On formal power series satisfying an ODE in a complex domain

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We consider an ordinary differential equation

$$F(z,u,\delta u,\dots,\delta^n u)=0 \tag{1}$$

of order n, where $F(z,y_0,y_1,\dots,y_n)\not\equiv 0$ is a holomorphic (in some domain) function of $n+2$ variables and $\delta = z\frac{d}{dz}$.

According to Maillet's theorem [6], if a formal power series $\hat\varphi=\sum_{k=0}^{\infty}c_kz^k\in\mathbb{C}[[z]]$ satisfies the equation (1), where F is a polynomial, then there is a real number $s\geqslant 0$ such that the power series $\sum_{k=0}^{\infty}(c_k/(k!)^s)z^k$ converges in some neighbourhood of zero. In other words,

$$|c_k|\leqslant A\,B^k\,(k!)^s \tag{2}$$

for some $A,B>0$. In this case one says that the formal power series $\hat\varphi$ has the *Gevrey type of order* s. Furthermore if there is no a real number $s'<s$ such that the series has the Gevrey type of order s', then the order s is called *precise*.

First precise estimates for the Gevrey order s were obtained by J.-P. Ramis [8] in the case of a linear equation

$$Lu=a_n(z)\delta^n u+a_{n-1}(z)\delta^{n-1}u+\ldots+a_0(z)u=0, \qquad a_i\in\mathbb{C}\{z\},$$

in terms of the Newton polygon $\mathcal{N}(L)$ of the linear operator L. He has shown that any formal power series satisfying the equation $Lu=0$ has the Gevrey type of precise order $s\in\{0,1/r_1,\dots,1/r_m\}$, where $0<r_1<\ldots<r_m<\infty$ are all of the positive slopes of the $\mathcal{N}(L)$ edges. In a general case of the equation (1) corresponding results were obtained by B. Malgrange [7] and Y. Sibuya [10, App. 2] (see also [3]) with the use of $\mathcal{N}(L_{\hat\varphi})$, where

$$L_{\hat\varphi}=\sum_{i=0}^{n}\frac{\partial F}{\partial y_i}(z,\hat\varphi,\dots)\,\delta^i$$

is a linearization of F along the formal solution $\hat\varphi$. In particular, if

$$\mathrm{ord}_0\,\frac{\partial F}{\partial y_i}(z,\hat\varphi,\dots)\geqslant \mathrm{ord}_0\,\frac{\partial F}{\partial y_n}(z,\hat\varphi,\dots), \qquad i=0,1,\dots,n-1, \tag{3}$$

then $s=0$, that is, the series $\hat\varphi$ converges in some neighbourhood of zero. The last assertion concerning the convergence has an alternative analytic proof proposed by A. D. Bruno and I. V. Goryuchkina [2] (see also [4]), which allows to estimate the radius of convergence.

We recall that a function f holomorphic in an open sector $V\subset\mathbb{C}$ with the vertex at the origin has an asymptotic expansion $\hat f=\sum_{i=0}^{\infty}a_kz^k\in\mathbb{C}[[z]]$ on V (in the classical sense of Poincaré), if for any proper subsector $W\subset V\cup\{0\}$ and $N\in\mathbb{N}$ there is a number $M=M(W,N)>0$ such that

$$\left|f(z)-\sum_{k=0}^{N-1}a_kz^k\right|<M|z|^N \qquad \forall z\in W.$$

If $M(W,N)=C(N!)^sA^N$ for some $C,A>0$ depending only on W, then the above asymptotic expansion is called the *asymptotic expansion in Gevrey sense of order* s.

The importance of Maillet's theorem is explained by the following fundamental result of Ramis–Sibuya [9] on asymptotic expansions in Gevrey sense: if a formal power series solution $\hat\varphi$ of the equation (1) has the Gevrey type of order s, then there exists a number $r>0$ such that for any open sector $V\subset\mathbb{C}$ with the vertex at the origin, of the opening $<\pi/r$ and sufficiently small radius there is an actual solution φ that has the asymptotic expansion $\hat\varphi$ on V in Gevrey sense of order s.

Now we pass to the *generalized* power series of the form

$$\hat\varphi=\sum_{k=0}^{\infty}c_kz^{s_k}, \qquad c_k\in\mathbb{C}, \quad s_0\prec s_1\prec\ldots\in\mathbb{C}, \quad \lim_{k\to\infty}\mathrm{Re}\,s_k=+\infty, \tag{4}$$

were $\prec$ is a usual ordering by first difference: $s_k \prec s_{k+1}$ iff $\operatorname{Re} s_k < \operatorname{Re} s_{k+1}$ or $\operatorname{Re} s_k = \operatorname{Re} s_{k+1}$, $\operatorname{Im} s_k < \operatorname{Im} s_{k+1}$.

Note that substituting the series (4) into the equation (1) makes sense, as only a finite number of terms in $\hat\varphi$ contribute to any term of the form cz^s in the expansion of $F(z, \hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi)$ in powers of z. Indeed, $\delta^j\hat\varphi = \sum_{k=0}^{\infty} c_k s_k^j z^{s_k}$ and an equation $s = s_{k_0} + s_{k_1} + \ldots + s_{k_m}$ has a finite number of solutions $(s_{k_0}, s_{k_1}, \ldots, s_{k_m})$, since $\operatorname{Re} s_k \to +\infty$. Furthermore, for any integer N an inequality $s_{k_0}+s_{k_1}+\ldots+s_{k_m} \preceq N$ has also a finite number of solutions, so powers of z in the expansion of $F(z, \hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi)$ are well ordered with respect to $\prec$.

Earlier in the paper [5] generalized power series of the form (4), with $s_0 < s_1 < \ldots \in \mathbb{R}$, were studied. There was proved (without the assumption $s_k \to +\infty$) that they form a differential ring, and if the series (4) satisfies the equation (1), then $\lim_{k\to\infty} s_k = +\infty$. Furthermore, the exponents $s_k \in \mathbb{R}$ of the formal solution (4) generate a finite $\mathbb{Z}$-module. One can prove the last fact also in the case of the complex exponents $s_k \in \mathbb{C}$.

For the generalized power series (4) one may naturally define the *order*

$$\operatorname{ord}\hat\varphi = s_0,$$

and this is also well defined for any polynomial in $\hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi$ with coefficients of the form $\alpha\, z^{\beta}$, $\alpha,\ \beta \in \mathbb{C}$. As a generalization of results of Malgrange and Sibuya concerning the convergence to the case of series (4), one has the following assertion.

Let the generalized power series (4) *formally satisfy the equation* (1), $\dfrac{\partial F}{\partial y_n}(z, \hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi) \neq 0$ *and*

$$\operatorname{ord}\frac{\partial F}{\partial y_i}(z, \hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi) \succeq \operatorname{ord}\frac{\partial F}{\partial y_n}(z, \hat\varphi, \delta\hat\varphi, \ldots, \delta^n\hat\varphi), \qquad i = 0, 1, \ldots, n-1.$$

Then for any sector S of sufficiently small radius with the vertex at the origin and of the opening less than 2π, the series $\hat\varphi$ converges uniformly in S.

This theorem in a somewhat different form has been formulated by A. D. Bruno [1, Th. 3.4] for the case of the *real* exponents $s_k \in \mathbb{R}$, and is proved in the present form by I. V. Goryuchkina. The proof is based on majorant methods and allows to estimate also the radius of the convergence sector S.

In the conclusion we would like to propose some questions for further investigations.

a) As we have estimates for the radius of convergence of the series $\hat\varphi = \sum_{k=0}^{\infty} c_k z^k$ when the conditions (3) hold, further one may try to estimate the number B from (2) for divergent series $\hat\varphi$ satisfying (1).

b) To obtain estimates of the form (2) for the growth of the coefficients of divergent generalized power series (4) satisfying (1) (a Maillet type theorem).

c) To propose a kind of theory of asymptotic expansions connected with generalized power series solutions of (1) (of a Ramis–Sibuya type).

References

[1] A. D. Bruno, *Asymptotic behaviour and expansions of solutions of an ordinary differential equation*, Russian Math. Surv., V. 59(3) (2004), 429–480.

[2] A. D. Bruno, I. V. Goryuchkina, *Asymptotic expansions of the solutions of the sixth Painlevé equation*, Trans. Moscow Math. Soc. (2010), 1–104.

[3] J. Cano, *On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations*, Analysis, V. 13 (1993), 103–119.

[4] R. R. Gontsov, I. V. Goryuchkina, *An analytic proof of the Malgrange theorem on the convergence of formal solutions of an ODE*, J. Dynam. Control Syst. (2014).

[5] D. Yu. Grigor'ev, M. F. Singer, *Solving ordinary differential equations in terms of series with real exponents*, Trans. Amer. Math. Soc., V. 327(1) (1991), 329–351.

[6] E. Maillet, *Sur les séries divergentes et les équations différentielles*, Ann. Sci. Ecole Norm. Sup., V. 3 (1903), 487–518.

[7] B. Malgrange, *Sur le théorème de Maillet*, Asympt. Anal., V. 2 (1989), 1–4.

[8] J.-P. Ramis, *Dévissage Gevrey*, Astérisque, V. 59/60 (1978), 173–204.

[9] J.-P. Ramis, Y. Sibuya, *Hukuhara's domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type*, Asympt. Anal., V. 2 (1989), 39–94.

[10] Y. Sibuya, *Linear Differential Equations in the Complex Domain: Problems of Analytic Continuation*, Transl. Math. Monographs, V. 82, A.M.S., 1990.