Example. We consider a (3×3)-system with the coefficient matrix

$$B(z) = \frac{M_1}{z^3} + \frac{M_4}{z} + \frac{M_2}{(z-1)^2} + \frac{M_5}{z-1} + \frac{M_3}{(z+1)^2} + \frac{M_6}{z+1},$$

where the matrices M_i are given by

$$M_1 = \begin{pmatrix} -5 & -4 & -4 \\ 17 & 14 & 13 \\ -10 & -8 & -7 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -6 & -5 & -5 \\ 23 & 17 & 15 \\ -14 & -9 & -7 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 1 & 1 \\ -11 & -7 & -6 \\ 8 & 4 & 3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 2a-c & a-c & a-c \\ -3-6a+5c & -2-3a+5c & -2-3a+5c \\ 3+4a-3c & 2+2a-3c & 2+2a-3c \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 0 & 0 \\ b+2 & -b+1 & -2b+1 \\ -b-2 & b-1 & 2b-1 \end{pmatrix},$$

$$M_6 = \begin{pmatrix} -2a+c & -a+c & -a+c \\ -b+6a+1-5c & b+3a+1-5c & 2b+3a+1-5c \\ b-4a-1+3c & -b-2a-1+3c & -2b-2a-1+3c \end{pmatrix}$$

and a, b and c are parameters. The point $z = \infty$ is non-singular, since $M_4 + M_5 + M_6 = 0$.

Our implementation checks that all the three singularities are non-resonant and gives the formal exponents for each of them.

```
> read "/Users/barkatou/Desktop/Spliting/split":
```

We apply the Splitting lemma at $z = 0$ up to $k = 2$.

```
> splitlemma(B, z=0, 2, T);
```

We get the equivalent matrix

$$\begin{bmatrix} -\frac{1}{z^3} + \frac{a}{z} & 0 & 0 \\ 0 & \frac{1}{z^3} & 0 \\ 0 & 0 & \frac{2}{z^3} + \frac{c}{z} \end{bmatrix} + \text{ regular part.}$$

The formal exponents at $z = 0$ are then $0, a$, and c.

We apply now the Splitting lemma at $z = 1$ up to $k = 1$.

```
> splitlemma(B, z=1, 1, T);
```

We get the equivalent matrix

$$\begin{bmatrix} -\frac{1}{(z-1)^2} & 0 & 0 \\ 0 & \frac{2}{(z-1)^2} + \frac{b}{z-1} & 0 \\ 0 & 0 & \frac{3}{(z-1)^2} \end{bmatrix} + \text{ regular part.}$$

The formal exponents at $z = 1$ are then $0, 0$, and b.

Finally, we apply the Splitting lemma at $z = -1$ up to $k = 1$.

```
> splitlemma(B, z=-1, 1, T);
```

We get the equivalent matrix

$$\begin{bmatrix} -\frac{a}{z+1} & 0 & 0 \\ 0 & -\frac{2}{(z+1)^2} - \frac{c}{z+1} & 0 \\ 0 & 0 & -\frac{1}{(z+1)^2} - \frac{b}{z+1} \end{bmatrix} + \text{ regular part.}$$

The formal exponents at $z = -1$ are then $-a, -c$, and $-b$.

Now let us check in which case the matrices M_i, $i = 1, \ldots, 6$, are simultaneously triangularizable. For this, we can first use the *LieAlgebras* package of Maple.

1. First one computes the Lie algebra L generated by the set $S = \{M_1, \ldots, M_6\}$.

2. Then one can use the *Query* function with the argument "*Solvable*" to check the solvability of L.

```
> with(DifferentialGeometry):
> with(LieAlgebras):
> S:=[seq(M_i, i = 1 .. 6)]:
> L := MatrixLieAlgebra(S);
```

$$\left[\begin{bmatrix} -5 & -4 & -4 \\ 17 & 14 & 13 \\ -10 & -8 & -7 \end{bmatrix}, \begin{bmatrix} -6 & -5 & -5 \\ 23 & 17 & 15 \\ -14 & -9 & -7 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ -11 & -7 & -6 \\ 8 & 4 & 3 \end{bmatrix}, \begin{bmatrix} -1 & -1 & -1 \\ 14 & 10 & 10 \\ -13 & -9 & -9 \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 2a-c & a-c & a-c \\ -3-6a+5c & -2-3a+5c & -2-3a+5c \\ 3+4a-3c & 2+2a-3c & 2+2a-3c \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b+2 & -b+1 & -2b+1 \\ -b-2 & b-1 & 2b-1 \end{bmatrix}\right]$$

```
> LD1:=LieAlgebraData(L,Alg1):
> DGsetup(LD1):
```

```
We check the solvability of  L
> Query("Solvable");
                                        true
```

This tells us that the Lie algebra L is sovable (apparently for all a, b, c) but it does not give a transformation P that simultaneously triangularizes the matrices M_i. Such a transformation can be directly obtained using a recent implementation by M. Barkatou and T. Cluzeau (the work in progress).

```
> read "/Users/barkatou/Desktop/Simul-Triang/
         utilitaires_reduction.mpl":
> read "/Users/barkatou/Desktop/Simul-Triang/SimultTriang.mpl":
> P := SimultaneousTriangularization([seq(M[i], i = 1 .. 6)]);
```

$$P := \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

One can check that indeed the matrices $P^{-1}M_iP$ are triangular:

```
> seq(1/P . M[i] . P, i = 1 .. 6);
```

$$\begin{bmatrix} 1 & -2 & 10 \\ 0 & -1 & 7 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -5 & 14 \\ 0 & -1 & 9 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -1 & 4 & -8 \\ 0 & 0 & -3 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1+2a & -3-4a+3c \\ 0 & a & -2a+2c \\ 0 & 0 & c \end{bmatrix}, \begin{bmatrix} b & -2b-1 & b+2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -b & 2b-2a & -b+4a+1-3c \\ 0 & -a & 2a-2c \\ 0 & 0 & -c \end{bmatrix}$$

In conclusion, the system in this example is solvable in the Liouvillian sense without any restriction on the parameters a, b, c.

On the other hand, the matrices M_i, $i = 1, \ldots, 6$, are never simultaneously diagonalizable since, for example, M_1, M_2 do not commute:

$$[M_1, M_2] = \begin{pmatrix} -1 & -1 & -1 \\ 14 & 10 & 10 \\ -13 & -9 & -9 \end{pmatrix}.$$

Hence, due to Theorem 2 we can assert that the system is not solvable by exponentials of integrals and algebraic functions whenever $|\mathrm{Re}\,\lambda| < 1/6$, for $\lambda \in \{a, b, c\}$.