A Maillet type theorem for generalized power series

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Let us consider an ordinary differential equation (ODE)

$$F(z,u,\delta u,\ldots,\delta^m u)=0 \tag{1}$$

of order m with respect to the unknown u, where $F(z,u_0,u_1,\ldots,u_m)\not\equiv 0$ is a polynomial of $m+2$ variables, $\delta = z\frac{d}{dz}$.

The classical Maillet theorem [1] asserts that any formal power series solution $\varphi=\sum_{n=0}^{\infty}c_n z^n\in\mathbb{C}[[z]]$ of (1) is a power series of Gevrey order $1/k$ for some $k\in\mathbb{R}_{>0}\cup\{\infty\}$. This means that the power series

$$f=\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(1+n/k)}\,z^n$$

converges in a neighbourhood of zero, where Γ is the Euler gamma-function. In other words,

$$|c_n|\leqslant A\,B^n\,(n!)^{1/k}$$

for some $A,B>0$.

First exact estimates for the Gevrey order of a power series formally satisfying an ODE were obtained by J.-P. Ramis [2] in the linear case,

$$Lu=a_m(z)\delta^m u+a_{m-1}(z)\delta^{m-1}u+\ldots+a_0(z)u=0,\qquad a_i\in\mathbb{C}\{z\}.$$

He has proved that such a power series is of the *exact* Gevrey order $1/k\in\{0,1/k_1,\ldots,1/k_s\}$, where $k_1<\ldots<k_s<\infty$ are all of the positive slopes of the Newton polygon $\mathcal{N}(L)$ of the operator L. This polygon is defined as the boundary curve of a convex hull of a union $\bigcup_{i=0}^{m}X_i$,

$$X_i=\{(x,y)\in\mathbb{R}^2\mid x\leqslant i,\ y\geqslant \mathrm{ord}_0 a_i(z)\},\qquad i=0,1,\ldots,m$$

(see Fig. 1). The exactness of the Gevrey order means that there is no $k'>k$ such that the power series is of the Gevrey order $1/k'$.

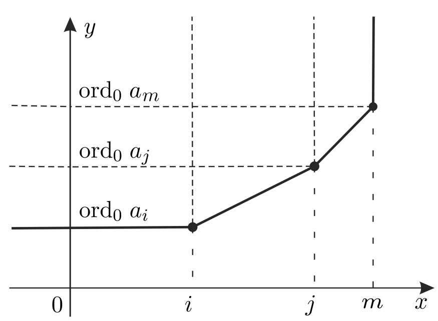

Figure 1: The Newton polygon $\mathcal{N}(L)$ with two positive slopes $k_1=(\mathrm{ord}_0\,a_j-\mathrm{ord}_0\,a_i)/(j-i)$, $k_2=(\mathrm{ord}_0\,a_m-\mathrm{ord}_0\,a_j)/(m-j)$.

The result of Ramis has been further generalized by B. Malgrange and Y. Sibuya for a non-linear ODE of the general form (1).

Theorem 1 (Malgrange [3]). *Let $\varphi\in\mathbb{C}[[z]]$ satisfy the equation (1), that is $F(z,\Phi)=0$, where $\Phi=(\varphi,\delta\varphi,\ldots,\delta^m\varphi)$, and $\frac{\partial F}{\partial u_m}(z,\Phi)\neq 0$. Then φ is a power series of Gevrey order $1/k$, where k is the least of all the positive slopes of the Newton polygon $\mathcal{N}(L_\varphi)$ of a linear operator*

$$L_\varphi=\sum_{i=0}^{m}\frac{\partial F}{\partial u_i}(z,\Phi)\,\delta^i$$

(*or* $k=+\infty$*, if* $\mathcal{N}(L_\varphi)$ *has no positive slopes*).

The refinement of Theorem 1 belongs to Sibuya [4, App. 2]. This claims that φ is a power series of the *exact* Gevrey order $1/k\in\{0,1/k_1,\ldots,1/k_s\}$, where $k_1<\ldots<k_s<\infty$ are all of the positive slopes of the Newton polygon $\mathcal{N}(L_\varphi)$.

In the talk we study *generalized* power series solutions of (1) of the form

$$\varphi=\sum_{n=0}^{\infty}c_n z^{s_n}, \qquad c_n\in\mathbb{C}, \qquad s_n\in\mathbb{C}, \tag{2}$$

with the power exponents satisfying conditions

$$0\leqslant \operatorname{Re}s_0\leqslant \operatorname{Re}s_1\leqslant\ldots, \qquad \lim_{n\to\infty}\operatorname{Re}s_n=+\infty$$

(the latter, in particular, implies that a set of exponents having a fixed real part is finite).

For the generalized power series (2) one may naturally define the *valuation*

$$\operatorname{val}\varphi=s_0,$$

and this is also well defined for any polynomial in $z,\varphi,\delta\varphi,\ldots,\delta^m\varphi$.

The main result of the present talk is an analog of the Maillet theorem (more precisely, of the Malgrange theorem) for generalized power series.

Theorem 2 ([5]). *Let the generalized power series* (2) *formally satisfy the equation* (1), $\frac{\partial F}{\partial u_m}(z,\Phi)\neq 0$, *and for each* $i=0,1,\ldots,m$ *one have*

$$\frac{\partial F}{\partial u_i}(z,\Phi)=A_i z^{\lambda}+B_i z^{\lambda_i}+\ldots, \qquad \operatorname{Re}\lambda_i>\operatorname{Re}\lambda, \tag{3}$$

where not all A_i *equal zero. Let* k *be the least of all the positive slopes of the Newton polygon* $\mathcal{N}(L_\varphi)$ (*or* $k=+\infty$*, if* $\mathcal{N}(L_\varphi)$ *has no positive slopes*). *Then for any sector* S *of sufficiently small radius with the vertex at the origin and of the opening less than* 2π*, the series*

$$\sum_{n=0}^{\infty}\frac{c_n}{\Gamma(1+s_n/k)}\,z^{s_n}$$

converges uniformly in S.

Remark 1. The Newton polygon of L_φ in the case of the generalized power series φ is defined similarly to the classical case: this is the boundary curve of a convex hull of a union

$$\bigcup_{i=0}^{m}\Bigl\{(x,y)\in\mathbb{R}^2 \mid x\leqslant i,\ y\geqslant \operatorname{Re}\operatorname{val}\frac{\partial F}{\partial u_i}(z,\Phi)\Bigr\}.$$

Note that $k=+\infty$ in Theorem 2 if and only if $A_m\neq 0$. In this case φ does converge in S, which has been already proved in [6] by the majorant method.

Remark 2. From Theorem 2 one deduces the following estimate for the coefficients c_n of the formal series solution (2) of (1):

$$|c_n|\leqslant A\,B^{\operatorname{Re}s_n}|\Gamma(1+s_n/k)|,$$

for some $A,B>0$.

Example 1. We consider an equation

$$z^3u\,u''-zu'+\sqrt{2}u+z^2=0,$$

or, written with use of the operator δ,

$$zu\,\delta^2u-(zu+1)\delta u+\sqrt{2}u+z^2=0.$$

It has a generalized power series solution of the form $\varphi=\sum_{s\in\mathrm{K}}c_s z^s$, where

$$\mathrm{K}=\{\sqrt{2}+\mathrm{m}_1(1+\sqrt{2})+\mathrm{m}_2(2-\sqrt{2})\mid \mathrm{m}_1,\mathrm{m}_2\in\mathbb{Z}_+\}.$$

The first terms of φ are

$$\varphi=z^{\sqrt{2}}+\Bigl(1+\frac{\sqrt{2}}{2}\Bigr)z^2+(3\sqrt{2}-4)z^{2\sqrt{2}+1}+\dots .$$

Since $F(z,u_0,u_1,u_2)=zu_0u_2-(zu_0+1)u_1+\sqrt{2}u_0+z^2$, we have

$$\begin{aligned}
\frac{\partial F}{\partial u_0}(z,\Phi) &= z\delta^2\varphi-z\delta\varphi+\sqrt{2}=\sqrt{2}+\dots,\\
\frac{\partial F}{\partial u_1}(z,\Phi) &= -z\varphi-1=-1+\dots,\\
\frac{\partial F}{\partial u_2}(z,\Phi) &= z\varphi=z^{\sqrt{2}+1}+\dots .
\end{aligned}$$

Therefore, the Newton polygon $\mathcal{N}(L_\varphi)$ is determined by the points $(0,0)$, $(1,0)$, and $(2,1+\sqrt{2})$. Hence, $k=1+\sqrt{2}$ is a unique positive slope of $\mathcal{N}(L_\varphi)$ and, by Theorem 2,

$$|c_s|\leqslant A\,B^s\,\Gamma\Bigl(1+\frac{s}{1+\sqrt{2}}\Bigr),$$

for some $A,B>0$.

References

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