Investigation of the Equilibrium Orientations of Two Connected Bodies in a Circular Orbit Using Computer Algebra Methods
(Исследование положений равновесия двух связанных тел на круговой орбите с применением методов компьютерной алгебры)

Gutnik S.A.

MGIMO University, Moscow Institute of Physics and Technology

Sarychev V.A.

Keldysh Institute of Applied Mathematics, RAS

Computer Algebra Seminar of Faculty of Computational Mathematics and Cybernetics of MSU, and Computing Centre of RAS, October 20 2022, Moscow

1. Equations of motion

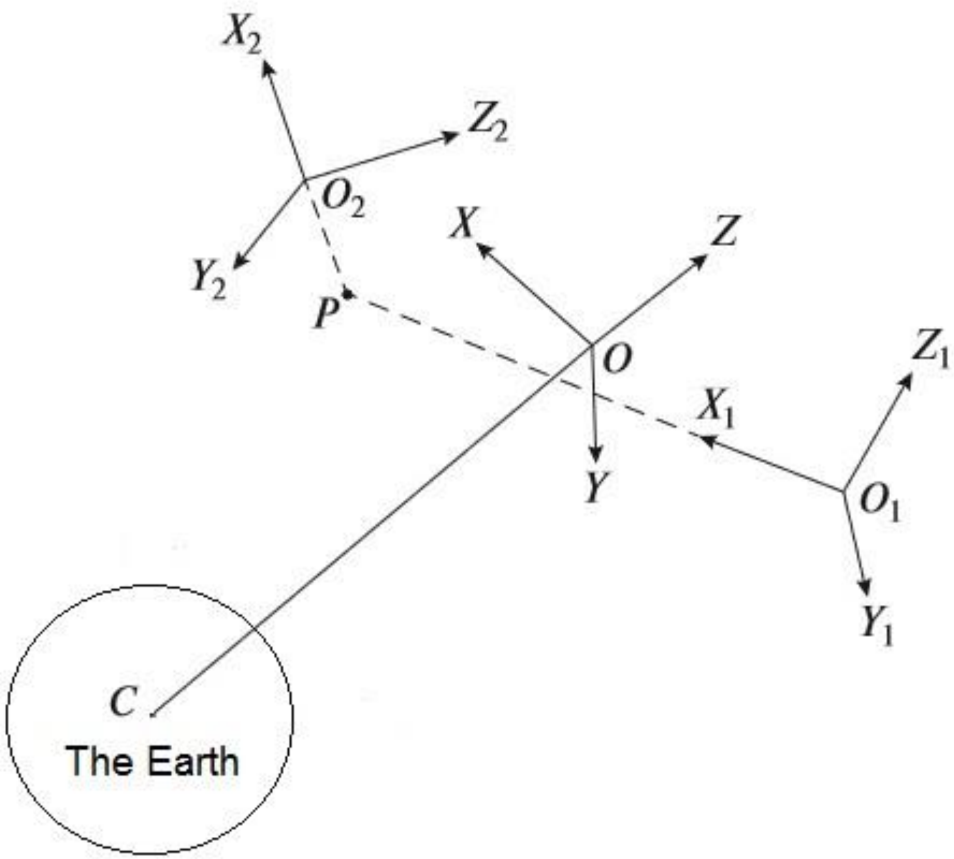

Fig. 1. Basic coordinate systems

$OXYZ$ - is the orbital coordinate system

(a_i, b_i, c_i) – are the coordinates of the spherical hinge P in the body coordinate system $O_i x_i y_i z_i$

1. Equations of motion

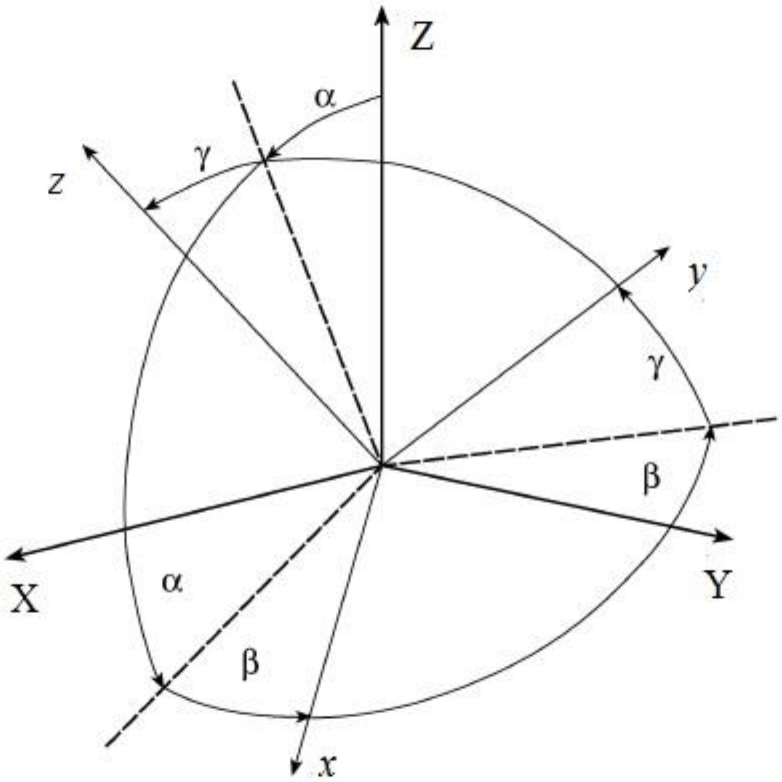

Fig. 1. Orientation of body–fixed axes with respect to the orbital coordinate system

International Space Station

.

International Space Station(1998-2024) in gravity orientation mode

Orbit parameters: h_a =418 km, h_p =413 km, i=51.63^0

1.1 Equations of motion in the orbital plane

Consider the motion of the two bodies system around its center of mass in the plane of a circular orbit when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, *two* aircraft angles $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = 0$. The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical hinge in the case $b_1 = b_2 = 0$ have the form:

$$
\begin{aligned}
U = {} & \frac{3}{2} M \omega_0^2 [(a_1 \sin\alpha_1 - c_1 \cos\alpha_1) - (a_2 \sin\alpha_2 - c_2 \cos\alpha_2)]^2 - \\
& -\frac{3}{2}\omega_0^2 (A_1 - C_1)\sin^2\alpha_1 - \frac{3}{2}\omega_0^2 (A_2 - C_2)\sin^2\alpha_2 + \\
& + M\omega_0^2[(a_1 a_2 + c_1 c_2)\cos(\alpha_1 - \alpha_2) - (a_1 c_2 - a_2 c_1)\sin(\alpha_1 - \alpha_2)].
\end{aligned}
\tag{1}
$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i-th body; $\alpha_i, \beta_i, \gamma_i$ - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(a_i, 0, c_i)$ - are the coordinates of the of the spherical hinge of the i-th body in reference frame; ω_0 – is the angular velocity of the orbital motion of the center of mass of the two-body system.

1.1 Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $b_1 = b_2 = 0$ have the form

$$T = \frac{1}{2}[B_1 + M(a_1^2 + c_1^2)](\dot\alpha_1 + \omega_0)^2 + \frac{1}{2}[B_2 + M(a_2^2 + c_2^2)](\dot\alpha_2 + \omega_0)^2 -$$
$$-M[(a_1 a_2 + c_1 c_2)\cos(\alpha_1 - \alpha_2) - \qquad (2)$$
$$-(a_1 c_2 - a_2 c_1)\sin(\alpha_1 - \alpha_2)](\dot\alpha_1 + \omega_0)(\dot\alpha_2 + \omega_0).$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation D in the Wolfram *Mathematica* 12.1

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\alpha_i} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = 1, 2.$$

1.1 Equations of motion

The Lagrange equations have the form

$$
\begin{aligned}
&(B_1 + M(a_1^2 + c_1^2))\ddot{\alpha}_1 - M((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) - \\
&-(a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2))\ddot{\alpha}_2 - M((a_1a_2 + c_1c_2)\sin(\alpha_1 - \alpha_2) + \\
&+(a_1c_2 - a_2c_1)\cos(\alpha_1 - \alpha_2))(\dot{\alpha}_2^2 + 2\dot{\alpha}_2) + 3(A_1 - C_1)\sin\alpha_1\cos\alpha_1 - \\
&-3M(a_1\cos\alpha_1 + c_1\sin\alpha_1)((a_1\sin\alpha_1 - c_1\cos\alpha_1) - (a_2\sin\alpha_2 - c_2\cos\alpha_2)) = 0,
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&-M((a_1a_2 + c_1c_2)\cos(\alpha_1 - \alpha_2) - (a_1c_2 - a_2c_1)\sin(\alpha_1 - \alpha_2))\ddot{\alpha}_1 + \\
&+(B_2 + M(a_2^2 + c_2^2))\ddot{\alpha}_2 + 3(A_2 - C_2)\sin\alpha_2\cos\alpha_2 + \\
&+M((a_1a_2 + c_1c_2)\sin(\alpha_1 - \alpha_2) + (a_1c_2 - a_2c_1)\cos(\alpha_1 - \alpha_2))(\dot{\alpha}_1^2 + 2\dot{\alpha}_1) + \\
&+3M(a_2\cos\alpha_2 + c_2\sin\alpha_2)((a_1\sin\alpha_1 - c_1\cos\alpha_1) - (a_2\sin\alpha_2 - c_2\cos\alpha_2)) = 0.
\end{aligned}
\tag{4}
$$

1.2 Equilibrium orientations

By assuming in (3) - (4) $\alpha_i = \text{const}$ we obtain the stationary equations

$$
\begin{aligned}
&((A_1 - C_1)/M)\sin\alpha_1\cos\alpha_1 - (a_1\cos\alpha_1 + c_1\sin\alpha_1)(a_1\sin\alpha_1 - c_1\cos\alpha_1) + \\
&+(a_1\cos\alpha_1 + c_1\sin\alpha_1)(a_2\sin\alpha_2 - c_2\cos\alpha_2) = 0, \\
&((A_2 - C_2)/M)\sin\alpha_2\cos\alpha_2 - (a_2\cos\alpha_2 + c_2\sin\alpha_2)(a_2\sin\alpha_2 - c_2\cos\alpha_2) + \\
&+(a_2\cos\alpha_2 + c_2\sin\alpha_2)(a_1\sin\alpha_1 - c_1\cos\alpha_1) = 0.
\end{aligned}
\tag{5}
$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system the plane of a circular orbit .
Equations (5) form a closed system of two equations with respect to the two aircraft angles.
Trigonometric system (5) cannot be solved directly.

1.3 Investigations of Equilibria

For system (5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\alpha_i)$

$$\sin\alpha_i = \frac{\tan(\alpha_i)}{\sqrt{1+\tan^2(\alpha_i)}} = \frac{t_i}{\sqrt{1+t_i^2}}, \quad \cos\alpha_i = \frac{1}{\sqrt{1+\tan^2(\alpha_i)}} = \frac{1}{\sqrt{1+t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$d_1 t_1 - (a_1 + c_1 t_1)(a_1 t_1 - c_1) = (a_1 + c_1 t_1)(c_2 - a_2 t_2)\frac{\sqrt{1+t_1^2}}{\sqrt{1+t_2^2}},$$

$$d_2 t_2 - (a_2 + c_2 t_2)(a_2 t_2 - c_2) = (a_2 + c_2 t_2)(c_1 - a_1 t_1)\frac{\sqrt{1+t_2^2}}{\sqrt{1+t_1^2}}. \qquad (6)$$

Where $\quad d_1 = (A_1 - C_1)/M, \;\; d_2 = (A_2 - C_2)/M.$

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2.

1.3 Investigations of Equilibria

$$\begin{aligned} &\bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + a_3 = 0, \\ &\bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 = 0. \end{aligned} \qquad (7)$$

Where

$$\bar{a}_0 = c_1(c_2(a_1^2 - a_2^2)t_2^3 + a_2(a_1^2 - a_2^2 + 2c_2^2 + d_2)t_2^2 +$$
$$+c_2(a_1^2 - c_2^2 + 2a_2^2 - d_2)t_2 + a_2(a_1^2 - c_2^2)),$$
$$\bar{a}_1 = a_1(c_2(a_1^2 - a_2^2 - 2c_1^2 - d_1)t_2^3 + a_2(a_1^2 - a_2^2 - 2c_1^2 + 2c_2^2 - d_1 + d_2)t_2^2 +$$
$$+c_2(a_1^2 + 2a_2^2 - 2c_1^2 - c_2^2 - d_1 - d_2)t_2 + a_2(a_1^2 - 2c_1^2 - c_2^2 - d_1)),$$
$$\bar{a}_2 = c_1(c_2(c_1^2 - a_2^2 - 2a_1^2 + d_1)t_2^3 + a_2(c_1^2 - a_2^2 - 2a_1^2 + 2c_2^2 + d_1 + d_2)t_2^2 -$$
$$-c_2(2a_1^2 - 2a_2^2 - c_1^2 + c_2^2 - d_1 + d_2)t_2 - a_2(2a_1^2 - c_1^2 + c_2^2 - d_1)),$$
$$\bar{a}_3 = a_1(c_2(c_1^2 - a_2^2)t_2^3 - a_2(a_2^2 - c_1^2 - 2c_2^2 - d_2)t_2^2 +$$
$$+c_2(2a_2^2 + c_1^2 - c_2^2 - d_2)t_2 + a_2(c_1^2 - c_2^2)),$$

1.3 Investigations of Equilibria

$$\begin{aligned}
\overline{b}_0 &= a_1 c_1 d_2 t_2, \\
\overline{b}_1 &= a_2 c_2 d_1 t_2^2 + ((a_2^2 - c_2^2) d_1 + (a_1^2 - c_1^2) d_2 - d_1 d_2) t_2 - a_2 c_2 d_1, \\
\overline{b}_2 &= -a_1 c_1 d_2 t_2.
\end{aligned} \tag{8}$$

Here

$$d_1 = \frac{(A_1 - C_1)}{M}, \quad d_2 = \frac{(A_2 - C_2)}{M}.$$

1.3 Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(7). Expanding the determinant of resultant matrix of Eq.(7) with the help of *Mathematica* matrix function **Resultant**, we obtain the 12th order algebraic equation in t_2. After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2)=0. \tag{9}$$

Here

$$P_1(t_2)=a_1c_1d_1(a_2t_2-c_2)^2=0,$$

$$P_2(t_2)=a_2c_2t_2^2+(a_2^2-c_2^2+d_2)t_2+a_2c_2=0,$$

$$P_3(t_2)=p_0t_2^8+p_1t_2^7+p_2t_2^6+p_3t_2^5+p_4t_2^4+p_5t_2^3+p_6t_2^2+p_7t_2+p_8,$$

$$p_0=c_2^4d_1^2(a_1^2-a_2^2)(a_2^2-c_1^2), \tag{10}$$

$$p_1=2a_2c_2^3d_1(d_2(a_1^2-c_1^2)(a_1^2-a_2^2+c_1^2)+d_1(2a_2^2(a_1^2-a_2^2)+$$

$$+c_1^2(c_2^2-a_1^2-d_2)+(2a_2^2-a_1^2)(c_1^2+c_2^2+d_2))),\ldots$$

Here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(9) and (7), for each set of the system parameters, we can determine numerically the angles α_2 and α_1, that is, all the equilibrium orientations of the satellite–stabilizer system.

1.3 Investigations of Equilibria

In studying the two-body system equilibrium orientations, we determine the domains with an equal number of real roots of Eq.(9) in the space of 6 parameters. The decomposition of the space of parameters into domains with an equal number of real roots is determined by the discriminant hypersurface. The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simpler case when $a_1 = a_2 = c_1 = c_2 = a$.

In this case from system (10) we will obtain the equation of the 4th degree

$$P_{10}(t_2) = p_{00}t_2^4 + p_{01}t_2^3 + p_{02}t_2^2 + p_{03}t_2 + p_{04} = 0, \tag{11}$$

where

$$p_{00} = 4(d_{01}d_{02} + d_{01} - d_{02})(d_{01}d_{02} + d_{01} + d_{02}),$$

$$p_{01} = -4(d_{02} + 2)(d_{01}^2 d_{02}^2 - d_{01}^2 d_{02} - 2d_{01}^2 + 2d_{02}^2),$$

$$p_{02} = (d_{01}^2 d_{02}^4 + 4d_{02}^4 - 16d_{01}^2 d_{02}^2 + 24(d_{01}^2 - d_{02}^2)),$$

$$p_{03} = 4(d_{02} - 2)(d_{01}^2 d_{02}^2 + d_{01}^2 d_{02} - 2d_{01}^2 + 2d_{02}^2),$$

$$p_{04} = 4(d_{01}d_{02} - d_{01} - d_{02})(d_{01}d_{02} - d_{01} + d_{02}).$$

1.3 Investigations of Equilibria

Here

$$d_{01} = (A_1 - C_1)/Ma^2,$$

$$d_{02} = (A_2 - C_2)/Ma^2;$$

Discriminant hypersurface of the polynomial (11) $P_1(t_2)$ (resultant of the two polynomials $P_1(t_2)$ and $P'_1(t_2)$) has the form

$$P_5(d_{01}, d_{02}) =$$

$$= 1024 d_{02}^{12} P_6(d_{01}, d_{02}) P_7(d_{01}, d_{02}) P_8(d_{01}, d_{02}) = 0.$$

Here

(12)

$$P_6(d_{01}, d_{02}) = (d_{01}^2 (d_{02} + 1)^2 - d_{02}^2),$$

$$P_7(d_{01}, d_{02}) = (((d_{01}^2 d_{02}^2 + 4(d_{01} - d_{02})^2)((d_{01}^2 d_{02}^2 + 4(d_{01} + d_{02})^2),$$

$$P_8(d_{01}, d_{02}) = (d_{01}^4 (d_{02}^2 + 8)^2 + 16 d_{01}^2 d_{02}^2 (d_{02}^2 + 10) + 64(d_{02}^4 - 108)).$$

1.3 Investigations of Equilibria

$P_6(d_{01},d_{02})$ and $P_7(d_{01},d_{02})$ are the polynomials of the 4th and 8th degree. Now we should check the change in the number of equilibria when the curve (12) is intersected. This can be done numerically by determining the number of equilibria at a single point of each domain at the plane (d_{01},d_{02}). Only the curve $P_8(d_{01},d_{02}) = 0$ separates the domains with different number of equilibria.

Fig.2 shows the distributions of domains with equal number of real roots of Eq.(11) and indicates the domains where 4 and 2 real solutions exist (16 and 8 equilibrium orientations).

In case when $a_1 = a_2 = c_1 = c_2 = a$

$$P_1(t_2) = a_1c_1d_1(a_2t_2 - c_2)^2 = a^3d_1(t_2 - 1)^2 = 0; \quad t_2 = \tan\alpha_2 = 1$$

$$P_2(t_2) = a_2c_2t_2^2 + (a_2^2 - c_2^2 + d_2)t_2 + a_2c_2 = t_2^2 + d_{02}t_2 + 1 = 0,$$

$$t_2 = \left(-d_{02} \pm \sqrt{d_{02}^2 - 4}\right)\Big/2.$$

Therefore, in the case when $a_1 = a_2 = c_1 = c_2 = a$, and $|d_{02}| \le 2$ there exist only 28 and 20 equilibrium orientations for the satellite–stabilizer system in the plane of a circular orbit.

1.3 Investigations of Equilibria

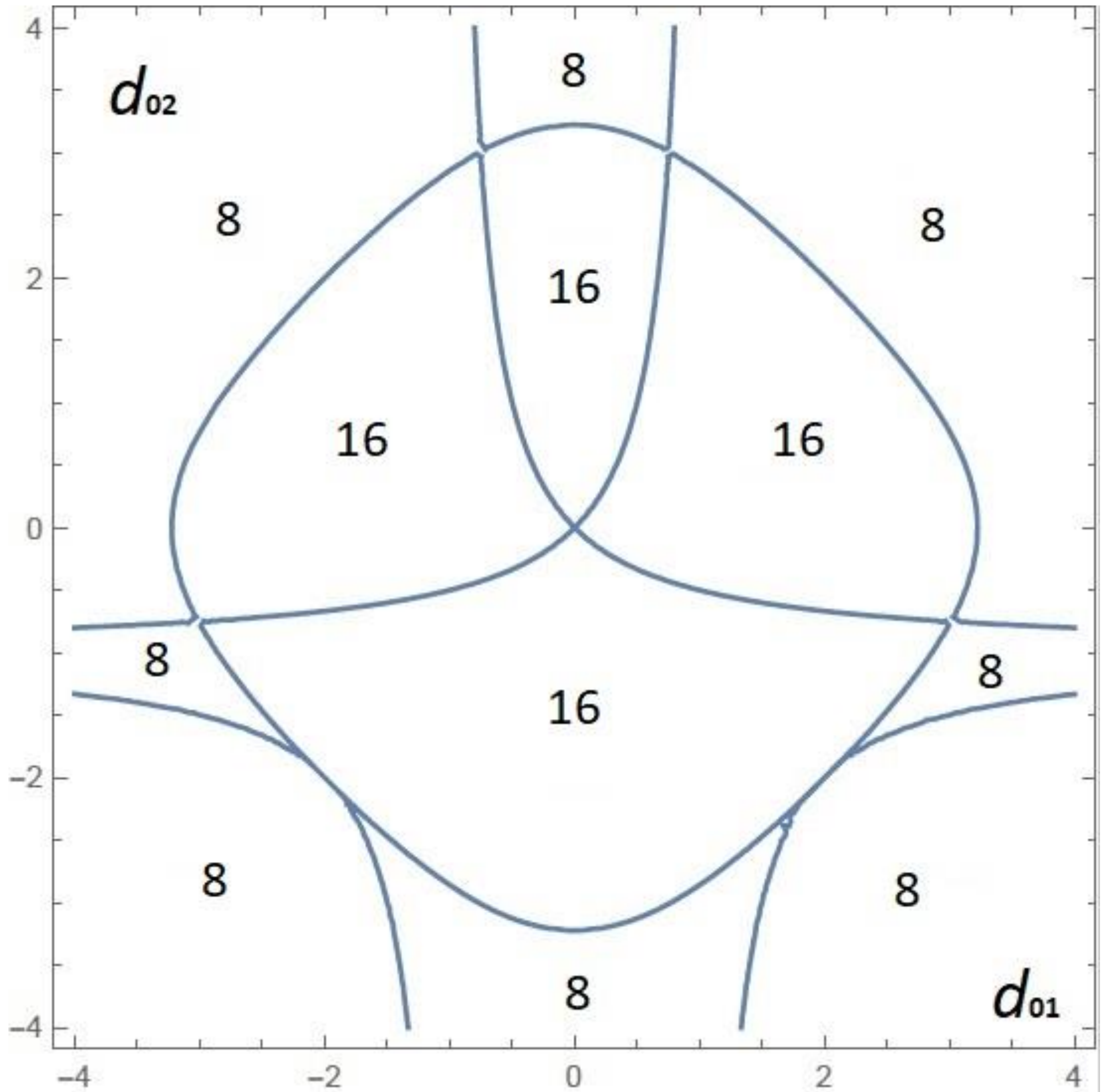

Fig. 2. The regions with the fixed number of equilibria

2.1. Equations of motion in the plane perpendicular to orbital plane

Consider the motion of the two bodies system around its center of mass in the plane perpendicular to the orbital plane of a circular orbit when $\beta_1 \neq 0, \ \beta_2 \neq 0$ *two* aircraft angles $\alpha_1 = \alpha_2 = 0$, $\gamma_1 = \gamma_2 = 0$. The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical hinge in the case $c_1 = c_2 = 0$ have the form:

$$U = M\omega_0^2((a_1 a_2 + b_1 b_2)\cos(\beta_1 - \beta_2) + \\ + (a_1 b_2 - a_2 b_1)\sin(\beta_1 - \beta_2)). \qquad (2.1)$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1 M_2 / (M_1 + M_2)$ and M_i is the mass of the i-th body; α_i, β_i, γ_i - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(a_i, b_i, 0)$ - are the coordinates of the of the spherical hinge of the i-th body in reference frame; ω_0 – is the angular velocity of the orbital motion of the center of mass of the two-body system.

2.1. Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when c_1=c_2=0 have the form

$$\begin{gathered}
T = \frac{1}{2}(C_1 + M(a_1^2 + b_1^2))\dot{\beta}_1^2 + \frac{1}{2}(C_2 + M(a_2^2 + b_2^2))\dot{\beta}_2^2 - \\
M((a_1 a_2 + b_1 b_2)\cos(\beta_1 - \beta_2) + (a_1 b_2 - a_2 b_1)\sin(\beta_1 - \beta_2))\dot{\beta}_1\dot{\beta}_2 + \\
+\frac{1}{2}((B_1 - A_1) - M(b_1^2 - a_1^2))\omega_0^2\cos^2\beta_1 - M a_1 a_2 \omega_0^2 \cos\beta_1 \cos\beta_2 + \\
+\frac{1}{2}((B_2 - A_2) - M(b_2^2 - a_2^2))\omega_0^2\cos^2\beta_2 - M b_1 b_2 \omega_0^2 \sin\beta_1 \sin\beta_2 - \\
-M\omega_0^2(b_1\sin\beta_1 - b_2\sin\beta_2)(a_1\cos\beta_1 - a_2\cos\beta_2).
\end{gathered} \tag{2.2}$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation D in the Wolfram *Mathematica* 12.1

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}_i} - \frac{\partial T}{\partial \alpha_i} - \frac{\partial U}{\partial \alpha_i} = 0, \quad i = 1, 2.$$

2.1. Equations of motion

The Lagrange equations have the form

$$
\begin{aligned}
&(C_1 + M(a_1^2 + b_1^2))\ddot{\beta}_1 - M((a_1 a_2 + b_1 b_2)\cos(\beta_1 - \beta_2) + \\
&+(a_1 b_2 - a_2 b_1)\sin(\beta_1 - \beta_2))\ddot{\beta}_2 - M((a_1 a_2 + b_1 b_2)\sin(\beta_1 - \beta_2) \\
&-(a_1 b_2 - a_2 b_1)\cos(\beta_1 - \beta_2))\dot{\beta}_2^{\,2} + (B_1 - A_1)\omega_0^2 \sin\beta_1 \cos\beta_1 + \\
&+M\omega_0^2 (a_1 \sin\beta_1 + b_1 \cos\beta_1)(a_1 \cos\beta_1 - b_1 \sin\beta_1) - \\
&-M\omega_0^2 (a_1 \cos\beta_1 - b_1 \sin\beta_1)(a_2 \sin\beta_2 + b_2 \cos\beta_2) = 0,
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
&(C_2 + M(a_2^2 + b_2^2))\ddot{\beta}_2 - M((a_1 a_2 + b_1 b_2)\cos(\beta_1 - \beta_2) + \\
&+(a_1 b_2 - a_2 b_1)\sin(\beta_1 - \beta_2))\ddot{\beta}_1 + M((a_1 a_2 + b_1 b_2)\sin(\beta_1 - \beta_2) - \\
&-(a_1 b_2 - a_2 b_1)\cos(\beta_1 - \beta_2))\dot{\beta}_1^{\,2} + (B_2 - A_2)\omega_0^2 \sin\beta_2 \cos\beta_2 + \\
&+M\omega_0^2 (a_2 \sin\beta_2 + b_2 \cos\beta_2)(a_2 \cos\beta_2 - b_2 \sin\beta_2) - \\
&-M\omega_0^2 (a_2 \cos\beta_2 - b_2 \sin\beta_2)(a_1 \sin\beta_1 + b_1 \cos\beta_1) = 0.
\end{aligned} \tag{2.4}
$$

2.2 Equilibrium orientations

By assuming in (2.3) - (2.4) $\beta_i = \mathrm{const}$ we obtain the stationary equations

$$((B_1 - A_1)/M)\sin\beta_1\cos\beta_1 + (a_1\sin\beta_1 + b_1\cos\beta_1)(a_1\cos\beta_1 - b_1\sin\beta_1) -$$
$$-(a_1\cos\beta_1 - b_1\sin\beta_1)(a_2\sin\beta_2 + b_2\cos\beta_2) = 0,$$

(2.5)

$$((B_2 - A_2)/M)\sin\beta_2\cos\beta_2 + (a_2\sin\beta_2 + b_2\cos\beta_2)(a_2\cos\beta_2 - b_2\sin\beta_2) -$$
$$-(a_2\cos\beta_2 - b_2\sin\beta_2)(a_1\sin\beta_1 + b_1\cos\beta_1) = 0.$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system the plane perpendicular to the circular orbit .
Equations (2.5) form a closed system of two equations with respect to the two aircraft angles.

2.3. Investigations of Equilibria

For system (2.5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\beta_i)$

$$\sin\alpha_i = \frac{\tan(\beta_i)}{\sqrt{1+\tan^2(\beta_i)}} = \frac{t_i}{\sqrt{1+t_i^2}}, \quad \cos\beta_i = \frac{1}{\sqrt{1+\tan^2(\beta_i)}} = \frac{1}{\sqrt{1+t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$\begin{gathered}
((B_1 - A_1)/M)\mathrm{tg}\beta_1 + (a_1\mathrm{tg}\beta_1 + b_1)(a_1 - b_1\mathrm{tg}\beta_1) - \\
-(a_1 - b_1\mathrm{tg}\beta_1)(a_2\mathrm{tg}\beta_2 + b_2)\frac{\sqrt{1+\mathrm{tg}^2\beta_1}}{\sqrt{1+\mathrm{tg}^2\beta_2}} = 0, \\
((B_2 - A_2)/M)\mathrm{tg}\beta_2 + (a_2\mathrm{tg}\beta_2 + b_2)(a_2 - b_2\mathrm{tg}\beta_2) - \\
-(a_2 - b_2\mathrm{tg}\beta_2)(a_1\mathrm{tg}\beta_1 + b_1)\frac{\sqrt{1+\mathrm{tg}^2\beta_2}}{\sqrt{1+\mathrm{tg}^2\beta_1}} = 0.
\end{gathered} \tag{2.6}$$

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2.

2.3. Investigations of Equilibria

$$\begin{aligned} &\bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + a_3 = 0, \\ &\bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 = 0. \end{aligned} \tag{2.7}$$

Where

$$\bar{a}_0 = b_1((a_2 t_2 + b_2)(a_2 b_2 + (d_2 + a_2^2 - b_2^2)t_2 - a_2 b_2 t_2^2) - a_1^2 (a_2 - b_2 t_2)(1 + t_2^2)),$$

$$\bar{a}_1 = -a_1((a_2 t_2 + b_2)\left(a_2 b_2 + \left(d_2 + a_2^2 - b_2^2\right)\left(t_2\right) - a_2 b_2 t_2^2\right) -$$

$$-(d_1 + a_1^2 - 2b_1^2)(a_2 - b_2 t_2)(1 + t_2^2)),$$

$$\bar{a}_2 = b_1((a_2 t_2 + b_2)(a_2 b_2 + (d_2 + a_2^2 - b_2^2)\,(t_2 - a_2 b_2 t_2^2)) +$$

$$+(d_1 + 2a_1^2 - b_1^2)(a_2 - b_2 t_2)(1 + t_2^2)),$$

$$\bar{a}_3 == -a_1((a_2 t_2 + b_2)\,(a_2 b_2 + (d_2 + a_2^2 - b_2^2)t_2 - a_2 b_2 t_2^2) -$$

$$-\,b_1^2 (a_2 - b_2 t_2)(1 + t_2^2)\Big),$$

2.3. Investigations of Equilibria

$$\begin{aligned}
\overline{b_0} &= a_1 b_1 d_2 t_2, \\
\overline{b_1} &= a_2 b_2 d_1 t_2^2 - d_1 d_2 t_2 + (b_1^2 - a_1^2) d_2 t_2 + (b_2^2 - a_2^2) d_1 t_2 - a_2 b_2 d_1, \\
\overline{b_2} &= -a_1 b_1 d_2 t_2.
\end{aligned} \tag{2.8}$$

Here

$$d_1 = \frac{(B_1 - A_1)}{M}, \quad d_2 = \frac{(B_2 - A_2)}{M}.$$

2.3. Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(2.7). Expanding the determinant of resultant matrix of Eq.(2.7) with the help of *Mathematica* matrix function **Resultant**, we obtain the 12th order algebraic equation in t_2. After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2)=0. \tag{2.9}$$

Here

$$P_1(t_2)=a_1b_1d_1(b_2+a_2t_2)^2=0,$$

$$P_2(t_2)=a_2b_2+(a_2^2-b_2^2+d_2)t_2-a_2b_2t_2^2=0,$$

$$P_3(t_2)=p_0t_2^8+p_1t_2^7+p_2t_2^6+p_3t_2^5+p_4t_2^4+p_5t_2^3+p_6t_2^2+p_7t_2+p_8,$$

$$p_0=d_1^2b_2^4(a_1^2-a_2^2)(a_2^2-b_1^2), \tag{2.10}$$

$$p_1=-2a_2b_2^3d_1(d_1(2a_1^2a_2^2-2a_2^4-2a_1^2b_1^2+2a_2^2b_1^2-a_1^2b_2^2+2a_2^2b_2^2-b_1^2b_2^2)$$
$$+d_2(a_1^4-a_1^2a_2^2+a_2^2b_1^2-b_1^4)+d_1d_2((a_1^2-2a_2^2+b_1^2)),\ldots$$

Here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(2.9) and (2.7), for each set of the system parameters, we can determine numerically the angles β_2 and β_1, that is, all the equilibrium orientations of the satellite–stabilizer system.

2.3. Investigations of Equilibria

The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simpler case when $a_1 = a_2 = b_1 = b_2 = b$.
In this case from system (2.10) we will obtain the equation of the 4th degree

$$P_{10}(t_2) = p_{00}t_2^4 + p_{01}t_2^3 + p_{02}t_2^2 + p_{03}t_2 + p_{04} = 0,$$

where

$$\begin{aligned}
p_{00} &= 4(d_{01}d_{02} + d_{01} - d_{02})(d_{01}d_{02} + d_{01} + d_{02}),\\
p_{01} &= -4(d_{02} + 2)(d_{01}^2 d_{02}^2 - d_{01}^2 d_{02} - 2d_{01}^2 + 2d_{02}^2),\\
p_{02} &= (d_{01}^2 d_{02}^4 + 4d_{02}^4 - 16d_{01}^2 d_{02}^2 + 24(d_{01}^2 - d_{02}^2)),\\
p_{03} &= 4(d_{02} - 2)(d_{01}^2 d_{02}^2 + d_{01}^2 d_{02} - 2d_{01}^2 + 2d_{02}^2),\\
p_{04} &= 4(d_{01}d_{02} - d_{01} - d_{02})(d_{01}d_{02} - d_{01} + d_{02}).
\end{aligned} \tag{2.11}$$

2.3. Investigations of Equilibria

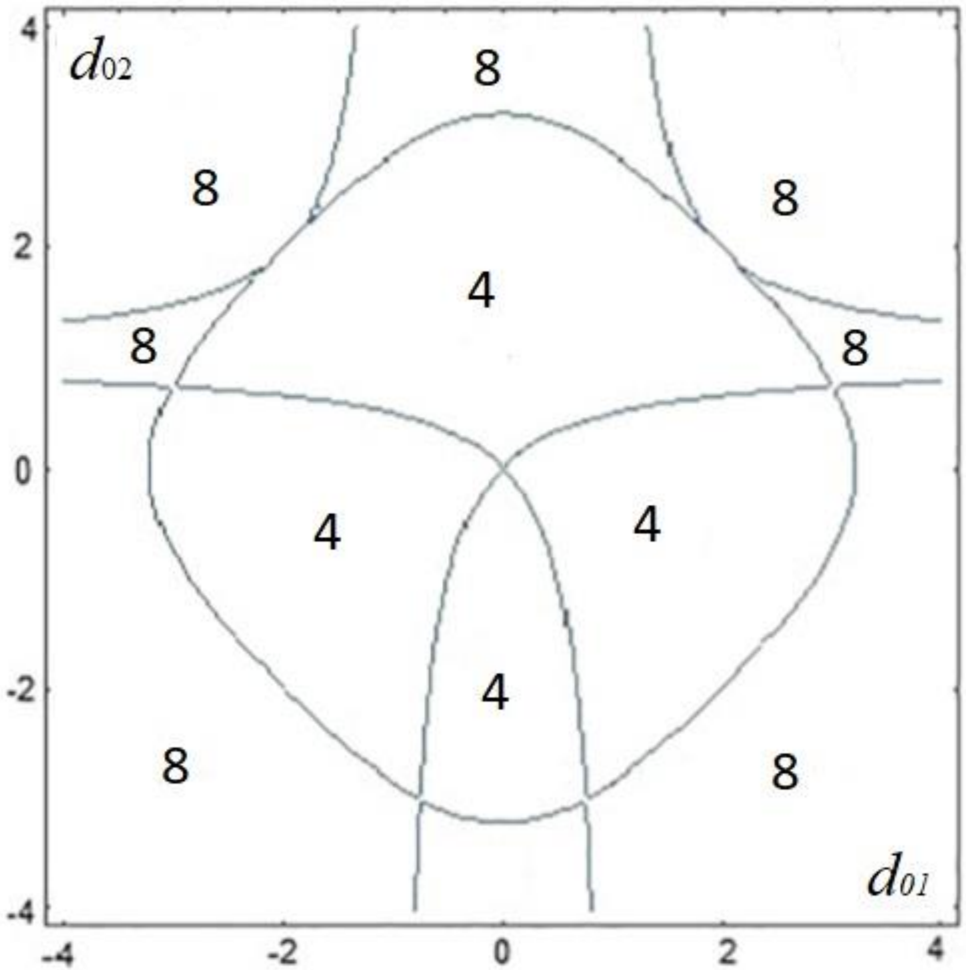

Fig. 2.2 The regions with the fixed number of equilibria

3.1. Equations of motion in the plane perpendicular to a circular orbit

Consider the motion of the two bodies system around its center of mass in the plane perpendicular to a circular orbit when two aircraft angles $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 0$ *and* $\gamma_1 \neq 0$, $\gamma_2 \neq 0$. The expressions for the force function, which determines the effect of the Earth's gravitational field on the system of two bodies connected by a spherical hinge in the case $a_1 = a_2 = 0$ have the form:

$$U = M\omega_0^2[(b_1b_2 + c_1c_2)\cos(\gamma_1 - \gamma_2) + (b_1c_2 - b_2c_1)\sin(\gamma_1 - \gamma_2)] +$$
$$\frac{3}{2}M\omega_0^2[(b_1\sin\gamma_1 - b_2\sin\gamma_2) + (c_1\cos\gamma_1 - c_2\cos\gamma_2)]^2 + \qquad (3.1)$$
$$+\frac{3}{2}\omega_0^2[(B_1 - C_1)\cos^2\gamma_1 + \frac{3}{2}\omega_0^2[(B_2 - C_2)\cos^2\gamma_2.$$

Here A_i, B_i, C_i are the principal central moments, $M = M_1M_2/(M_1 + M_2)$ and M_i is the mass of the *i*-th body, p_i, q_i, r_i are the projections of the angular velocity of the *i*-th body on the axes O_ix_i, O_iy_i O_iz_i ; $\alpha_i, \beta_i, \gamma_i$ - are the angles of pitch, yaw and roll; a_{ij}, b_{ij} - the direction cosines of the axis in the orbital reference frame, $(0, b_i, c_i)$ - are the coordinates of the of the spherical hinge of the *i*-th body in reference frame; ω_0 – is the angular velocity of the orbital motion of the for the center of mass of the two-body system.

3.1. Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $a_1 = a_2 = 0$ have the form

$$
\begin{aligned}
T = {} & \frac{1}{2}[A_1 + M(b_1^2 + c_1^2)]\dot{\gamma}_1^2 + \frac{1}{2}[A_2 + M(b_2^2 + c_2^2)]\dot{\gamma}_2^2 + \\
& + \frac{1}{2}[(B_1 - C_1) - M(b_1^2 - c_1^2)]\omega_0^2 \cos^2\gamma_1 + \frac{1}{2}[(B_2 - C_2) - M(b_2^2 - c_2^2)]\omega_0^2 \cos^2\gamma_2 - \\
& - M[(b_1 b_2 + c_1 c_2)\cos(\gamma_1 - \gamma_2) + (b_1 c_2 - b_2 c_1)\sin(\gamma_1 - \gamma_2)]\dot{\gamma}_1\dot{\gamma}_2 - \\
& - M b_1 b_2 \omega_0^2 \sin\gamma_1 \sin\gamma_2 - M c_1 c_2 \omega_0^2 \cos\gamma_1 \cos\gamma_2 + \\
& + M\omega_0^2 (b_1 \sin\gamma_1 - b_2 \sin\gamma_2)(c_1 \cos\gamma_1 - c_2 \cos\gamma_2).
\end{aligned} \tag{3.2}
$$

The equations of motion for this system can be written in Lagrange form the second kind by symbolic differentiation in the *Mathematica*

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot{\gamma}_i} - \frac{\partial T}{\partial \gamma_i} - \frac{\partial U}{\partial \gamma_i} = 0, \quad i = 1, 2.
$$

3.1. Equations of motion

The Lagrange equations have the form

$$
\begin{aligned}
&[A_1 + M(b_1^2 + c_1^2)]\ddot{\gamma}_1^2 - \\
&-M[(b_1 b_2 + c_1 c_2)\cos(\gamma_1 - \gamma_2) + (b_1 c_2 - b_2 c_1)\sin(\gamma_1 - \gamma_2)]\ddot{\gamma}_2 - \\
&-M[(b_1 b_2 + c_1 c_2)\sin(\gamma_1 - \gamma_2) - (b_1 c_2 - b_2 c_1)\cos(\gamma_1 - \gamma_2)]\dot{\gamma}_2^2 + \\
&+4[(B_1 - C_1) - M(b_1^2 - c_1^2)]\omega_0^2 \sin\gamma_1 \cos\gamma_1 - 4Mb_1 c_1 \omega_0^2 \cos 2\gamma_1 + \\
&+M\omega_0^2 (b_1 \sin\gamma_1 + c_1 \cos\gamma_1)(b_2 \cos\gamma_2 - c_2 \sin\gamma_2) + \\
&+3M\omega_0^2 (b_1 \cos\gamma_1 - c_1 \sin\gamma_1)(b_2 \sin\gamma_2 + c_2 \cos\gamma_2) = 0,
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
&[A_2 + M(b_2^2 + c_2^2)]\ddot{\gamma}_2^2 - \\
&-M[(b_1 b_2 + c_1 c_2)\cos(\gamma_1 - \gamma_2) + (b_1 c_2 - b_2 c_1)\sin(\gamma_1 - \gamma_2)]\ddot{\gamma}_1 + \\
&+M[(b_1 b_2 + c_1 c_2)\sin(\gamma_1 - \gamma_2) - (b_1 c_2 - b_2 c_1)\cos(\gamma_1 - \gamma_2)]\dot{\gamma}_1^2 + \\
&+4[(B_2 - C_2) - M(b_2^2 - c_2^2)]\omega_0^2 \sin\gamma_2 \cos\gamma_2 - 4Mb_2 c_2 \omega_0^2 \cos 2\gamma_2 + \\
&+M\omega_0^2 (b_2 \sin\gamma_2 + c_2 \cos\gamma_2)(b_1 \cos\gamma_1 - c_1 \sin\gamma_1) + \\
&+3M\omega_0^2 (b_2 \cos\gamma_2 - c_2 \sin\gamma_2)(b_1 \sin\gamma_1 + c_1 \cos\gamma_1) = 0.
\end{aligned}
\tag{3.4}
$$

3.2 Equilibrium orientations

By assuming in (3.3) - (3.4) $\gamma_i = \text{const}$ we obtain the stationary equations

$$\begin{aligned}
&4[(C_1 - B_1)/M + b_1^2 - c_1^2]\sin\gamma_1\cos\gamma_1 + b_1c_1(\cos^2\gamma_1 - \sin^2\gamma_1)] - \\
&-(b_1\sin\gamma_1 + c_1\cos\gamma_1)(b_2\cos\gamma_2 - c_2\sin\gamma_2) - \\
&-3(b_1\cos\gamma_1 - c_1\sin\gamma_1)(b_2\sin\gamma_2 + c_2\cos\gamma_2) = 0, \\
&4[(C_2 - B_2)/M + b_2^2 - c_2^2]\sin\gamma_2\cos\gamma_2 + b_2c_2(\cos^2\gamma_2 - \sin^2\gamma_2) + \\
&-(b_2\sin\gamma_2 + c_2\cos\gamma_2)(b_1\cos\gamma_1 - c_1\sin\gamma_1) + \\
&+3\omega_0^2(b_2\cos\gamma_2 - c_2\sin\gamma_2)(b_1\sin\gamma_1 + c_1\cos\gamma_1) = 0.
\end{aligned} \tag{3.5}$$

Which allow us to determine the equilibrium orientation for the system of two bodies connected by a spherical hinge in the orbital coordinate system.

Equations (3.5) form a closed system of two equations with respect to the two aircraft angles.

Trigonometric system (3.5) cannot be solved directly.

3.3 Investigations of Equilibria

For system (3.5) we used the universal change of sines and cosines through the tangent $t_i = \tan(\gamma_i)$

$$\sin\gamma_i = \frac{\tan(\gamma_i)}{\sqrt{1+\tan^2(\gamma_i)}} = \frac{t_i}{\sqrt{1+t_i^2}},\ \cos\gamma_i = \frac{1}{\sqrt{1+\tan^2(\gamma_i)}} = \frac{1}{\sqrt{1+t_i^2}}.$$

and obtain from (5) two equations with two unknowns t_1 and t_2

$$\begin{aligned}
&4\sqrt{(1+t_1^2)(1+t_2^2)}[((C_1-B_1)/M+b_1^2-c_1^2)t_1+b_1c_1(1-t_1^2)] = \\
&=(1+t_1^2)[(b_1t_1+c_1)(b_2-c_2t_2)+3(b_1-c_1t_1)(b_2t_2+c_2)], \\
&4\sqrt{(1+t_1^2)(1+t_2^2)}[((C_2-B_2)/M+b_2^2-c_2^2)t_2+b_2c_2(1-t_2^2)] = \\
&=(1+t_2^2)[(b_2t_2+c_2)(b_1-c_1t_1)+3(b_2-c_2t_2)(b_1t_1+c_1)].
\end{aligned} \tag{3.6}$$

Then, we first divide and second multiply the left-hands and the right-hands sides of these equations and obtain two algebraic equations with two unknowns t_1 and t_2.

3.3 Investigations of Equilibria

$$\begin{aligned} &a_0 t_1^3 + a_1 t_1^2 + a_2 t_1 + a_3 = 0, \\ &b_0 t_1^2 + b_1 t_1 + b_2 = 0. \end{aligned} \tag{3.7}$$

Where

$$\begin{aligned}
a_0 &= (b_1 b_2 c_1^2 + 3b_1^2 c_1 c_2 - 3\, b_2^2 c_1 c_2 - b_1 b_2 c_2^2) t_2^3 + \\
&+[3b_2 c_1 (b_2^2 - 2c_2^2 + d_2) - 3b_1^2 b_2 c_1 + b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2^2 + \\
&+[3b_1^2 c_1 c_2 - b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2) + 3c_1 c_2 (2b_2^2 - c_2^2 + d_2)] t_2 \; + \\
&+3b_2 c_1 c_2^2 - 3b_1^2 b_2 c_1 + \; b_1 c_2 (c_1^2 - b_2^2), \\
a_1 &= \; [3b_1 c_2 (b_2^2 + 2c_1^2 - d_1) - 2b_1^2 b_2 c_1 - 3b_1^3 c_2 + b_2 c_1 (c_1^2 - c_2^2 - d_1)] t_2^3 + \\
&+[3b_1^3 b_2 - 2b_1^2 c_1 c_2 - 3b_1 b_2 (b_1^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) \; + \\
&+c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)] \; t_2^2 - [2b_1^2 b_2 c_1 + 3b_1^3 c_2 + \\
&+b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + 3b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2 + \\
&+3b_1^3 b_2 - 2b_1^2 c_1 c_2 + \; c_1 c_2 (\, c_1^2 - b_2^2 - d_1) - 3b_1 b_2 (2c_1^2 + c_2^2 - d_1),
\end{aligned}$$

3.3 Investigations of Equilibria

$$
\begin{aligned}
&a_0 t_1^3 + a_1 t_1^2 + a_2 t_1 + a_3 = 0, \\
&b_0 t_1^2 + b_1 t_1 + b_2 = 0.
\end{aligned}
\qquad (3.7)
$$

Where

$$
\begin{aligned}
a_0 = {} & (b_1 b_2 c_1^2 + 3b_1^2 c_1 c_2 - 3\, b_2^2 c_1 c_2 - b_1 b_2 c_2^2) t_2^3 + \\
& +[3b_2 c_1 (b_2^2 - 2c_2^2 + d_2) - 3b_1^2 b_2 c_1 + b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2^2 + \\
& +[3b_1^2 c_1 c_2 - b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2) + 3c_1 c_2 (2b_2^2 - c_2^2 + d_2)] t_2 + \\
& +3b_2 c_1 c_2^2 - 3b_1^2 b_2 c_1 + b_1 c_2 (c_1^2 - b_2^2), \\
a_1 = {} & [3b_1 c_2 (b_2^2 + 2c_1^2 - d_1) - 2b_1^2 b_2 c_1 - 3b_1^3 c_2 + b_2 c_1 (c_1^2 - c_2^2 - d_1)] t_2^3 + \\
& +[3b_1^3 b_2 - 2b_1^2 c_1 c_2 - 3b_1 b_2 (b_1^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) + \\
& +c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)]\, t_2^2 - [2b_1^2 b_2 c_1 + 3b_1^3 c_2 + \\
& +b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + 3b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2 + \\
& +3b_1^3 b_2 - 2b_1^2 c_1 c_2 + c_1 c_2 (c_1^2 - b_2^2 - d_1) - 3b_1 b_2 (2c_1^2 + c_2^2 - d_1),
\end{aligned}
$$

3.3 Investigations of Equilibria

$$
\begin{aligned}
a_2 = {} & [b_1^3 b_2 - 3b_1^2 c_1 c_2 + 3c_1 c_2 (c_1^2 - b_2^2 - d_1) + b_1 b_2 (d_1 - 2c_1^2 - c_2^2)] t_2^3 + \\
& + [6b_1^2 b_2 c_1 + b_1^3 c_2 + 3b_2 c_1 (b_2^2 - c_1^2 - 2c_2^2 + d_1 + d_2) + \\
& + b_1 c_2 (2b_2^2 - 2c_1^2 - c_2^2 + d_1 + d_2)] t_2^2 + [b_1^3 b_2 - 6b_1^2 b_2 c_1 - \\
& - b_1 b_2 (b_2^2 + 2c_1^2 - 2c_2^2 - d_1 + d_2) + 3c_1 c_2 (2b_2^2 + c_1^2 - c_2^2 - d_1 + d_2)] t_2 + \qquad (3.8) \\
& + b_1^2 b_2 c_1 + b_1^3 c_2 + b_1 c_2 (d_1 - b_2^2 - 2c_1^2), \\
a_3 = {} & [b_2 c_1 (b_1^2 - c_2^2) + 3b_1 (b_2^2 - c_1^2)] t_2^3 + \\
& + [c_1 c_2 (b_1^2 + 2b_2^2 - c_2^2 + d_2) - 3b_1 b_2 (b_2^2 - c_1^2 - 2c_2^2 + d_2)] t_2^2 + \\
& + [b_2 c_1 (b_1^2 - b_2^2 - c_1^2 + 2c_2^2 - d_2) - 3b_1 c_2 (2b_2^2 + c_1^2 - c_2^2 + d_2)] t_2 + \\
& + c_1 c_2 (b_1^2 - b_2^2) + 3b_1 b_2 (c_1^2 - c_2^2),
\end{aligned}
$$

3.3 Investigations of Equilibria

$$\begin{aligned}
b_0 &= -3(b_2c_1 - b_1c_2)^2 t_2^2 + \\
&+[6b_2c_2(b_1^2 - c_1^2) - 2b_1c_1(3b_2^2 - 3c_2^2 + 8d_2)]t_2 - 3(b_1b_2 - c_1c_2)^2, \\
b_1 &= [6b_1c_1(b_2^2 - c_2^2) - 6b_1^2b_2c_2 + 2b_2c_2(3c_1^2 + 8d_1)]t_2^2 + \\
&+2[5b_2^2(c_1^2 - b_1^2) + 5c_2^2(b_1^2 - c_1^2) + 12b_1b_2c_1c_2 + \\
&+8(b_1^2 - c_1^2 + d_1)(b_2^2 - c_2^2 + d_2)]t_2 + \\
&+6b_2c_2(b_1^2 - c_1^2) + 6b_1c_1(c_2^2 - b_2^2) + 16b_2c_2d_1, \\
b_2 &= -3(b_1b_2 + c_1c_2)^2 t_2^2 + \\
&+[6b_2c_2(c_1^2 - b_1^2) + 2b_1c_1(3b_2^2 - 3c_2^2 + 8d_2)]t_2 - 3(b_2c_1 - b_1c_2)^2.
\end{aligned} \tag{3.8}$$

Here

$$d_1 = \frac{(C_1 - B_1)}{M}, \quad d_2 = \frac{(C_2 - B_2)}{M}.$$

3.3 Investigations of Equilibria

Using the Resultant concept we eliminate the variable t_1 from Eq.(3.7). Expanding the determinant of resultant matrix of Eq.(3.7) with the help of *Mathematica* matrix function, we obtain the 12th order algebraic equation in t_2. After factorization this equation has the form

$$P_1(t_2)P_2(t_2)P_3(t_2)=0. \qquad (3.9)$$

Here

$$P_1(t_2)=b_2c_2t_2^2-(b_2^2-c_2^2+d_2)t_2-b_2c_2,$$

$$P_2(t_2)=[b_1c_1(c_2^2-9b_2^2)d_1+3c_1^2b_2c_2(d_1-c_1^2)-3b_1^2b_2c_2(2c_1^2+d_1)-$$
$$-3b_1^4b_2c_2]t_2^2+[3(b_2^2-c_2^2)(b_1^4+c_1^2(c_1^2-d_1)+b_1^2(2c_1^2+d_1)-20b_1b_2c_1c_2d_1]t_2+$$
$$+3b_1^4b_2c_2+3c_1^2b_2c_2(c_1^2-d_1)+b_1c_1(b_2^2-9c_2^2)d_1+3b_1^2b_2c_2(2c_1^2+d_1),$$

$$P_3(t_2)=p_0t_2^8+p_1t_2^7+p_2t_2^6+p_3t_2^5+p_4t_2^4+p_5t_2^3+p_6t_2^2+p_7t_2+p_8, \qquad (3.10)$$

Here p_i — are rather complicated coefficients, depending on 6 parameters. Using Eqs.(3.9) and (3.7), for each set of the system parameters, we can determine numerically the angles γ_2 and γ_1, that is, all the equilibrium orientations of the satellite–stabilizer system.

3.3 Investigations of Equilibria

In studying the two-body system equilibrium orientations, we determine the domains with an equal number of real roots of Eq.(3.9) in the space of 6 parameters. The decomposition of the space of parameters into domains with an equal number of real roots is determined by the discriminant hypersurface. The form of the discriminant of the polynomial $P_3(t_2)$ is a very cumbersome expression.

Let us consider a simpler case when $b_1 = b_2 = c_1 = c_2 = b$.

In this case from system (3.10) we will obtain the equation of the 6th degree

$$P_1(t_2) = p_{00}t_2^6 + p_{01}t_2^5 + p_{02}t_2^4 + p_{03}t_2^3 + p_{04}t_2^2 + p_{05}t_2 + p_{06} = 0, \qquad (3.11)$$

where

$$d_{01} = (A_1 - C_1)/Mb^2;\ d_{02} = (A_2 - C_2)/Mb^2;$$

$$p_{00} = 12(1 + d_{01})(2d_{01}d_{02} - d_{01} - d_{02});$$

$$p_{01} = d_{01}^2(23 + 64d_{02} - 76d_{02}^2) + d_{02}(96 - 89d_{02}) + 6d_{01}(16 - 19d_{02}),$$

$$p_{02} = 4[d_{01}^2(16d_{02}^3 - 8d_{02}^2 - 58d_{02} + 23) + 4d_{02}(44d_{02}^2 - 32d_{02} - 51) +$$

$$+3d_{01}(12d_{02}^2 - 5d_{02} - 17)],$$

3.3 Investigations of Equilibria

$$p_{03} = -2[d_{01}^2(8d_{02}^4 - 116d_{02}^2 + 105) + d_{02}^2(32d_{02}^2 - 231) + 6d_{01}d_{02}(4d_{02}^2 - 29)],$$

$$p_{04} = -4[d_{01}^2(16d_{02}^3 + 8d_{02}^2 - 58d_{02} - 23) + d_{02}(44d_{02}^2 + 32d_{02} - 51) +$$

$$+3d_{01}(12d_{02}^2 + 5d_{02} - 17)],$$

$$p_{05} = d_{01}^2(23 - 64d_{02} - 76d_{02}^2) - d_{02}(96 + 89d_{02}) - 6d_{01}(16 + 19d_{02}),$$

$$p_{06} = -12(d_{01} - 1)(2d_{01}d_{02} + d_{01} + d_{02}).$$

Discriminant hypersurface of the polynomial (3.11) $P_1(t_2)$ (resultant of the two polynomials $P_1(t_2)$ and $P'_1(t_2)$) has the form

$$P_2(d_{01}, d_{02}) = \tag{3.12}$$

$$= (d_{01}+1)(2d_{01}d_{02} - d_{01} - d_{02})P_3(d_{01}, d_{02})P_4(d_{01}, d_{02})P_5^2(d_{01}, d_{02})P_6(d_{01}, d_{02}) = 0.$$

Here

$$P_3(d_{01}, d_{02}) = (d_{01}^2 d_{02}^2 + 4d_{01}^2 + 4d_{02}^2 - 2d_{01}d_{02} + 9),$$

$$P_4(d_{01}, d_{02}) = 4(d_{01} + d_{02})^2 + d_{01}^2 d_{02}^2.$$

3.3 Investigations of Equilibria

$P_5(d_{01}, d_{02})$ and $P_6(d_{01}, d_{02})$ are the polynomials of the 12th and 16th degree.

Now we should check the change in the number of equilibria when the curve (3.12) is intersected. This can be done numerically by determining the number of equilibria at a single point of each domain at the plane (d_{01}, d_{02}).

Only the curve $(2d_{01}d_{02}-d_{01}-d_{02})P_6(d_{01},d_{02}) = 0$ separates the domains with different number of equilibria.

Fig.3.2 shows the distributions of domains with equal number of real roots of Eq.(3.11) and indicates the domains where 6 and 4 and 2 real solutions exist (12, 8 and 4 equilibrium orientations).

Therefore, in the case when $b_1 = b_2 = c_1 = c_2 = b$, there exist only 12, 8 and 4 equilibrium orientations for the satellite–stabilizer system.

3.3 Investigations of Equilibria

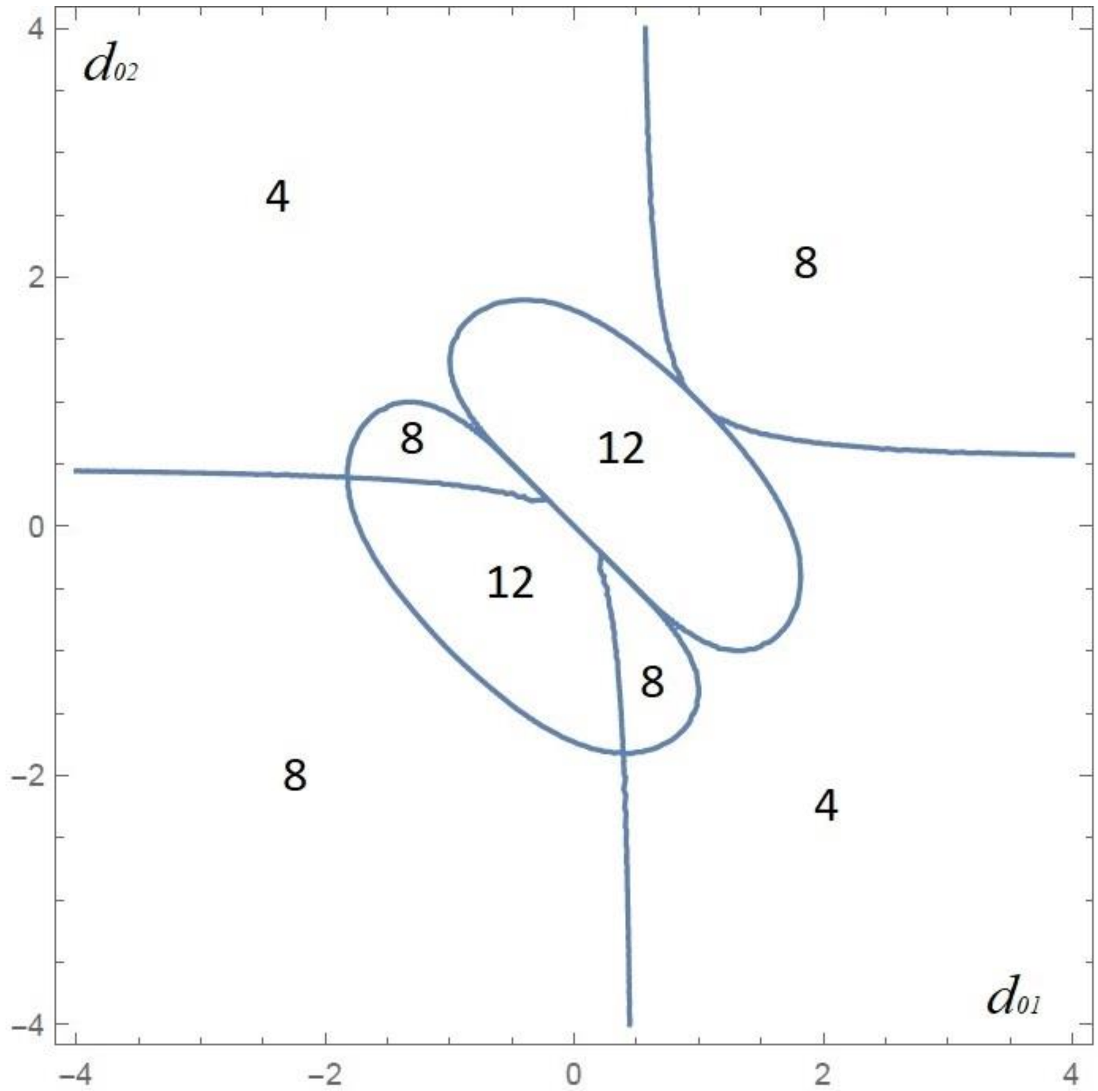

Fig. 3.2. The regions with the fixed number of equilibria

4. Conclusion

- We have obtained all equilibrium orientations of satellite–stabilizer system in the plane of a circular orbit and in the plane perpendicular in the orbital reference frame, using computer algebra method (based on the resultant calculations)
- The conditions for the existence of these equilibria were obtained
- An analysis of the evolution of domains of existence of equilibrium orientations in the plane of system parameters d_{01} and d_{02} for the special case when the coordinates of the spherical hinge in the satellite body coordinate system $Ox_1y_1z_1$ and stabilizer body coordinate system $Ox_2y_2z_2$ are equal have made using discriminant hypersurface approach
- To find equilibrium orientations, the computer algebra methods were used

5. References

- 1. *Sarychev V.A.* Problems of orientation of satellites. *Itogi Nauki I Tekhniki.* Ser. Space Research. VINITI, Moscow 11(1978) (in Russian)
- 2. Gutnik S.A., Sarychev V. A., Application of computer algebra methods to investigate the dynamics of the system of two connected bodies moving along a circular orbit. Programming and Computer Software. 2019. Vol. 45, P. 51-57.
- 3. Gutnik, S.A., Sarychev V.A.: Symbolic investigation of the dynamics of a system of two connected bodies moving along a circular orbit. In: England, M., Koepf, W., Sadykov T.M., Seiler, W.M., Vorozhtsov, E.V. (eds.) CASC 2019. LNCS, vol. 11661, pp. 164 -178. Springer, Cham (2019)
- 4. Gutnik, S.A., Sarychev, V.A.: Symbolic Computations of the Equilibrium Orientations of a System of Two Connected Bodies Moving on a Circular Orbit around the Earth. Math.Comput.Sci. 15 (3) P. 407-417 (2021)
- 5. Meiman, N.N.: Some problems on the distribution of the zeros of polynomials. Uspekhi Mat. Nauk 34, 154–188 (1949). (in Russian)
- 6. Gantmacher, F.R.: The Theory of Matrices. Chelsea Publishing Company, New York (1959).
- 7. Wolfram, S.: The Mathematica Book, 5-th edn. Wolfram media, Inc. Champaign (2003)
- 8. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic Geometry. Springer-Verlag, Berlin Heidelberg New York (2006)