On properties of formal solutions to an algebraic ODE

We consider an algebraic ordinary differential equation of n-th order

$$f(x, y, y', \ldots, y^{(n)}) = 0 \tag{1}$$

(i.e. f is a polynomial of its arguments) and its formal solution of the form

$$y = \sum_{k=0}^{\infty} \varphi_k(x) x^{s_k}, \tag{2}$$

where $s_k \in \mathbb{R}$, $s_{k+1} > s_k$, $\lim\limits_{k\to\infty} s_k = \infty$, $\varphi_k(x)$ are zero order functions or Laurent series in zero order functions with the finite principal part. Furthermore these functions are assumed to have the property that their m-th derivative has the order $-m$ (if these derivatives do not equal zero).

The examples of the series (2) are (generalized) power series ($\varphi_k(x)$ are Laurent polynomials in several $x^{\mathrm{i}\alpha_\ell}$), power-logarithmic series ($\varphi_k(x)$ are Laurent polynomials in several $x^{\mathrm{i}\alpha_\ell}$ and $\log x$). But there exist much more complicated examples which arise in solving the Painlevé equations and which we will talk about on the seminar.

The terms of the formal series (2) are solutions of the approximate equations. These equations are selected from the initial equation by means of the Newton-Bryuno polygon. Moreover starting with some k the coefficients $\varphi_k(x)$ satisfy the (approximate) equations which are linear inhomogeneous ODEs. In some particular cases these equations have explicit solutions. In general case one can find their formal series solutions.

In the talk we will describe the algorithm for calculating $\varphi_k(x)$ calculations and the method of studing their analytical properties. Also there will be given examples of the sixth Painlevé equation formal solutions and there will be formulated statements on asymptotical and analytical properties of coefficients $\varphi_k(x)$ of these solutions.