COMPUTER INTERVAL ALGEBRA OF MINIMIZATION PROBLEMS

Н. М. Глазунов
Институт кибернетики им. В. М. Глушкова НАНУ, Киев
Glushkov Institute of Cybernetics NAS Ukraine, Kiev

glanm@yahoo.com

Науки следует классифицировать проблемами, которыми они задаются, нежели методами, которыми они располагают.

Н.Г. Чеботарев

Sciences should be classified by the problems with which they pose, rather than by the methods at their disposal.

N.G. Chebotarev

Благодарности

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A brief overview of algebraic and number theoretic foundations of computer interval arithmetics, methods of their implementation and their applications to problems of (nondifferentiable) optimization is given. The implementation of interval arithmetic on various classes of computers is presented. Interval extensions of computational methods are briefly discussed. Selected methods of interval analysis are presented and discussed. We consider their applications to the solution of a problem (related to a problem by H. Minkowski) of parametrization of the set of minima of a non-compact real convex surface with a boundary when it is embedded in a three-dimensional real space. A brif presentation of categorical-algebraic foundations of interval calculations is given.

Plan

0. Introduction

In computer algebra systems (CAS), processed constants, variables, coefficients are polynomials, matrix elements and other objects allowed by the CAS language are usually integers or rational numbers (arbitrary or unlimited bit width), or algebraic numbers. Real numbers in most cases should be reduced to this form. That is, we are talking about the approximation of real numbers by rational ones. Therefore, our first problem of minimizing will be associated with such an approximation, which is called Diophantine approximation.

0.1. Elements of Diophantine Approximations and Minimization Problems [1, 2, 3, 4]

According to Khinchin (1935) [2] the rational number p/q *is a best Diophantine approximation of the real number* θ *if*

$$\left|\theta - \frac{p}{q}\right| < \left|\theta - \frac{p_1}{q_1}\right|, \tag{1}$$

for every rational number p_1/q_1 *different from* p/q *s.t.* $0 < q_1 \leq q$.

By other words we have to

$$minimize\left|\theta - \frac{x}{y}\right|$$
$$subject\ to\ x, y \in \mathbb{Z}, 0 < y \leq q.$$

Nowadays some other definition is currently used.

Recall some notations:

$\lfloor\theta\rfloor, \{\theta\}, \|\theta\|$ ***denote respectively integer, fractional parts of the real number*** θ ***and the minimum distance from*** θ ***to the nearest integer. So***

$$\|q\theta\| = \min(\{q\theta\}, 1 - \{q\theta\})$$

.

Theorem 1 ***(L. Dirichlet) Let*** θ ***and*** Q ***be real numbers. Then there exists integer*** q ***such that***

$$0 < q < Q, \|q\theta\| \leq Q^{-1}.$$

According to Cassels and others

Definition 1 ([3, 4]). A fraction

$$\frac{p}{q}, \qquad q > 0$$

is called the best (optimal) approximation of the real number θ ***if***

$$\|q\theta\| = |q\theta - p|$$

i.e.

$$\min(\{q\theta\}, 1 - \{q\theta\}) = |q\theta - p|$$

and if

$$\|q'\theta\| > \|q\theta\| \qquad \textbf{\textit{for}}\ 0 < q' < q.$$

The formula (1): $\left|\theta - \frac{p}{q}\right| < \left|\theta - \frac{p_1}{q_1}\right|$ ***follows from this definition .***

Theorem 2 ([1, 2, 3, 4]) . Let $0 < \theta < 1$ ***and let*** p_n, q_n, a_n ***be defiined as***

$$p_0 = 1, q_0 = 0,$$
$$p_1 = 0, q_1 = 1,$$

$$for\ n \geq 1$$
$$p_{n+1} = a_n p_n + p_{n-1},$$
$$q_{n+1} = a_n q_n + q_{n-1}$$

where $a_n = \left\lfloor \frac{|q_{n-1}\theta - p_{n-1}|}{|q_n\theta - p_n} \right\rfloor$

Theorem 2 gives the method for optimal approximation of a real number.

0.2. Equidistribution

This is important approach to approximation of real numbers but I will present general review only

The concept of equidistribution was introduced by H. Weyl at the beginning of the last century in connection with the problem of small denominators in problems of celestial mechanics.

Example 1 . ***The equidistribution of the sequence*** $(\{\sqrt{2}n\}),\ \ n \in \mathbb{Z}$ ***on the interval*** $[0,1]$. ***This is the equidistribution with density 1.***

0.2.1. Methods for constructing sequences equidistributed with non-unit density

Let X ***be a compact topological space and*** $C(X)$ ***be the Banach space of continuous complex-valued functions on*** X. ***For*** $f \in X$ ***let its norm*** $||f|| = sup_{x\in X}|f(x)|$. ***Let*** δ_x ***be the Dirac measure associated to*** $x \in X : \delta_x(f) = f(x)$. ***For a sequence*** $(x_n), n \geq 1$ ***let*** $\mu_n = \frac{\delta_{x_1}+\cdots+\delta_{x_n}}{n}$. ***Let*** μ ***be a Radon measure on*** X.

Definition 2 . ***The sequence*** (x_n) ***is said to be equidistributed with respect to measure*** μ ***(or*** μ***-equidistributed) if*** $\mu_n \to \mu$ ***(weakly) as*** $n \to \infty$.

Let G ***be a compact group and let*** X ***be the space of conjugacy classes of G Let*** $x_v,\ v \in \Sigma$ ***be a family of elements of*** X ***indexed by a countable (denumerable) set*** Σ.

Theorem 3 (Chebotarev, Artin, Serre). The elements $x_v,\ v \in \Sigma$ ***are equidistributed for the normalized Haar measure of*** G ***if and only if the*** $L-$***functions relative to the non trivial irreducible characters of*** G ***are holomorphic and non zero at*** $s = 1$.

Sato-Tate conjecture.
Geometric analogous of Sato-Tate conjecture (Sato-Tate conjecture over function fields) are investigated and proved by B. Birch [5] and by H. Yoshida [6].

Let

$$E : y^2 = x^3 + ax + b,\ a, b \in \mathbb{Z}$$

be the elliptic curve without complex multiplication.

$$\#E_p(\mathbb{F}_p) = 1 + p - a_p$$

$$a_p = 2\sqrt{p}\cos\varphi_p$$

Sato-Tate conjecture: Angles φ_p ***are distributed on the interval*** $[0, \pi)$ ***with the Sato-Tate density*** $\frac{2}{\pi}\sin^2 t$ ***.***
Sato-Tate conjecture, now Clozel–Harris–Shepherd-Barron–Taylor Theorem

Theorem 4 (Clozel, Harris, Shepherd-Barron, Taylor). Suppose E ***is an elliptic curve over Q with non-integral*** j ***invariant. Then for all*** $n > 0; L(s; E; Sym^n)$ ***extends to a meromorphic function which is holomorphic and non-vanishing for*** $Re(s) \geq 1 + n/2.$

These conditions and statements are sufficient to prove the Sato-Tate conjecture.
Under the prove of the Sato-Tate conjecture the Taniyama-Shimura-Weil conjecture (Fermat last theorem) oriented methods of A. Wiles and R. Taylor are used.

Kloosterman sums

Let

$$T_p(c,d) = \sum_{x=1}^{p-1} e^{2\pi i (\frac{cx + \frac{d}{x}}{p})}$$

$$1 \le c, d \le p-1; \; x, c, d \in \mathbf{F}_p^*$$

be a Kloosterman sum.
By A. Weil estimate

$$T_p(c,d) = 2\sqrt{p} \cos \theta_p(c,d)$$

There are possible two distributions of angles $\theta_p(c,d)$ ***on semiinterval*** $[0,\pi)$:

a) *p is fixed and c and d varies over* $\mathbf{F}_p^*$; *what is the distribution of angles* $\theta_p(c,d)$ *as* $p \to \infty$;

b) *c and d are fixed and p varies over all primes not dividing c and d.*

In seventies the author of the communication (by the request of S.A. Stepanov and J.-P. Serre) have computed ($\approx$ 1980***) the distribution of angles*** $\theta_p(c,d)$ ***(mainly for the case*** $c = d = 1$, ***prime*** p ***runs from 2 in interval*** $[2, 10000]$, ***and unsystematically for some prime from the interval with constant*** p ***and varying*** $1 \le c, d \le p-1$).

There are possible two distributions of angles $\theta_p(c,d)$ ***on semiinterval*** $[0,\pi)$:
a) *p is fixed and c and d varies over* $\mathbf{F}_p^*$; *what is the distribution of angles* $\theta_p(c,d)$ *as* $p \to \infty$;

b) *c and d are fixed and p varies over all primes not dividing c and d.*

Conjecture 1 . In the case b) when** p **varies over all primes then angles** $\theta_p(1,1)$ **are distributed on the interval $[0, \pi)$ ***with the Sato-Tate density*** $\frac{2}{\pi}\sin^2 t$ ***.***

For the case $a)$ ***N. Katz [7] (1988) and A. Adolphson [8] (1989) proved that*** θ ***are distributed on*** $[0, \pi)$ ***with density*** $\frac{2}{\pi}\sin^2 t$.
It is interesting to compare results of computer experiments in cases $a)$ ***and*** $b)$. ***Such computations [9] and [10] demonstrated that though in case*** $b)$ ***equidistribution is possible but results of computation shows not so good compatibility with equidistribution as in (proved) case*** $a)$.

0.3. Elements of Interval Algebra (of Validated numerics)

The compact closed interval $I = [a, b]$ is the set of all real numbers x such that $a \leq x \leq b$.

Let $\mathbb{I}R$ be the set of all intervals over $\mathbb{R}$. Arithmetic interval operations $+, -, +, /$ define interval algebra on the set of intervals.

Let x be a variable over $\mathbb{R}$, $\mathrm{x} = ([\underline{x}, \overline{x}]$ be a compact closed interval from $\mathbb{I}R$, let $\underline{x}$ and $\overline{x}$ be the lower and upper values of x on x.

Let's say that $\mathrm{x} > 0$, ($\mathrm{x} \geq 0$) if $\underline{x} > 0$, ($\underline{x} \geq 0$) and $\mathrm{x} < 0$, ($\mathrm{x} \leq 0$) if $\overline{x} < 0$, ($\overline{x} \leq 0$).

We call such intervals constant sign. Non-constant intervals are called alternating

When calculating the lower and upper estimates of the values of functions on a computer, it is convenient to express the arithmetic of intervals in terms of the boundary points of the intervals.

For the sum, assuming $\mathrm{x}_i = ([\underline{x}_i, \overline{x}_i]$, we have

$$\sum_i \mathrm{x}_i = \left[\sum_i \underline{x}_1, \sum_i \overline{x}_1\right]$$

Difference $\mathrm{x} - \mathrm{y}$, is reduced to the sum of $\mathrm{x} + \mathrm{z}$ when $\mathbf{z} = -\mathbf{y} = [-\overline{y}, -\underline{y}]$.

For rational interval arithmetics and some other types of interval arithmetics see ([14]: Николай Глазунов. **Разработка методов обоснования гипотез формальных теорий.***)*

Let

$$\mathbf{X} = ([\underline{x}_1, \overline{x}_1], \cdots, [\underline{x}_n, \overline{x}_n]$$

be the n-dimensional real interval vector with

$$\underline{x}_i \leq x_i \leq \overline{x}_i$$

("rectangle" or "box").

Let f ***be a real continuous function of*** n ***variables defined on*** $\mathbf{X}$***. The*** *interval evaluation* ***of*** f ***on the interval*** $\mathbf{X}$ ***is the interval*** $[\underline{f}, \overline{f}]$ ***such that for any*** $x \in \mathbf{X}$, $f(x) \in [\underline{f}, \overline{f}]$. ***The interval evaluation is called*** *optimal* ***[11] if*** $\underline{f} = \min f$, ***and*** $\overline{f} = \max f$ ***on the interval*** $\mathbf{X}$***. Let*** Of ***be the optimal interval evaluation of*** f ***on*** $\mathbf{X}$***.***

Definition 3 . The pair $(\mathbf{X}, Of)$ ***is called the*** *interval functional element.* ***If*** Ef ***is an interval that contains*** Of ***then we will call the pair*** $(\mathbf{X}, Ef)$ ***an*** *extension* ***of*** $(\mathbf{X}, Of)$ ***or*** *eif−element.* ***Let*** f ***be a constant sign function on*** $\mathbf{X}$. ***If*** $f > 0$ ***(respectively*** $f < 0$***) on*** $\mathbf{X}$ ***and*** $Of > 0$ ***(respectively*** $Of < 0$***) then we will call*** $(\mathbf{X}, Of)$ *the correct interval functional element* ***(or*** *c−element****).***

More generally we will call *the correct interval functional element* ***an extension*** $(\mathbf{X}, Ef)$ ***of*** $(\mathbf{X}, Of)$ ***that has the same sign as*** Of.

1. Problems and methods of (nondifferentiable) minimization

A general discussion in subsections 1.2-1.4 to approaches to the nondifferentiable optimization as well as specific technical results at the field on can refer to the l book by N. Shor [Nondifferentiable Optimization and Polynomial Problems, Kluwer Acad. Publ. 1998], papers [17, 18, 19, 20] and in references therein.

1.1. Minkowski's conjecture concerning the critical determinant

Let

$$|\alpha x + \beta y|^p + |\gamma x + \delta y|^p \leq c|\det(\alpha\delta - \beta\gamma)|^{p/2},$$

be a diophantine inequality defined for a given real $p > 1$***; hear*** $\alpha, \beta, \gamma, \delta$ ***are real numbers with*** $\alpha\delta - \beta\gamma \neq 0$.

H. Minkowski in his monograph [15] raise the question about minimum constant c ***such that the inequality has integer solution other than origin. Minkowski with the help of his theorem on convex body has found a sufficient condition for the solvability of Diophantine inequalities in integers not both zero:***

$$c = {\kappa_p}^p, \kappa_p = \frac{\Gamma(1+\frac{2}{p})^{1/2}}{\Gamma(1+\frac{1}{p})}.$$

But this result is not optimal, and Minkowski also raised the issue of not improving constant c***. For this purpose Minkowski has proposed to use the critical determinant.***

Given any set $\mathcal{R} \subset \mathbb{R}^n$***, a lattice*** Λ ***is*** *admissible* ***for*** $\mathcal{R}$ ***(or is*** $\mathcal{R}$-*admissible*) ***if*** $\mathcal{R} \bigcap \Lambda = \emptyset$ ***or*** $\{0\}$***. The infimum*** $\Delta(\mathcal{R})$ ***of the determinants (the determinant of a lattice*** Λ ***is written*** $d(\Lambda)$***) of all lattices admissible for*** $\mathcal{R}$ ***is called the*** *critical determinant* ***of*** $\mathcal{R}$***. A lattice*** Λ ***is*** *critical* ***for*** $\mathcal{R}$ ***if*** $d(\Lambda) = \Delta(\mathcal{R})$***.***

The Minkowski's problem can be reformulated as a conjecture concerning the critical determinant of the region $\mid x \mid^p + \mid y \mid^p \leq 1, \ p > 1$. ***Recall once more that** mentioned mathematical problems are closely connected with Diophantine Approximation.*

For the given 2-dimension region $D_p \subset \mathbf{R}^2 = (x, y), \ p > 1$ *:*

$$|x|^p + |y|^p < 1,$$

let $\Delta(D_p)$ *be the critical determinant of the region.*

I will present the (interval) computation of the critical determinant at the next section (Section 2. Critical determinants and critical lattices of the region $|x|^p + |y|^p < 1$ *for* $p > 1$.*)*

1.2. Matrix nondifferentiable optimization and Compressed sensing (with Kuzik O.V.)

The problem of restoring a matrix from a sample of its elements, which can be correlated with source coding, is formulated as a convex optimization problem [16, 21, 22, 23]. An initially inherent feature of the considered problem recovery of a matrix from a sample of its elements is the non-differentiability of this problem, which causes problematic application of classical methods of differentiable optimization. In connection with this circumstance to solve it, the application of $r-$ *algorithms [16] is proposed [23].*

Постановки задачи и применения. *В рамках кодирования источника сжатое опознавание интерпретируется как восстановление информации источника по неполным данным, кодирующим элементы этой информации. Хотя ниже речь идет о вещественных матрицах, фактически при вычислениях матрицы целочисленны, или имеют рациональные коэффициенты. Задача восстановления мат-*

рицы по выборке её элементов возникает во многих математических и прикладных исследованиях. Упомянем следующие прикладные задачи: Базы данных; Триангуляция по неполным данным; Сжатое опознавание (Compressed Sensing); Машинное обучение (Machine Learning).

Пусть X есть искомая матрица, $M_{i,j}$ известные значения. Одна из математических формулировок вышеперечисленных задач имеет следующее представление:

$$\begin{gathered} minimize\; rank(X) \\ subject\ to\ X_{i,j} = M_{i,j}, (i,j) \in \Omega, \end{gathered}$$

где (i,j) есть множество индексов, $M_{i,j} \in \Omega$ наблюдаемые значения. К сожалению, как доказано в [22], в такой постановке задача суперэкспоненциальна по сложности.

Задача полуопределенного программирования состоит [16] в минимизации линейной функции от m вещественных переменных относительно матричного неравенства

$$\begin{gathered} minimize\; c^T x \\ subject\ to\ F(x) \geq 0, \end{gathered}$$

где $F(x) = F_0 + \sum_{i=1}^{m} x_i F_i$ и $F_0, F_1, \ldots, F_m$ есть симметрические матрицы. Задача полуопределенного программирования является задачей выпуклой оптимизации, так как целевая функция и ограничения выпуклы: если $F(x) \geq 0$ и $F(y) \geq 0$, то для всех $\lambda, 0 \leq \lambda \leq 1$ $F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y) \geq 0$.

Пусть X есть матрица размера $n \times m$, X^* есть матрица, сопряженная к X . Тогда собственные значения матриц XX^* и X^*X совпадают и являются положительными. Арифметические значе-

ния квадратных корней общих собственных значений матриц XX^* и X^*X называют сингулярными значениями матрицы X. Далее полагаем, что σ_k есть $k-$ое сингулярное значение матрицы X, и что эти сингулярные значения занумерованы в порядке убывания $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ где σ_n есть наименьшее сингулярное значение. Сингулярные значения $\sigma_{n+1} \cdots$ полагают нулевыми.

Математическая модель. ***Скалярное произведение (X, Y) матриц X и Y размера $n \times m$ определяют как след $\mathrm{Tr}(X^*Y)$ произведения указанных матриц. Напомним, что субградиентом матричной выпуклой функции f называют матрицу $g_f(X_0)$, удовлетворяющую неравенству $f(X) - f(X_0) \geq (g_f(X_0), X - X_0)$ для всех вещественных матриц размера $n \times m$. Ядерной нормой матрицы X называют величину $\|X\|_* = \sum_{k=1}^n \sigma_k(X)$ где $\sigma_k(X)$ $k-$ое сингулярное значение X. Исследуется задача оптимизации (с ядерной нормой):***

$$\begin{gathered} minimize \, \|(X)\|_* \\ subject \; to \; X_{i,j} = M_{i,j}, (i,j) \in \Omega. \end{gathered}$$

Метод решения. ***Метод решения вышеприведенной оптимизационной задачи основывается на матричном расширении $r-$ алгоритма Н.З. Шора [16]. Для сингулярного разложения матрицы ранга s выражение для субградиента ядерной нормы этой матрицы известно. В процессе выполнения $r-$ алгоритма преобразуется пространство поиска и выполняются ортогональные проектирования.***

1.3. Shape optimization of elastic bodies(with Nagornyak T.)

Shape optimization problems for elastic bodies, that can withstand extreme stress have been considered by Bernoulli and Euler [25]. By elastic body we understand

a solid body for which the additional deformation produced by an increment of stress completely disappears when the increment is removed. Consider the column (rod) with variable, but geometrically similar and equally oriented cross sections, loaded longitudinal force. Evolving the work [25], researches of J. L Lagrange, T. Clausen, E. Nicolai, and others, N. Olhoff and S. Rasmunssen have found [26], that the shape optimization problems can be reduced to problems of non-differentiable optimization. The author of review [27] believes, that the modality situation may occur in many optimization problems in terms of stability criteria. At first we formulate the problem of shape optimization of elastic bodies on the example of following problem of Euler, Lagrange, Pearson and others:

To find the curve which by its revolution about an axes in its plane determine the column of the structures resistance to buckling under axial compression.

INTERVAL ANALYSIS
s - interval calculations
interval mathematics
interval arithmetic
v - COMPUTATIONAL MATHEMATICS
a - INTERVAL ANALYSIS METHODS

INTERVAL ARITHMETIC
n - REAL INTERVAL ARITHMETIC
RATIONAL INTERVAL ARITHMETIC
COMPLEX INTERVAL ARITHMETIC

INTERVAL
s - interval number
closed real interval
n - CONSTANT-SIGN INTERVAL

SIGN-VARIABLE INTERVAL
DEGENERATE INTERVAL
a - ARITHMETICAL OPERATIONS OVER INTERVALS

DEGENERATE INTERVAL
s - point interval number
v - INTERVAL

ARITHMETICAL OPERATIONS OVER INTERVALS
n - BINARY OPERATIONS OVER INTERVALS
UNARY OPERATIONS OVER INTERVALS

BINARY OPERATIONS OVER INTERVALS
n - ADDITION OF INTERVALS

```
s -  [a,b] + [c,d] = [a + c, b + d]
```

SUBTRACTION OF INTERVALS

MULTIPLICATION OF INTERVALS
DIVISION OF INTERVALS

```
s [a,b]/[c,d] = [min (a/c, a/d, b/c, b/d), max (a/c, a/d, b/
```

INTERVAL ANALYSIS METHODS
n - INTERVAL METHODS IN THE LINEAR ALGEBRA
METHODS FOR NARROWING OF THE INTERVALS CONTAINING RANGE OF THE FUNCTION
INTERVAL METHODS FOR THE SOLVING NON-LINEAR ALGEBRAIC EQUATIONS
INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS
a - INTERVAL EXTENSION

INTERVAL ASSESSMENT

INTERVAL METHODS IN THE LINEAR ALGEBRA
v - INTERVAL ANALYSIS METHODS
n - DIRECT METHODS FOR SYSTEMS OF THE LINEAR ALGEBRAIC EQUATIONS
INTERVAL AND ANALYTICAL METHOD OF DRIVING
INTERVAL ITERATIVE METHODS FOR SLAE

DIRECT METHODS FOR SYSTEMS OF THE LINEAR ALGEBRAIC EQUATIONS v - INTERVAL METHODS IN THE LINEAR ALGEBRA n - INTERVAL METHOD OF GAUSS

METHODS FOR NARROWING OF THE INTERVALS CONTAINING RANGE OF THE FUNCTION v - INTERVAL ANALYSIS METHODS n - SKELBOU'S METHOD APPLICATION OF THE GENERALIZED INTERVAL ARITHMETICS METHOD COMPUTATION METHOD BY MEANS OF NON-STANDARD ARITHMETICS

INTERVAL METHODS FOR THE SOLVING NON-LINEAR ALGEBRAIC EQUATIONS v - INTERVAL ANALYSIS METHODS n - MOORE'S METHOD ANALOGUE OF CHEBYSHEV'S METHOD HANSEN'S METHOD KRAVCHIK'S METHOD

INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS v - INTERVAL ANALYSIS METHODS n - EXPLICIT METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS IMPLICIT METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS INTERVAL METHODS FOR THE SOLUTION OF THE CAUCHY PROBLEM INTERVAL METHODS FOR THE SOLUTION OF BOUNDARY VALUE

PROBLEMS FOR THE ODE INTERVAL METHODS FOR THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH PRIVATE DERIVATIVES AND INTEGRABLE EQUATIONS

EXPLICIT METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS** v **- INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS** n **- MOORE'S METHODS S-TH ORDER INTERVAL METHODS LIKE RUNGE-KUTT'S METHOD INTERVAL METHODS LIKE ADAMS' METHOD INTERVAL METHOD ON THE BASIS OF THE RECTANGULAR FORMULA THE INTERVAL METHOD USING THE TRAPEZOID FORMULA THE FOURTH ORDER INTERVAL METHOD BASED ON THE "THREE EIGHTH"QUADRATURE FORMULA THE BASES ON APPLICATION OF THE SIMPSON FORMULA METHOD

IMPLICIT METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS** v **- INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS** n **- ANALOGUE OF THE IMPLICIT EULER'S METHOD IMPLICIT INTERVAL AND ANALYTICAL METHOD OF THE SECOND ORDER IMPLICIT INTERVAL AND ANALYTICAL METHODS OF HIGHER ORDERS IMPLICIT METHOD BASED ON THE SIMPSON'S FORMULA

***INTERVAL METHODS FOR THE SOLUTION OF THE CAUCHY PROBLEM** v **- INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS** n **- KRYUKEBERG'S METHOD METOD OF BAUKH INTERVAL METHOD FOR BAUKH'S EQUATION INTERVAL METHOD FOR THE EQUATION** $y' = f(y)$ **USING CHAPLYGIN THEOREM METHOD FOR SOLVING EQUATION** $' = f(x,)$*

INTERVAL METHODS FOR THE SOLUTION OF

BOUNDARY VALUE PROBLEMS FOR THE ODE
v - INTERVAL METHODS FOR THE SOLVING DIFFERENTIAL EQUATIONS
n - ANALOGUE OF FINITE-DIFFERENCE METHODS
OLIVEIRA'S METHOD

1.4. Jet engine nozzle profiles (Stetysuk P.I. with colleagues)

By P.I. Stetsyk and others [A set of programs for constructing theoretical contours of the outer and inner surfaces nozzles with a central body according to a given law of changes in areas (Glushkov Institute of cybernetics NASU, stage 2, 2020].

Introduction

Laval nozzle
Frenkl nozzle

1. General provisions and problem statement

2. Constructing the outer contour of the nozzle

3. Constructing the contour of the central body

4. Constructing the contour of the central body by geometric averaging

5. Optimization of parameters in the generalized Delambert formula for a function of two variables

2. Critical determinants and critical lattices of the region $|x|^p + |y|^p < 1$ for $p > 1$

2.1 Minkowsk analytic conjecture

In considering the question of the minimum value taken by the expression $|x|^p + |y|^p$, with $p \geq 1$, at points, other that the origin, of a lattice Λ of determinant $d(\Lambda)$, Minkowski [15] shows that the problem of determining the maximum value of the minimum for different lattices may be reduced to that of finding the minimum possible area of a parallelogram with one vertex at the origin and the three remaining vertices on the curve $|x|^p + |y|^p = 1$. The problem with $p = 1, 2$ and ∞ is trivial: in these cases the minimum areas are $1/2$, $\sqrt{3}/2$ and 1 respectively. Let $D_p \subset \mathbf{R}^2 = (x, y)$, $p > 1$ be the 2-dimension region:

$$|x|^p + |y|^p < 1.$$

Let $\Delta(D_p)$ be the critical determinant of the region. Recall considerations of the previous section. For $p > 1$, let

$$D_p = \{(x, y) \in \mathbb{R}^2 \mid |x|^p + |y|^p < 1\}.$$

Minkowski [15] raised a question about critical determinants and critical lattices of regions D_p for varying $p > 1$. Let $\Lambda_p^{(0)}$ and $\Lambda_p^{(1)}$ be two D_p-admissible lattices each of which contains three pairs of points on the boundary of D_p and with the property that $(1, 0) \in \Lambda_p^{(0)}$, $(-2^{-1/p}, 2^{-1/p}) \in \Lambda_p^{(1)}$, (under these conditions the lattices are uniquely defined). Using analytic parameterization Cohn [28] gives analytic formulation of Minkowski's conjecture.

Let

$$\Delta(p, \sigma) = (\tau + \sigma)(1 + \tau^p)^{-\frac{1}{p}}(1 + \sigma^p)^{-\frac{1}{p}}, \quad (1)$$

be the function defined in the domain

$$\mathcal{M}: \ \infty > p > 1, \ 1 \le \sigma \le \sigma_p = (2^p - 1)^{\frac{1}{p}},$$

of the $\{p, \sigma\}$ ***plane, where*** σ ***is some real parameter; here*** $\tau = \tau(p, \sigma)$ ***is the function uniquely determined by the conditions***

$$A^p + B^p = 1, \ 0 \le \tau \le \tau_p,$$

where

$$A = A(p, \sigma) = (1 + \tau^p)^{-\frac{1}{p}} - (1 + \sigma^p)^{-\frac{1}{p}}$$

$$B = B(p, \sigma) = \sigma(1 + \sigma^p)^{-\frac{1}{p}} + \tau(1 + \tau^p)^{-\frac{1}{p}},$$

τ_p ***is defined by the equation***

$$2(1 - \tau_p)^p = 1 + \tau_p^p, \ 0 \le \tau_p \le 1.$$

In this case needs to extend the notion of parameter variety to parameter manifold. The function $\Delta(p, \sigma)$ ***in region*** $\mathcal{M}$ ***determines the parameter manifold.***

Let $\Delta_p^{(1)} = \Delta(p, 1) = 4^{-\frac{1}{p}} \frac{1+\tau_p}{1-\tau_p}$***,***

$\Delta_p^{(0)} = \Delta(p, \sigma_p) = \frac{1}{2}\sigma_p$.

Minkowski's analytic $(p, \sigma)-$**conjecture:**

For any real p with conditions $p > 1, \ p \ne 2, \ 1 < \sigma < \sigma_p$,

$$\Delta(p, \sigma) > min(\Delta_p^{(1)}, \Delta_p^{(0)}).$$

In the vicinity of the point $p = 1$ ***and in the vicinity of the point*** $(2, \sigma_2)$ ***the*** (p, τ) ***variant of the Minkowski's analytic conjecture is used.***

Minkowski's analytic $(p,\tau)-$conjecture:

Let $\tilde{\Delta}(p,\tau)=\Delta(p,\sigma),\ \sigma=\sigma(p,\tau):\ A^p+B^p=1$***.***

For any real p and τ with conditions $p>1,\ p\neq 2,\ 0<\tau<\tau_p$ ***,***

$$\tilde{\Delta}(p,\tau)>min(\Delta_p^{(1)},\Delta_p^{(0)}).$$

For investigation of properties of function $\Delta(p,\sigma)$ ***which are need for proof of Minkowski's conjecture [15, 28] we considered the value of*** $\Delta=\Delta(p,\sigma)$ ***and its derivatives*** $\Delta_{\sigma}^{'}$, $\Delta_{\sigma^2}^{''}$, $\Delta_{p}^{'}$, $\Delta_{\sigma p}^{''}$, $\Delta_{\sigma^2 p}^{'''}$ ***on some subdomains of the domain*** $\mathcal{M}$ ***[32].***

2.2. Interval evaluation of functions, algorithms, software and computations

Let $\mathbf{X}=(\mathbf{x}_1,\cdots,\mathbf{x}_n)=([\underline{x}_1,\overline{x}_1],\cdots,[\underline{x}_n,\overline{x}_n]$ ***be the n-dimensional real interval vector with*** $\underline{x}_i\leq x_i\leq\overline{x}_i$ ***("rectangle"or "box"). The*** *interval evaluation* ***of a function*** $G(x_1,\cdots,x_n)$ ***on an interval*** $\mathbf{X}$ ***is the interval*** $[\underline{G},\overline{G}]$ ***such that for any*** $x\in\mathbf{X},\ G(x)\in[\underline{G},\overline{G}]$***. The interval evaluation is called*** *optimal* ***if*** $\underline{G}=\min G,$ ***and*** $\overline{G}=\max G$ ***on the interval*** $\mathbf{X}$***.***
Let D ***be a subdomain of*** $\mathcal{M}$***. Under evaluation in*** D ***a mentioned function the domain is covered by rectangles of the form***

$$[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}].$$

In the case of the formula that expressing $\Delta_{\sigma}^{'}$, $\Delta_{\sigma^2}^{''}$, $\Delta_{p}^{'}$, $\Delta_{\sigma p}^{''}$, $\Delta_{\sigma^2 p}^{'''}$
in terms of a sum of derivatives of "atoms" $s_i=\sigma^{p-i},\ t_i=\tau^{p-i},\ a_i=(1+\sigma^p)^{-i-\frac{1}{p}},\ b_i=(1+\tau^p)^{-i-\frac{1}{p}},\ A=$

$b_0 - a_0,\ B = \tau b_0 + \sigma a_0,\ \alpha_i = A^{p-i},\ \beta_i = B^{p-i}\ (i = 0, 1, 2, \ldots)$ ***one applies the rational interval evaluation to construct formulas for lower bounds and upper bounds of the functions, which in the end can be expressed in terms of*** $\underline{p},\ \overline{p},\ \underline{\sigma},\ \overline{\sigma},\ \underline{\tau},\ \overline{\tau},\ ;$ ***here the bounds*** $\underline{\tau},\ \overline{\tau},$ ***are obtained with the help of the iteration process:***

$$\underline{t}_{i+1} = (1+\underline{t}_i^{\overline{p}})^{\frac{1}{\overline{p}}}((1-((1+\underline{t}_i^{\overline{p}})^{-\frac{1}{\overline{p}}}-(1+\overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}})^{\underline{p}})^{\frac{1}{\underline{p}}}-\overline{\sigma}(1+\overline{\sigma}^{\underline{p}})^{-\frac{1}{\underline{p}}}),$$

$$\overline{t}_{i+1} = (1+\overline{t}_i^{\underline{p}})^{\frac{1}{\underline{p}}}((1-((1+\overline{t}_i^{\underline{p}})^{-\frac{1}{\underline{p}}}-(1+\underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}})^{\overline{p}})^{\frac{1}{\overline{p}}}-\underline{\sigma}(1+\underline{\sigma}^{\overline{p}})^{-\frac{1}{\overline{p}}}).$$

$$i = 0, 1, \cdots$$

As interval computation is the enclosure method, we have to put:

$$[\underline{\tau},\ \overline{\tau}] = [\underline{t}_N,\ \overline{t}_N] \bigcap [\underline{\tau}_0,\ \overline{\tau}_0]\ .$$

N ***is computed on the last step of the iteration.***

For initial values we may take : $[\underline{t}_0,\ \overline{t}_0] = [\underline{\tau}_0,\ \overline{\tau}_0] = [0,\ 0.36]$.

Algorithms and software modules

Here we give names, input and output of algorithms and and software modules for interval evaluation only. All these algorithms and and software modules are implemented, tested and applied under the computer-assisted proof of Minkowski's conjecture [29, 30, 33, 34, 32] .

First two algorithms are auxiliary and described in [35].

Algorithm MonotoneFunction

Input: ***A real function*** $F(x,y)$ ***monotonous by*** x ***and by*** y.
Interval $[\underline{x},\overline{x};\underline{y},\overline{y}]$.

Output: ***The interval evaluation of*** F.

Algorithm RationalFunction

Input: ***A rational function*** $R(x,y)$. ***Interval*** $[\underline{x},\overline{x};\underline{y},\overline{y}]$.

Output: ***The interval evaluation of*** R.

Next algorithms and software modules compute functions of Malyshev's method.

Algorithm TAUPV

Input: ***An implicitly defined function*** τ_p ***from Section 4.***
Interval $[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}]$.

Method: ***Iterative interval computation.***

Output: ***The interval evaluation of*** τ_p***.***

Algorithm TAUV
Input: ***Implicitly defined function*** τ ***from this Section.***

Interval $[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}]$.

Method: ***Described in this Section.***

Output: ***The interval evaluation of*** τ***.***

Algorithm L0V

Input: ***Function*** $l^0 = \Delta(p,\sigma) - \Delta_p^{(0)}$***.***

Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.

Method: ***Interval computations.***

Output: ***The interval evaluation of*** l^0.

Algorithm L1V

Input: ***Function*** $l^1 = \Delta(p, \sigma) - \Delta_p^{(1)}$***.***

Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.

Method: ***Interval computations.***

Output: ***The interval evaluation of*** l^1.

Algorithm GV

Input: ***A function*** $g(p, \sigma)$ ***which has the same sign as function*** $\Delta_\sigma^{'}$***.***

Interval $[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.

Method: ***Interval computations.***

Output: ***The interval evaluation of*** $g(p, \sigma)$.

Algorithm HV

Input: ***A function*** $h(p, \sigma)$ ***which is the partial derivative by*** σ ***the function*** $g(p, \sigma)$***.***

$[\underline{p}, \overline{p};\ \underline{\sigma}, \overline{\sigma}]$.

Method: ***Interval computations.***
Output: ***The interval evaluation of*** $h(p, \sigma)$.

Algorithm DHV

This is the most complicated function and there are several algorithms and software modules for its computation [?].

Input: *A function* $Dh(p,\sigma)$ *which is the partial derivative by* p *the function* $h(p,\sigma)$*.*

$[\underline{p},\overline{p};\ \underline{\sigma},\overline{\sigma}]$.

Method: *Interval computations.*
Output: *The interval evaluation of* $Dh(p,\sigma)$.

Remark 1 $(p,\tau)-$***method implemented using algorithms and software modules SIG and SIGV with input parameters respectively*** $(p,\tau,E1)$ ***and*** $(\underline{p},\underline{\tau},\delta_p,\delta_\tau,E1)$***. Here*** $E1$ ***is the accuracy of computations of*** σ ***and*** $(\underline{\sigma},\overline{\sigma})$***;*** $\overline{p}=\underline{p}+\delta_p,\overline{\tau}=\underline{\tau}+\delta_\tau$***.***

2.3. Results of Computations

It is important to note that Malyshev's method gives possibility to prove that a value of the target minimum is an analytic function but is not a point. Ordinary numerical methods do not allow to obtain results of the kind.

In notations [32] next result have proved:

Theorem 5 [32].

$$\Delta(D_p)=\begin{cases}\Delta(p,1),\ 1<p\le 2,\ p\ge p_0,\\ \Delta(p,\sigma_p),\ 2\le p\le p_0;\end{cases}$$

here p_0 ***is a real number that is defined unique by conditions*** $\Delta(p_0,\sigma_p)=\Delta(p_0,1),\ 2,57\le p_0\le 2,58.$

Corollary 1

$$\kappa_p=\Delta(D_p)^{-\frac{p}{2}}.$$

2.4. Minkowski's problem generalizations

Strengthened Minkowski's analytic conjecture

Strengthened Minkowski's analytic (MAS) conjecture:
A.V. Malishev and the author on the base of some theoretical evidences and results of mentioned computation have proposed the strengthened Minkowski's analytic conjecture (MAS) [33].

For given $p > 1$ *and increasing* σ *from* 1 *to* σ_p *the function* $\Delta(p, \sigma)$

1) increase strictly monotonous if $1 < p < 2$ *and* $p \geq p^{(1)}$*,*

2) decrease strictly monotonous if $2 \leq p \leq p^{(2)}$*,*

3) has a unique maximum on the segment $(1, \sigma_p)$*; until the maximum* $\Delta(p, \sigma)$ *increase strictly monotonous and then decrease strictly monotonous if* $p^{(2)} < p < p^{(1)}$*;*

4) constant, if $p = 2$*;*

here

$p^{(1)} > 2$ *is a root of the equation* $\Delta''_{\sigma^2}|_{\sigma=\sigma_p} = 0$*;*
$p^{(2)} > 2$ *is a root of the equation* $\Delta''_{\sigma^2}|_{\sigma=1} = 0$ *.*

It seems that the conjecture (MAS) has not been proved for any parameter except the trivial $p = 2$***.***

3. Computer Interval Algebra

3.1. Computer Interval Algebraic Systems

A set of intervals with the inclusion relation forms a category CIP ***of preorder.***

Definition 4 . A contravariant functor from CIP ***to the category of sets is called the interval presheaf.***

For a finite set $FS = \{\mathbf{X}_i\}$ ***of*** m***-dimensional intervals in*** $\mathbf{R}^n$, $m \leq n$, ***the union*** V ***of the intervals forms a piecewise-linear manifold in*** $\mathbf{R}^n$. ***Let*** G ***be the graph of the adjacency relation of intervals from*** FS***. The manifold*** V ***is connected if*** G ***is a connected graph. In this paper we are considering connected manifolds. Let*** f ***be a constant sign function on*** $\mathbf{X} \in FS$. ***The set*** $\{(\mathbf{X}_j, Of)\}$ ***of*** $c-$***elements (if it exists) is called*** *a constant sign continuation of* f ***on*** $\{\mathbf{X}_j\}$. ***If*** $\{\mathbf{X}_j\}$ ***is the maximal subset of*** FS ***relative to constant signs function*** f ***then*** $\{(\mathbf{X}_j, Of)\}$ ***is called the*** *constant signs continuation of* f ***on*** FS***.***

3.2. Interval Cellular Covering

Definition 5 . For any n ***and any*** j, $0 \leq j \leq n$, ***a*** *j-dimensional interval cell*, ***or*** *j-I-cell*, ***in*** $\mathbf{R}^n$ ***is a subset*** Ic ***of*** $\mathbf{R}^n$ ***that (possibly after permutation of variables) has the form***

$$Ic = \{x \in \mathbf{R}^n \mid \underline{a}_i, \overline{a}_i, r_k \in \mathbf{R}, \underline{a}_i \leq x_i \leq \overline{a}_i, 1 \leq i \leq j, x_{j+1} = r_1, \ldots, x_n =$$

Here $\underline{a}_i \leq \overline{a}_i$.

If $j = n$ ***then we have an*** n***-dimensional interval vector. Let*** $\mathcal{P}$ ***be the hyperplane that contains*** Ic. ***The dimension of*** Ic ***is equal to the minimal dimension of hyperplanes that contain*** Ic.

Let $\mathcal{P}$ ***be such a hyperplane of the minimal dimension,*** $Int\ Ic$ ***the set of interior points of*** Ic ***in*** $\mathcal{P}$, $Bd\ Ic =$

$Ic \setminus Int\ Ic$. For an m-dimensional I-cell Ic let d_i be an $(m-1)$-dimensional I-cell from $Bd\ Ic$. Then d_i is called an $(m-1)-$dimensional *face* ***of the I-cell Ic.***

Definition 6 . Let D be a bounded set in $\mathbb{R}^n$. By an interval cellular covering Cov we will understand any finite set of $n-$dimensional I-cells such that their union contains D and adjacent I-cells intersect only in faces. By $|Cov|$ we will denote the union of all I-cells from Cov.

Let Cov be the interval covering. By its *subdivision* ***we will understand an interval covering Cov' such that $|Cov| = |Cov'|$ and each I-cell from Cov' is contained in an I-cell from Cov. In our computations we have used mainly bounded horizontal and vertical strips in R^2, their interval coverings and subdivisions.***

3.3. Categories and Functors of Interval Mathematics

A contravariant functor from $\mathcal{CIP}$ to the category of sets is called the interval presheaf.

Let $A = I\mathbf{R}$ be the interval algebra [11] with interval arithmetic operations. In many cases extra interval operations are required. So we have to extend the notion of interval algebra. Let us define the "operator domain"Ω of interval computations as sequence of sets Ω_0 (interval constants and variables), Ω_1 (unary interval operations), Ω_2 (binary interval operations). .... In these notations the set T_Ω of all "non-branching"programs in Ω is defined as the least subset of $(\bigcup_{n=0}^{\infty} \Omega_n \bigcup \{()\})^*$ such that following axioms are satisfied:
(t) $\Omega_0 \subseteq T_\Omega$;
(tt) for $n \geq 1$, $\omega \in \Omega_n$ and $t_1, \ldots, t_n \in T_\Omega$, $\omega(t_1, \ldots, t_n) \in T_\Omega$.
$\Omega-$interval algebra is constructed from the interval algebra A and functions $\omega_A : A^n \to A$, $\omega \in \Omega_n$. Below in the section our results follows Gougen [36] who discussed the non-interval case.

Proposition 1 *T_Ω **is an initial object in the category of $\Omega-$ interval algebras.***

. Let IP be the interval program that implements an interval computation, $G = G(IP)$ the graph of the flow diagram of IP, $\mathcal{G}^\otimes$ the category of all paths in G. Let $\mathcal{IPF}$ be the category of interval sets with partial interval functions.

Proposition 2 . Interval program IP defines a functor $\overline{IP} : \mathcal{G}^\otimes \to \mathcal{IPF}$.

3.4. On Interval Operads

Operads was introduced by J. May [37]. Operadic language is useful for investigation of many problems in mathematics and physics. Here we give a short description of interval operad. The space of continuous interval functions of j variables forms the topological space $I\mathcal{C}(j)$. Its points are operations $I^j\mathbf{R} \to I\mathbf{R}$ of arity j. $I\mathcal{C}(0)$ is a single point $*$. The class of interval spaces $I^n\mathbf{R}$, $n \geq 0$ forms the category [14]. We will consider the spaces with base points and denote the category of those spaces by $I\mathcal{U}$. Let $X \in I\mathcal{U}$ and for $k \geq 0$ let $I\mathcal{E}(k)$ be the space of maps $M(X^k, X)$. There is the action (by permuting the inputs) of the symmetric group S_k on $I\mathcal{E}(k)$. The identity element $1 \in I\mathcal{E}(1)$ is the identity map of X.

Proposition-Definition 1 . In the above mentioned conventions let $k \geq 0$ and $j_1, \ldots, j_k \geq 0$ be integers. Let for each choice of k and $j_1, \ldots, j_k$ there is a map

$$\gamma : I\mathcal{E}(k) \times I\mathcal{E}(j_1) \ldots \times I\mathcal{E}(j_k) \to I\mathcal{E}(j_1 + \ldots + j_k)$$

given by multivariable composition. If maps γ satisfy associativity, equivalence and unital properties then $I\mathcal{E}$ is the endomorphism interval operad $I\mathcal{E}_X$ of X.

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Thank you for your attention!