:

$$\chi_{\mathrm{P}}(\lambda) = \det(\mathrm{P} - \lambda\mathrm{I}) = (\lambda - 1)^{k_1}\left(\lambda^2 - 1\right)^{k_2} \cdots \left(\lambda^{\mathrm{N}} - 1\right)^{k_{\mathrm{N}}}$$

  array $[k_1, k_2, \ldots, k_{\mathrm{N}}]$ is called cycle type of permutation
  $k_i$ is number of cycles of length $i$

# Example: Group of Permutations of Three Things $S_3$

application in physics: "tribimaximal mixing" in neutrino oscillations

Faithful action on $\Omega = S_2 \backslash S_3 = \{1, 2, 3\}$

$$S_3 = \{\overbrace{g_1 = ()}^{K_1},\ \overbrace{g_2 = (23)\,, g_3 = (13)\,, g_4 = (12)}^{K_2},\ \overbrace{g_5 = (123)\,, g_6 = (132)}^{K_3}\}$$

can be generated by two generators $g_2$ and $g_6$ (one of many possible choices)

Permutation matrices of generators

$$P_2 = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{pmatrix}, \quad P_6 = \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}$$

$$\mathrm{T}^{-1} P \mathrm{T} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}, \text{ where } \mathrm{T} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & r^2 \\ 1 & r^2 & 1 \\ 1 & r & r \end{pmatrix}, \mathrm{T}^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & r & r^2 \\ r & 1 & r^2 \end{pmatrix}$$

r is primitive 3d root of unity embedding into $\mathbb{C}$: $\frac{-1 \pm i\sqrt{3}}{2}$ or $\mathrm{e}^{\pm 2\pi i/3}$

Matrices of 2D faithful representation for generators

$$U_2 = \begin{pmatrix} 0 & r^2 \\ r & 0 \end{pmatrix}, \quad U_6 = \begin{pmatrix} r & 0 \\ 0 & r^2 \end{pmatrix}$$

## $S_3$. Projecting States into Invariant 2D Subspace

- State vectors in:
  - "permutation basis"

$$|n\rangle = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \quad |m\rangle = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

  - "quantum basis"

$$\left|\widetilde{\psi}\right\rangle = \mathrm{T}^{-1}\,|n\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2 + n_3 \\ n_1 + n_2\mathrm{r} + n_3\mathrm{r}^2 \\ n_1\mathrm{r} + n_2 + n_3\mathrm{r}^2 \end{pmatrix}, \quad \left|\widetilde{\phi}\right\rangle = \mathrm{T}^{-1}\,|m\rangle = \cdots$$

- Projections onto $U$:

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} n_1 + n_2\mathrm{r} + n_3\mathrm{r}^2 \\ n_1\mathrm{r} + n_2 + n_3\mathrm{r}^2 \end{pmatrix}, \quad |\phi\rangle = \cdots$$

# $S_3$. Quantum Interference in Invariant Subspace

- Born's probability for $2D$ state vectors in terms of $3D$ parameters

$$\mathbf{P}(\phi,\psi)=\frac{|\langle\phi\mid\psi\rangle|^2}{\langle\phi\mid\phi\rangle\langle\psi\mid\psi\rangle}=\frac{\left(\mathrm{Q}_3\left(m,n\right)-\frac{1}{3}\mathrm{L}_3\left(m\right)\mathrm{L}_3\left(n\right)\right)^2}{\left(\mathrm{Q}_3\left(m,m\right)-\frac{1}{3}\mathrm{L}_3\left(m\right)^2\right)\left(\mathrm{Q}_3\left(n,n\right)-\frac{1}{3}\mathrm{L}_3\left(n\right)^2\right)}$$

- $\mathrm{L}_\mathrm{N}\left(n\right)=\sum_{i=1}^{\mathrm{N}} n_i$ and $\mathrm{Q}_\mathrm{N}\left(m,n\right)=\sum_{i=1}^{\mathrm{N}} m_i n_i$ are (common to all groups) linear and quadratic invariants of N-dimensional permutation representations
- Condition for destructive quantum interference

$$3\left(m_1n_1+m_2n_2+m_3n_3\right)-\left(m_1+m_2+m_3\right)\left(n_1+n_2+n_3\right)=0$$

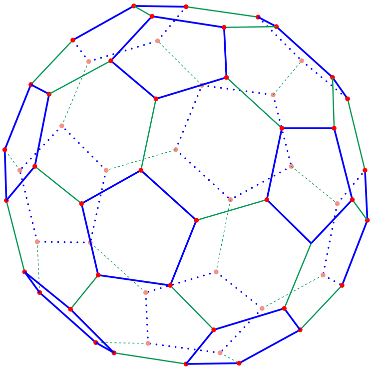
has infinitely many solutions in natural numbers, e.g., $|n\rangle=\begin{pmatrix}1\\1\\2\end{pmatrix},\ |m\rangle=\begin{pmatrix}1\\3\\2\end{pmatrix}$

**Thus, we obtained essential features of quantum behavior**

from "permutation dynamics" and "natural" interpretation of quantum amplitude by simple transition to invariant subspace

# Icosahedral Group $A_5$. Main Properties

- Smallest simple non-commutative group
- Very important in mathematics and applications:
  F. Klein devoted a whole book to it *“Vorlesungen über das Ikosaeder”*, 1884
- “Physical incarnation”: carbon molecule fullerene $C_{60}$ “is” Cayley graph of $A_5$



- Presentation by generators and relators:

$$\langle a, b \mid a^5(\text{pentagons}), b^2, (ab)^3(\text{hexagons}) \rangle$$

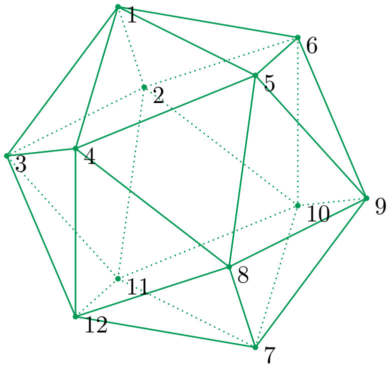
- 5 irreducible representations (4 faithful):

$$\mathbf{1}, \mathbf{3}, \mathbf{3}', \mathbf{4}, \mathbf{5}$$

- 3 primitive permutation representations:

$$\overline{\mathbf{5}} \cong \mathbf{1} \oplus \mathbf{4}, \ \overline{\mathbf{6}} \cong \mathbf{1} \oplus \mathbf{5}, \ \overline{\mathbf{10}} \cong \mathbf{1} \oplus \mathbf{4} \oplus \mathbf{5}$$

# Action of $A_5$ on Icosahedron




- Permutation action on 12 vertices
  $\overline{\mathbf{12}} \cong \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}$ is transitive
  but imprimitive
- Imprimitivity (block) system:
  $\{| B_1 | \cdots | B_i | \cdots | B_6 | \} =$
  $\{| 1, 7 | \cdots | i, i+6 | \cdots | 6, 12 |\}$
  Blocks are pairs of opposite vertices
- Notations for further use:
  "Complementarity":
  $q = p^{\mathrm{C}}$ and $p = q^{\mathrm{C}}$ if $p, q \in B_i$
  Example: $1 = 7^{\mathrm{C}}$ and $7 = 1^{\mathrm{C}}$
  "Neighborhood" of vertex:
  $\mathrm{N}(p)$ is set of vertices adjacent to $p$
  Example: $\mathrm{N}(1) = \{2, 3, 4, 5, 6\}$

# Transformation Matrix Decomposing Action on Icosahedron

Unitary matrix $\mathrm{T}$ such that $\mathrm{T}^{-1}\left(\overline{\mathbf{12}}\right)\mathrm{T} = \mathbf{1} \oplus \mathbf{3} \oplus \mathbf{3}' \oplus \mathbf{5}$

$$\mathrm{T} = \begin{pmatrix}
\frac{\sqrt{3}}{6} & \alpha & \beta & 0 & \alpha & \beta & 0 & \frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{15}}{12} \\
\frac{\sqrt{3}}{6} & 0 & \alpha & \beta & -\beta & 0 & \alpha & -\frac{\phi}{4} & 0 & -\frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & \beta & 0 & \alpha & 0 & -\alpha & -\beta & \frac{\phi-1}{4} & 0 & 0 & -\frac{1}{2} & \delta \\
\frac{\sqrt{3}}{6} & 0 & \alpha & -\beta & -\beta & 0 & -\alpha & -\frac{\phi}{4} & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & -\beta & 0 & \alpha & 0 & \alpha & -\beta & \frac{\phi-1}{4} & 0 & 0 & \frac{1}{2} & \delta \\
\frac{\sqrt{3}}{6} & \alpha & -\beta & 0 & -\alpha & \beta & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{\sqrt{15}}{12} \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & \beta & \beta & 0 & \alpha & -\frac{\phi}{4} & 0 & \frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & \beta & 0 & -\alpha & 0 & -\alpha & \beta & \frac{\phi-1}{4} & 0 & 0 & \frac{1}{2} & \delta \\
\frac{\sqrt{3}}{6} & -\alpha & \beta & 0 & \alpha & -\beta & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & \frac{\sqrt{15}}{12} \\
\frac{\sqrt{3}}{6} & -\alpha & -\beta & 0 & -\alpha & -\beta & 0 & \frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{\sqrt{15}}{12} \\
\frac{\sqrt{3}}{6} & 0 & -\alpha & -\beta & \beta & 0 & -\alpha & -\frac{\phi}{4} & 0 & -\frac{1}{2} & 0 & \gamma \\
\frac{\sqrt{3}}{6} & -\beta & 0 & -\alpha & 0 & \alpha & \beta & \frac{\phi-1}{4} & 0 & 0 & -\frac{1}{2} & \delta
\end{pmatrix}$$

$\phi = \dfrac{1+\sqrt{5}}{2}$ is "golden ratio", $\alpha = \dfrac{\phi}{4}\sqrt{10-2\sqrt{5}}$, $\beta = \dfrac{\sqrt{5}\sqrt{10-2\sqrt{5}}}{20}$,

$$\gamma = \frac{\sqrt{3}}{8}\left(1-\frac{\sqrt{5}}{3}\right), \quad \delta = -\frac{\sqrt{3}}{8}\left(1+\frac{\sqrt{5}}{3}\right)$$

# Invariant Inner Products in Invariant Subspaces in Terms of Permutation Invariants

$n = (n_1, \ldots, n_{12})^T$, $m = (m_1, \ldots, m_{12})^T$ are natural vectors

1. $\langle \Phi_1 \mid \Psi_1 \rangle = \frac{1}{12} \mathrm{L}_{12}(m) \mathrm{L}_{12}(n)$
2. $\langle \Phi_{3\oplus3'} \mid \Psi_{3\oplus3'} \rangle = \frac{1}{2} \left( \mathrm{Q}_{12}(m,n) - \mathrm{A}(m,n) \right)$
   1. $\langle \Phi_3 \mid \Psi_3 \rangle = \frac{1}{20} \left( 5\mathrm{Q}_{12}(m,n) - 5\mathrm{A}(m,n) + \sqrt{5} \left( \mathrm{B}(m,n) - \mathrm{C}(m,n) \right) \right)$
   2. $\langle \Phi_{3'} \mid \Psi_{3'} \rangle = \frac{1}{20} \left( 5\mathrm{Q}_{12}(m,n) - 5\mathrm{A}(m,n) - \sqrt{5} \left( \mathrm{B}(m,n) - \mathrm{C}(m,n) \right) \right)$

   Here irrationality is consequence of imprimitivity:
   one can not move vertex without simultaneous moving of its opposite
3. $\langle \Phi_5 \mid \Psi_5 \rangle = \frac{1}{12} \left( 5\mathrm{Q}_{12}(m,n) + 5\mathrm{A}(m,n) - \mathrm{B}(m,n) - \mathrm{C}(m,n) \right)$

$$\mathrm{A}(m,n) = \mathrm{A}(n,m) = \sum_{k=1}^{12} m_k n_{k^{\mathrm{C}}}$$

$$\mathrm{B}(m,n) = \mathrm{B}(n,m) = \sum_{k=1}^{12} m_k \sum_{i \in \mathrm{N}(k)} n_i$$

$$\mathrm{C}(m,n) = \mathrm{C}(n,m) = \sum_{k=1}^{12} m_k \sum_{i \in \mathrm{N}\left(k^{\mathrm{C}}\right)} n_i$$

Identity: $\mathrm{A}(m,n) + \mathrm{B}(m,n) + \mathrm{C}(m,n) + \mathrm{Q}_{12}(m,n) = \mathrm{L}_{12}(m) \mathrm{L}_{12}(n)$

# Conclusions

1. Quantum mechanics is *a priori* mathematical scheme based on fundamental impossibility to trace identity of indistinguishable objects in their evolution — some kind of "*calculus of indistinguishables*"
2. Any quantum mechanical problem can be reduced to permutations
3. Quantum interferences are appearances observable in invariant subspaces of permutation representation and expressible in terms of permutation invariants
4. Interpretation of quantum amplitudes ("*waves*") as vectors of "population numbers" of underlying entities ("*particles*") leads to rational quantum probabilities — in line with frequency interpretation of probability
   - Idea of natural quantum amplitudes is very promising. It requires verification — evidences may be expected in particle physics. If it is valid quantum phenomena in different invariant subspaces are different manifestations — visible in different "*observational set-ups*" — of single process of permutations of underlying things
   - Otherwise, trivial assumption of arbitrary amplitudes leads — up to physically inessential difference between "finite" and "infinite" — to usual quantum mechanics reformulated in terms of permutations