# Computer algebra aided numerical solving nonlinear PDEs

Vladimir P. Gerdt[1] Yuri A. Blinkov[2]

[1]Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980, Dubna, Russia

[2]Department of Mathematics and Mechanics
Saratov University
410012, Saratov, Russia

Talk at MSU 2013, October 16, Moscow, Russia

# Contents

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

⇓

Discretization (FDM, FEM, FVM)

⇓

Algebraic (difference) equations

⇓

Numerical solving

⇓

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \dots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# Numerical solving PDEs

## Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)

$\Downarrow$

Discretization (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings $\mathbf{h} := \{h_1, \ldots, h_n\}$.

$$\text{PDE(s)} \Longrightarrow \text{FDA}$$

The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$\Downarrow$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$\Downarrow$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$\Downarrow$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$\Downarrow$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \rightarrow 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

**S-consistency**

Definition. FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

Theorem. For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$$\Downarrow$$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$$\Downarrow$$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

## S-consistency

**Definition.** FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

**Theorem.** For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|\mathbf{h}| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

$\Downarrow$

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

$\Downarrow$

For polynomially nonlinear PDE(s) s(trong)-consistency of FDA (Gerdt'12).

## S-consistency

Definition. FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|\mathbf{h}| \to 0$ is reduced to a differential consequence of PDE(s).

Theorem. For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.

# Contents

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + su = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.
2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + su = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.

2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + s u = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.
2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + s u = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.

2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + su = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.
2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + su = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

They possess infinitely many conservation laws and symmetries.

2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + s u = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.
2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + s u = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

They possess infinitely many conservation laws and symmetries.

2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic ($s < 0$), living organisms ($s > 0$), rubber ($s = 0$).

# Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

$$\Phi := \left\{ u_t + \left(f_1 u + f_2 u^2 + f_3 u^3\right)_x + u_{xxx} + s_2 u_{xx} + s u = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \right\} .$$

## Motivation

1. Korteveg-de Vries (KdV) and modified KdV (MKdV) equations are contained in $\Phi$

$$u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2 u_x = 0 \in \Phi .$$

   They possess infinitely many conservation laws and symmetries.
2. Equations in $\Phi$ admit a wide class of exact solutions.
3. Equations in $\Phi$ describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov,Ivanov,Mogilevich'2012). The sign of $s$ characterizes the shell material: nonorganic $(s < 0)$, living organisms $(s > 0)$, rubber $(s = 0)$.

# Contents

# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F + u_{xx} + s_2 u_x)\, dt + u\, dx + s \iint_{\Omega} u\, dt\, dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t, x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\, dx = u(t, x_{j+1}) - u(t, x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx}\, dx = u_x(t, x_{j+1}) - u_x(t, x_j).$$

# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F+u_{xx}+s_2u_x)\,dt+u\,dx+s\iint_{\Omega} u\,dt\,dx=0\,,\quad F:=f_1u+f_2u^2+f_3u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx=u(t,x_{j+1})-u(t,x_j),\ \int_{x_j}^{x_{j+1}} u_{xx}\,dx=u_x(t,x_{j+1})-u_x(t,x_j).$$

# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F+u_{xx}+s_2 u_x)\,dt + u\,dx + s\iint_{\Omega} u\,dt\,dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

$\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j).$$

# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F + u_{xx} + s_2 u_x)\,dt + u\,dx + s\iint_{\Omega} u\,dt\,dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j).$$

# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F+u_{xx}+s_2 u_x)\,dt + u\,dx + s\iint_{\Omega} u\,dt\,dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j),\quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j).$$

# Generation of FDA

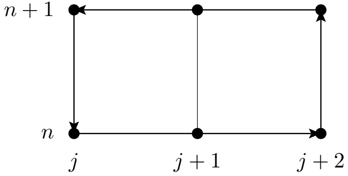
We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F+u_{xx}+s_2u_x)\,dt+u\,dx+s\iint_{\Omega} u\,dt\,dx = 0\,, \quad F := f_1u+f_2u^2+f_3u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j).$$

# Generation of FDA

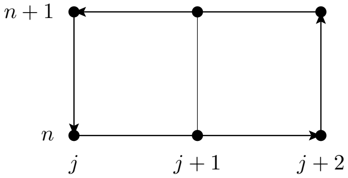
We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F + u_{xx} + s_2 u_x)\, dt + u\, dx + s \iint_{\Omega} u\, dt\, dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t, x)$ bounded by $\partial\Omega$.

2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\, dx = u(t, x_{j+1}) - u(t, x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx}\, dx = u_x(t, x_{j+1}) - u_x(t, x_j).$$

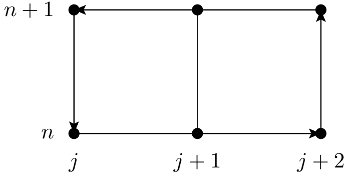
# Generation of FDA

We use our algorithmic approach (Gerdt,Blinkov,Mozzhilkin'06) based on FVM combined with numerical integration and difference elimination

1. Convert into integral form (Green's theorem)

$$\oint_{\partial\Omega} -(F+u_{xx}+s_2 u_x)\,dt + u\,dx + s\iint_{\Omega} u\,dt\,dx = 0\,, \quad F := f_1 u + f_2 u^2 + f_3 u^3\,.$$

   $\Omega$ is arbitrary region in the plane $(t,x)$ bounded by $\partial\Omega$.
2. Choose of a "control volume" $\Omega$




3. Add the integral relations

$$\int_{x_j}^{x_{j+1}} u_x\,dx = u(t,x_{j+1}) - u(t,x_j),\ \int_{x_j}^{x_{j+1}} u_{xx}\,dx = u_x(t,x_{j+1}) - u_x(t,x_j).$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$-\left(\left(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\right) + \left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right) + s_2\left({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\right)\right) \cdot \frac{\tau}{2} + (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0,$$

$$({u_x}_{j+1}^n + {u_x}_j^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$-\left(\left(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\right) + \left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right) + s_2\left({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\right)\right) \cdot \frac{\tau}{2} + (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0,$$

$$({u_x}_{j+1}^n + {u_x}_j^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$-\Big(\Big(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\Big) + \Big({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\Big) +$$

$$s_2\Big({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\Big)\Big) \cdot \frac{\tau}{2} +$$

$$+ (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0,$$

$$({u_x}_{j+1}^n + {u_x}_j^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$-\left(\left(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\right) + \left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right) + s_2\left({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\right)\right) \cdot \frac{\tau}{2} + + (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0,$$

$$({u_x}_{j+1}^n + {u_x}_j^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$-\left(\left(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\right) + \left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right) + s_2\left({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\right)\right)\cdot\frac{\tau}{2} + \\ + (u_{j+1}^{n+1} - u_{j+1}^n)\cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n)\cdot h\tau = 0,$$

$$({u_x}_{j+1}^n + {u_x}_j^n)\cdot\frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.$$

# Discretization

4. Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply
   - the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
   - the midpoint rule for the other integrations

   In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$
\begin{gathered}
-\left(\left(F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}\right) + \left({u_{xx}}_j^n + {u_{xx}}_j^{n+1} - {u_{xx}}_{j+2}^n - {u_{xx}}_{j+2}^{n+1}\right) + \right. \\
\left. s_2\left({u_x}_j^n + {u_x}_j^{n+1} - {u_x}_{j+2}^n - {u_x}_{j+2}^{n+1}\right)\right) \cdot \frac{\tau}{2} + \\
+ (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0, \\
({u_x}_{j+1}^n + {u_x}_j^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad {u_{xx}}_{j+1}^n \cdot 2h = {u_x}_{j+2}^n - {u_x}_j^n.
\end{gathered}
$$

# Difference elimination I

5. Elimination of $u_x$ and $u_{xx}$ by computing a difference Gröbner basis for an elimination monomial ordering extending the ranking $u_{xx} \succ u_x \succ u \succ F$. The input for the Maple package LDA (Gerdt,Robertz'12)

```
> restart:
> libname:=libname, "/usr/local/lib/LDA"":

> L:=[-((F(n,j)+F(n+1,j)-F(n,j+2)-F(n+1,j+2)) +
> (uxx(n,j)+uxx(n+1,j)-uxx(n,j+2)-uxx(n+1,j+2))+
> s2(ux(n,j)+ux(n+1,j)-ux(n,j+2)-ux(n+1,j+2)) )tau/2+
> (u(n+1,j+1)-u(n,j+1))2h,
> s(ux(n,j+1)+ux(n,j))h/2-(u(n,j+1)-u(n,j)),
> 2uxx(n,j+1)h-(ux(n,j+2)-ux(n,j))];

>  JanetBasis(L, [n,j], [uxx,ux,u,F],2):
>  collect(%[1,1]/(4*tau*h^3),[s,s2,tau,h]);
```

# Difference elimination II

$$s\frac{u(n+1,j+2)+u(n,j+2)}{2}+s2\frac{1}{2h^2}(-2u(n+1,j+2)+u(n,j+3)+$$
$$+u(n,j+1)+u(n+1,j+1)-2u(n,j+2)+u(n+1,j+3))+$$
$$+\frac{F(n+1,j+3)-F(n+1,j+1)+F(n,j+3)-F(n,j+1)}{4h}+$$
$$+\frac{1}{4h^3}(-u(n,j)-2u(n,j+3)+2u(n+1,j+1)-u(n+1,j)+$$
$$+2u(n,j+1)+u(n+1,j+4)+u(n,j+4)-2u(n+1,j+3))+$$
$$+\frac{u(n+1,j+2)-u(n,j+2)}{\tau}$$

# Strong consistency

## FDA

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{(F_{j+1}^{n+1}-F_{j-1}^{n+1})+(F_{j+1}^n-F_{j-1}^n)}{4h}+$$
$$+\frac{(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+(u_{j+2}^n-2u_{j+1}^n+2u_{j-1}^n-u_{j-2}^n)}{4h^3}+$$
$$+s_2\frac{(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+(u_{j+1}^n-2u_j^n+u_{j-1}^n)}{2h^2}+s\frac{u_j^{n+1}+u_j^n}{2}=0$$

## S-consistency

If one chooses an admissible difference monomial ordering such that $u_{j+2}^{n+1}$ is the leading monomial in the above FDA, then its left-hand side is a Gröbner basis. In the limit $\tau, h \longrightarrow 0$ it is reduced to the original PDE.

# Strong consistency

## FDA

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{(F_{j+1}^{n+1}-F_{j-1}^{n+1})+(F_{j+1}^{n}-F_{j-1}^{n})}{4h}+$$
$$+\frac{(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1})+(u_{j+2}^{n}-2u_{j+1}^{n}+2u_{j-1}^{n}-u_{j-2}^{n})}{4h^3}+$$
$$+s_2\frac{(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+(u_{j+1}^{n}-2u_j^{n}+u_{j-1}^{n})}{2h^2}+s\frac{u_j^{n+1}+u_j^n}{2}=0$$

## S-consistency

If one chooses an admissible difference monomial ordering such that $u_{j+2}^{n+1}$ is the leading monomial in the above FDA, then its left-hand side is a Gröbner basis. In the limit $\tau, h \longrightarrow 0$ it is reduced to the original PDE.

# Contents

# Exact solutions

Let $s = 0$, $U, n_0, n_1, d_0, d_1, k, \omega, \in \mathbb{R}$ and solution be the form

$$u = U\frac{n_0 + n_1\exp(\zeta) + \exp(2\zeta)}{d_0 + d_1\exp(\zeta) + \exp(2\zeta)}, \quad \zeta = kx - \omega t$$

Then the method of indefinite coefficients gives the following multi-parametric solution

$$f_1 = \frac{\omega}{k} + 2k^2 + \frac{6k^2 d_0^2(2n_0 - n_1^2)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2},$$

$$f_2 = -\frac{6(d_0 + n_0 - n_1 d_1)d_0^2 k^2}{U(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)},$$

$$f_3 = \frac{2k^2 d_0^2(2d_0 - d_1^2)}{U^2(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)},$$

$$s_2 = -\frac{3kd_1(d_0 - n_0)(d_0 d_1 - 2n_1 d_0 + n_0 d_1)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2}.$$

# Exact solutions

Let $s = 0$, $U, n_0, n_1, d_0, d_1, k, \omega, \in \mathbb{R}$ and solution be the form

$$u = U\frac{n_0 + n_1\exp(\zeta) + \exp(2\zeta)}{d_0 + d_1\exp(\zeta) + \exp(2\zeta)}, \quad \zeta = kx - \omega t$$

Then the method of indefinite coefficients gives the following multi-parametric solution

$$f_1 = \frac{\omega}{k} + 2k^2 + \frac{6k^2 d_0^2(2n_0 - n_1^2)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2},$$

$$f_2 = -\frac{6(d_0 + n_0 - n_1 d_1)d_0^2 k^2}{U(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)},$$

$$f_3 = \frac{2k^2 d_0^2(2d_0 - d_1^2)}{U^2(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)},$$

$$s_2 = -\frac{3kd_1(d_0 - n_0)(d_0 d_1 - 2n_1 d_0 + n_0 d_1)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2}.$$

# Exact solutions

Let $s = 0$, $U, n_0, n_1, d_0, d_1, k, \omega, \in \mathbb{R}$ and solution be the form

$$u = U\frac{n_0 + n_1\exp(\zeta) + \exp(2\zeta)}{d_0 + d_1\exp(\zeta) + \exp(2\zeta)}, \quad \zeta = kx - \omega t$$

Then the method of indefinite coefficients gives the following multi-parametric solution

$$f_1 = \frac{\omega}{k} + 2k^2 + \frac{6k^2 d_0^2(2n_0 - n_1^2)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2} \ ,$$

$$f_2 = -\frac{6(d_0 + n_0 - n_1 d_1)d_0^2 k^2}{U(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)} \ ,$$

$$f_3 = \frac{2k^2 d_0^2(2d_0 - d_1^2)}{U^2(d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)} \ ,$$

$$s_2 = -\frac{3kd_1(d_0 - n_0)(d_0 d_1 - 2n_1 d_0 + n_0 d_1)}{d_0^2(n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2} \ .$$

# Exact solutions with $u \neq \text{const}$

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)},$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6,$
3. $d_1 = \dfrac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0},$
4. $d_0 = -d_1^2,\ n_0 = d_1\left(\dfrac{1 \pm \sqrt{5}}{2}n_1 - \dfrac{3 \pm \sqrt{5}}{2}d_1\right).$
   This solution is blowup.

# Exact solutions with $u \neq$ const

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}\,,$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6\,,$
3. $d_1 = \dfrac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}\,,$
4. $d_0 = -d_1^2,\ n_0 = d_1\left(\dfrac{1 \pm \sqrt{5}}{2}n_1 - \dfrac{3 \pm \sqrt{5}}{2}d_1\right).$
   This solution is blowup.

# Exact solutions with $u \neq$ const

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}\,,$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6\,,$
3. $d_1 = \dfrac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}\,,$
4. $d_0 = -d_1^2,\ n_0 = d_1\left(\dfrac{1 \pm \sqrt{5}}{2}n_1 - \dfrac{3 \pm \sqrt{5}}{2}d_1\right).$

   This solution is blowup.

# Exact solutions with $u \neq$ const

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}\,,$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6\,,$
3. $$d_1 = \frac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}\,,$$
4. $$d_0 = -d_1^2,\ n_0 = d_1\left(\frac{1 \pm \sqrt{5}}{2}n_1 - \frac{3 \pm \sqrt{5}}{2}d_1\right).$$
   This solution is blowup.

# Exact solutions with $u \neq$ const

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}\,,$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6\,,$
3. $$d_1 = \frac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}\,,$$
4. $d_0 = -d_1^2,\ n_0 = d_1\left(\frac{1 \pm \sqrt{5}}{2}n_1 - \frac{3 \pm \sqrt{5}}{2}d_1\right).$

   This solution is blowup.

# Exact solutions with $u \neq \text{const}$

There are 4 types of such solutions:

1. $\{d_0 = 0,\ n_0 = -d_1^2 + n_1 d_1\} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\}$
   In this case the equation is linear and its solution is given by
   $$u = U\frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}\,,$$
2. $d_0 = d_1^2/6,\ n_0 = d_1^2/6\,,$
3. $d_1 = \dfrac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}\,,$
4. $d_0 = -d_1^2,\ n_0 = d_1\left(\dfrac{1 \pm \sqrt{5}}{2}n_1 - \dfrac{3 \pm \sqrt{5}}{2}d_1\right).$
   This solution is blowup.

# Contents

# Linearization

The above FDA contains nonlinear terms for the grid function on the next time layer. To treat them in construction of a numerical solution we used the following linearization

$$\begin{aligned} v_{k+1}^3 &= v_{k+1}^3 - v_k^3 + v_k^3 = (v_{k+1} - v_k)(v_{k+1}^2 + v_{k+1}v_k + v_k^2) + v_k^3 \approx \\ &\approx v_{k+1} \cdot 3v_k^2 - 2v_k^3 , \\ v_{k+1}^2 &= v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx \\ &\approx v_{k+1} \cdot 2v_k - v_k^2 . \end{aligned}$$

We implemented numerical procedure for construction of a solution in Python with the use of package SciPy. In doing so, we fixed $\tau := h/2$.

# Linearization

The above FDA contains nonlinear terms for the grid function on the next time layer. To treat them in construction of a numerical solution we used the following linearization

$$
\begin{aligned}
v_{k+1}^3 &= v_{k+1}^3 - v_k^3 + v_k^3 = (v_{k+1} - v_k)(v_{k+1}^2 + v_{k+1}v_k + v_k^2) + v_k^3 \approx \\
&\approx v_{k+1} \cdot 3v_k^2 - 2v_k^3\,, \\
v_{k+1}^2 &= v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx \\
&\approx v_{k+1} \cdot 2v_k - v_k^2\,.
\end{aligned}
$$

We implemented numerical procedure for construction of a solution in Python with the use of package SciPy. In doing so, we fixed $\tau := h/2$.

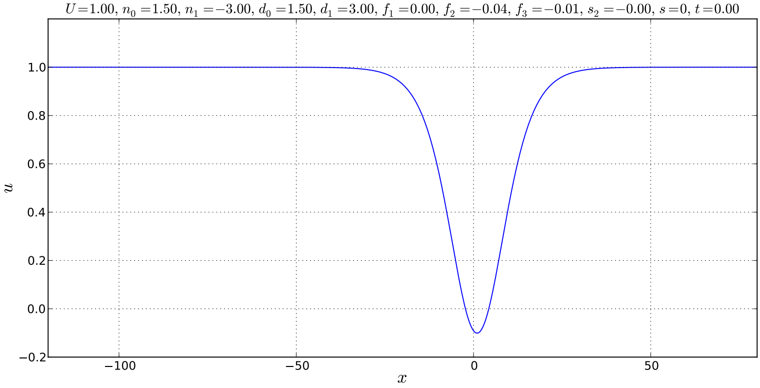
# Linearization

The above FDA contains nonlinear terms for the grid function on the next time layer. To treat them in construction of a numerical solution we used the following linearization

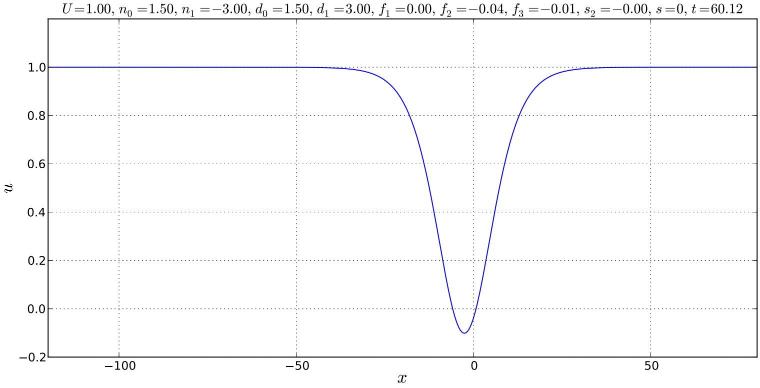
$$\begin{aligned} v_{k+1}^3 &= v_{k+1}^3 - v_k^3 + v_k^3 = (v_{k+1} - v_k)(v_{k+1}^2 + v_{k+1}v_k + v_k^2) + v_k^3 \approx \\ &\approx v_{k+1} \cdot 3v_k^2 - 2v_k^3\,, \\ v_{k+1}^2 &= v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx \\ &\approx v_{k+1} \cdot 2v_k - v_k^2\,. \end{aligned}$$

We implemented numerical procedure for construction of a solution in Python with the use of package SciPy. In doing so, we fixed $\tau := h/2$.

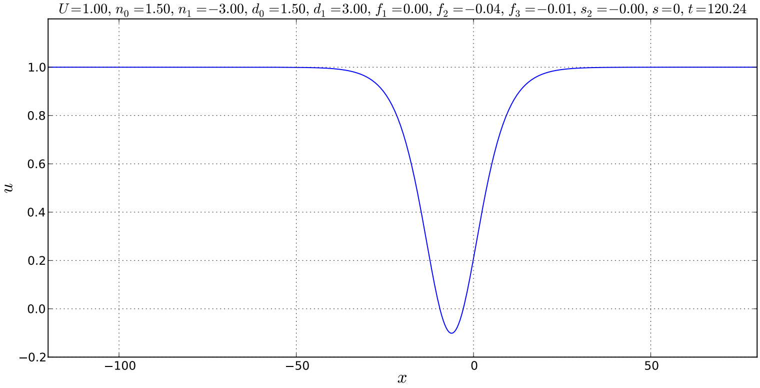
# Exact solution of type 2



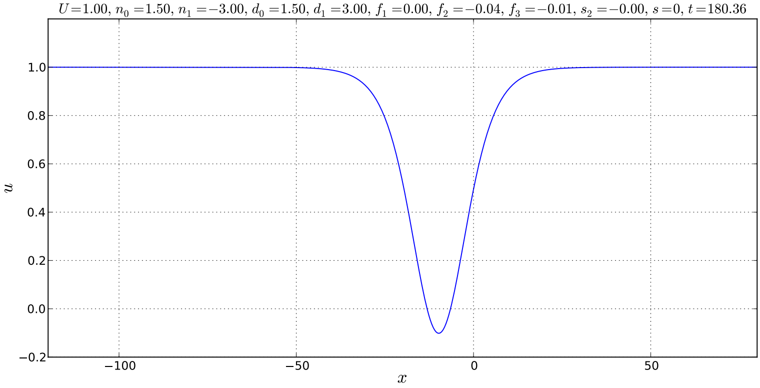
# Exact solution of type 2



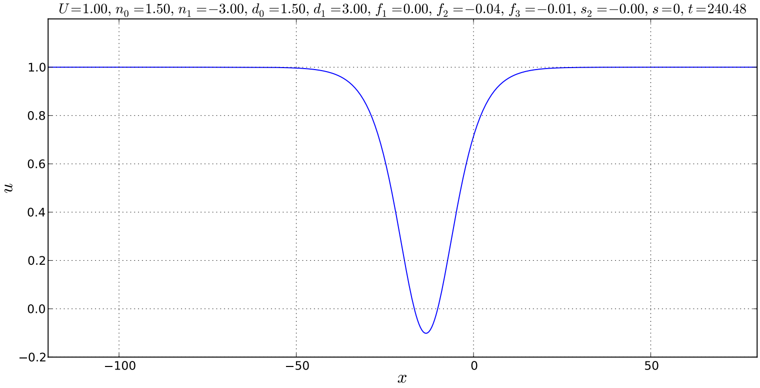
# Exact solution of type 2



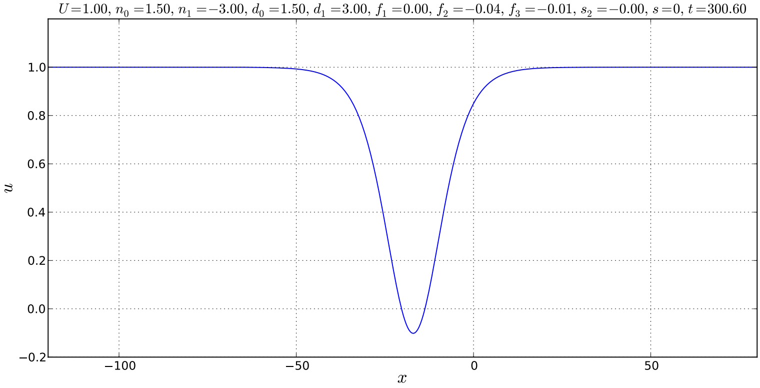
# Exact solution of type 2



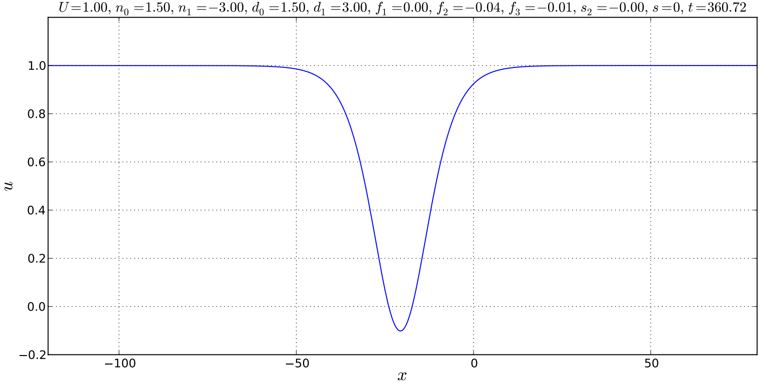
# Exact solution of type 2



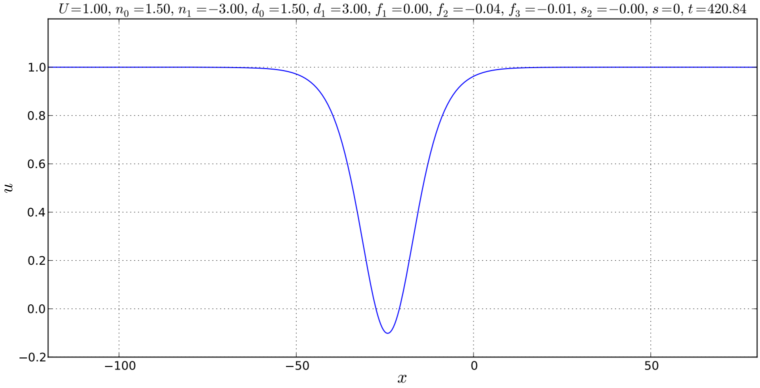
# Exact solution of type 2



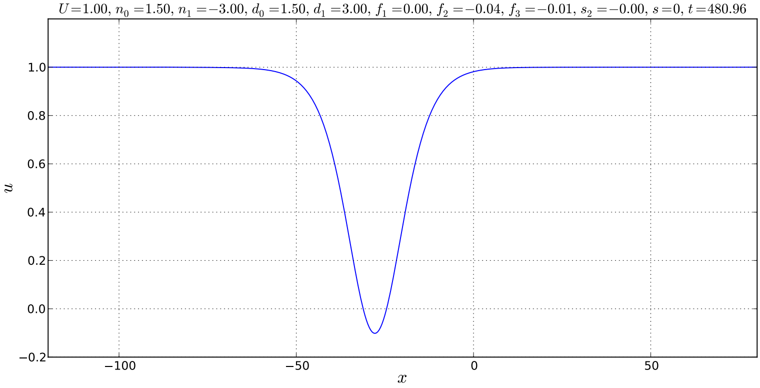
# Exact solution of type 2



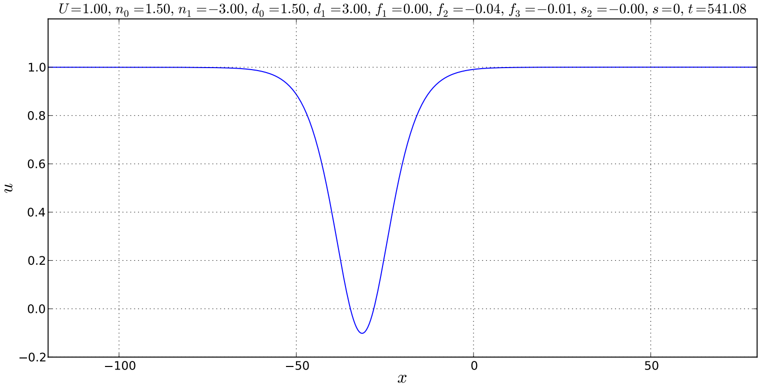
# Exact solution of type 2



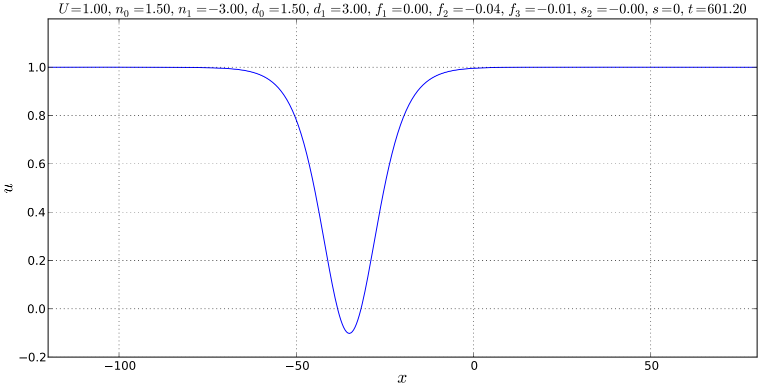
# Exact solution of type 2



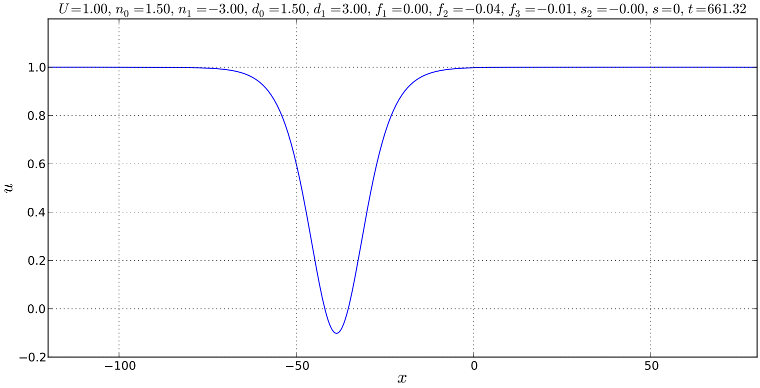
# Exact solution of type 2



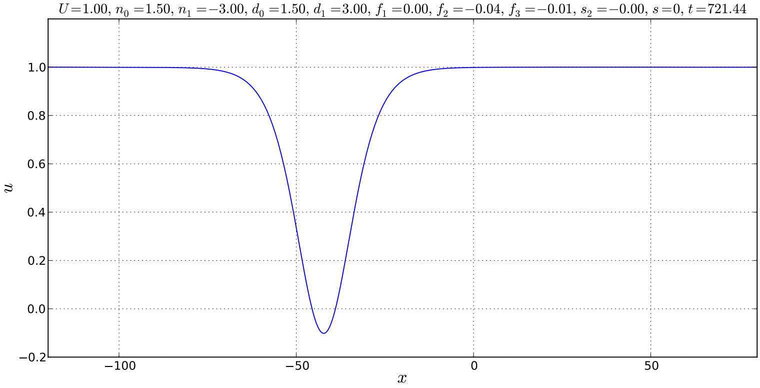
# Exact solution of type 2



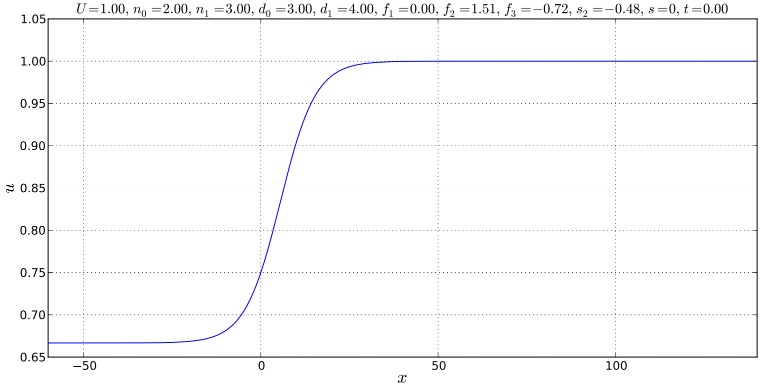
# Exact solution of type 2



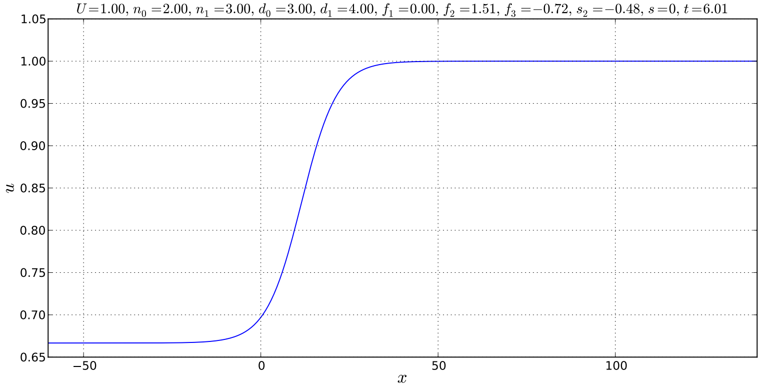
# Exact solution of type 2



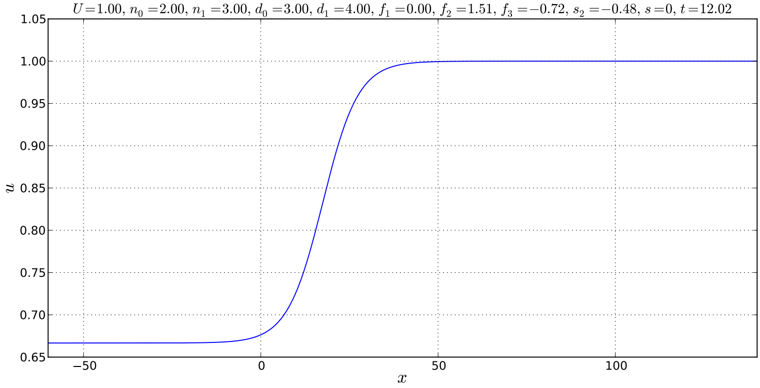
# Exact solution of type 3



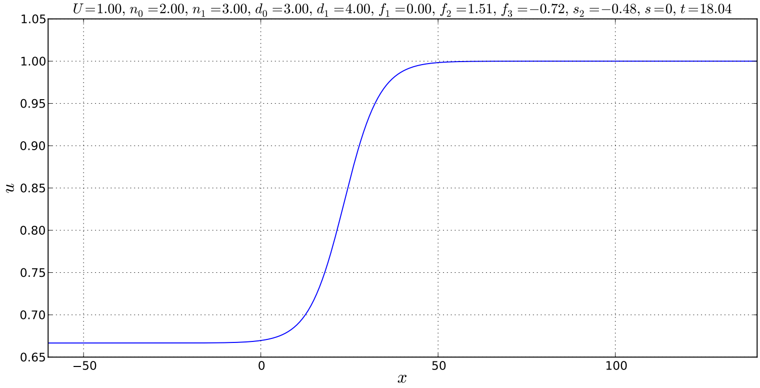
# Exact solution of type 3



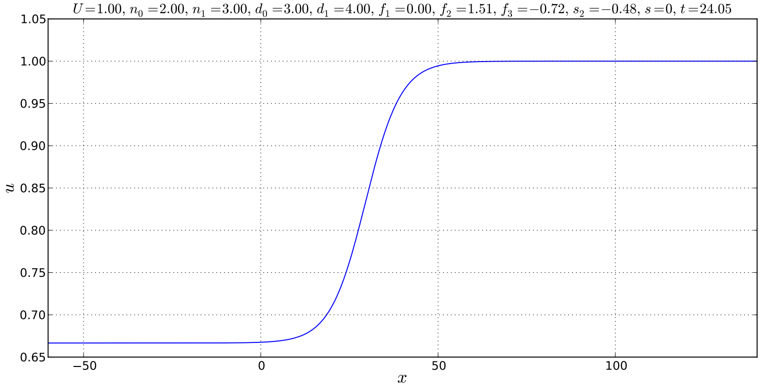
# Exact solution of type 3



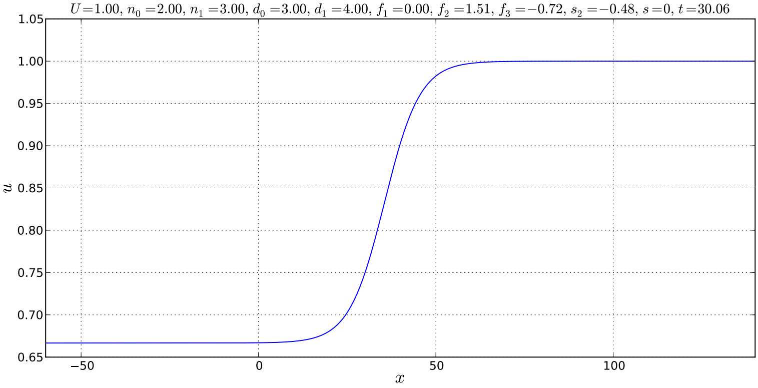
# Exact solution of type 3



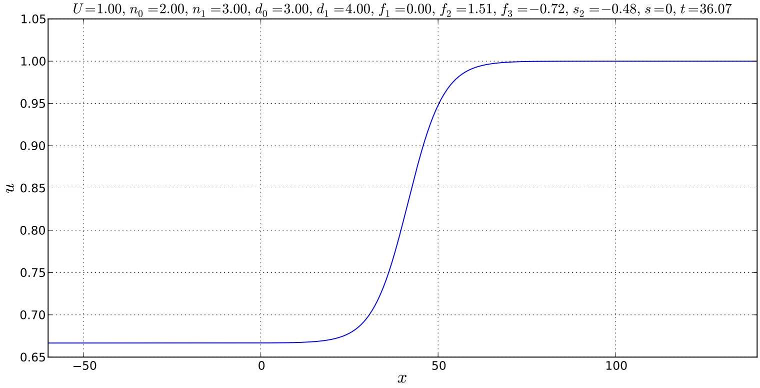
# Exact solution of type 3



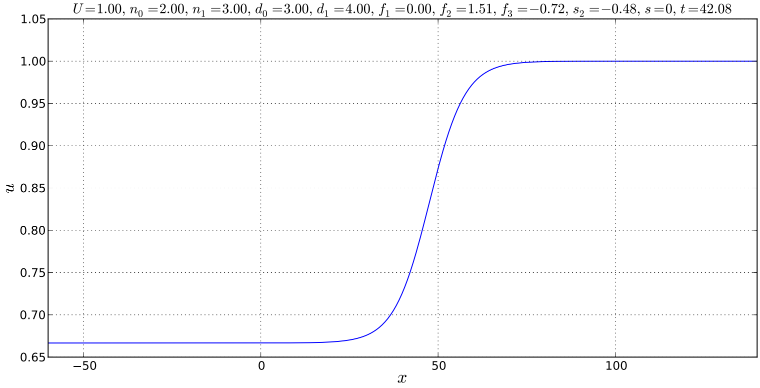
# Exact solution of type 3



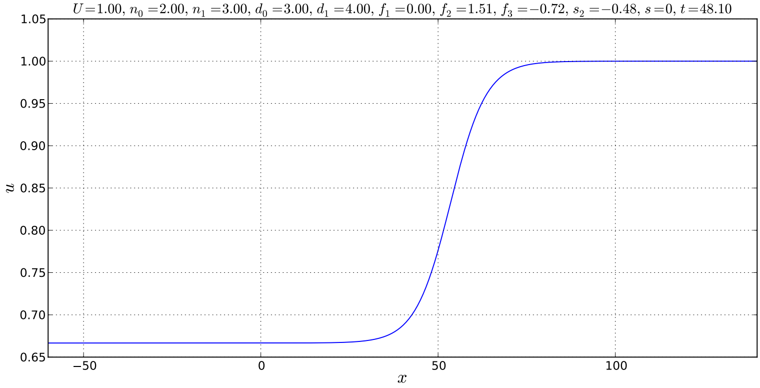
# Exact solution of type 3



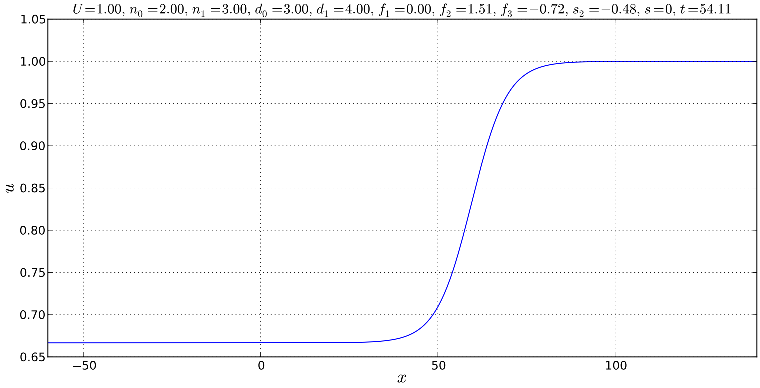
# Exact solution of type 3



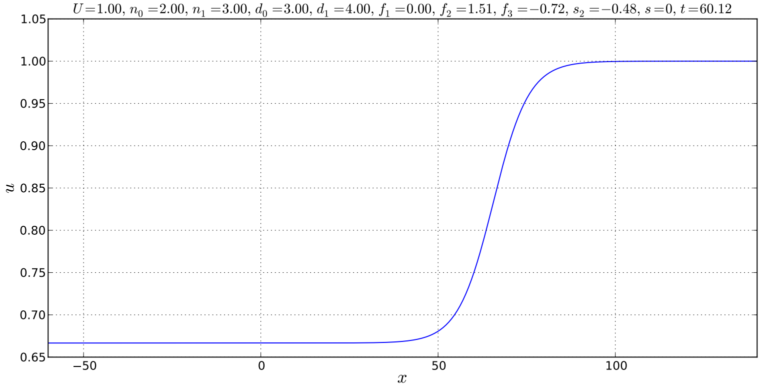
# Exact solution of type 3



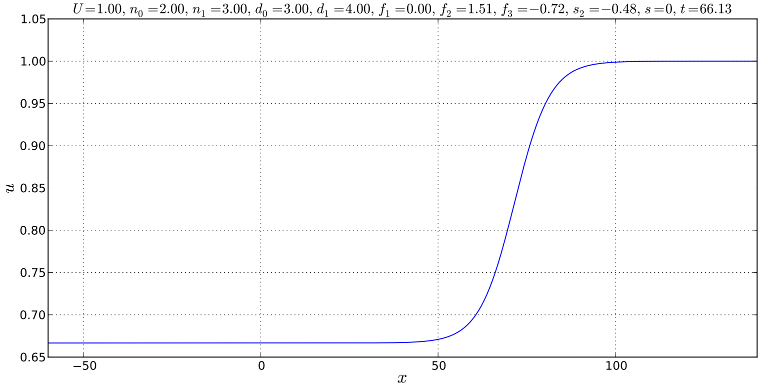
# Exact solution of type 3



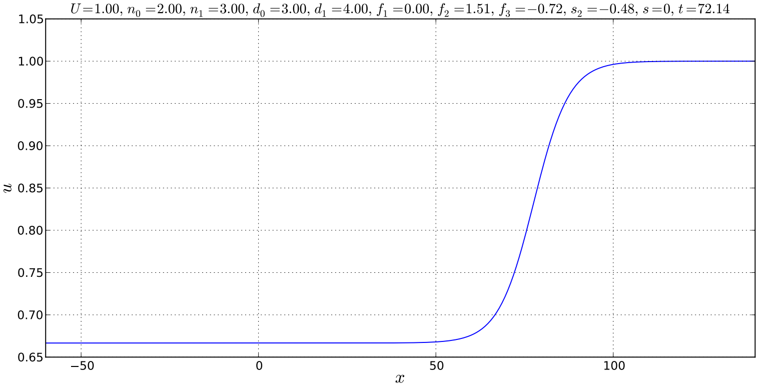
# Exact solution of type 3



# Exact solution of type 3



# Exact solution of type 3



# Contents

# Navier-Stokes PDE system

Involutive PDE system of the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity can be written in the following form (G.,Blinkov'09) obtained by the method suggested in (G.,Blinkov, Mozzhilkin'06)

$$F := \begin{cases} f_1 := u_x + v_y = 0\,, \\ f_2 := u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}) = 0\,, \\ f_3 := v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) = 0\,, \\ f_4 := u_x^2 + 2v_x u_y + v_y^2 + p_{xx} + p_{yy} = 0\,. \end{cases}$$

Here

- $f_1$ - the continuity equation,
- $f_2, f_3$ - the proper Navier-Stokes equations,
- $f_4$ - the pressure Poisson equation which is the integrability condition for $\{f_1, f_2, f_3\}$,
- $(u, v)$ - the velocity field,
- $p$ - the pressure,
- Re - the Reynolds number.

# Navier-Stokes PDE system

Involutive PDE system of the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity can be written in the following form (G.,Blinkov'09) obtained by the method suggested in (G.,Blinkov, Mozzhilkin'06)

$$F := \begin{cases} f_1 := u_x + v_y = 0\,, \\ f_2 := u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy}) = 0\,, \\ f_3 := v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) = 0\,, \\ f_4 := u_x^2 + 2v_x u_y + v_y^2 + p_{xx} + p_{yy} = 0\,. \end{cases}$$

Here

- $f_1$ - the continuity equation,
- $f_2, f_3$ - the proper Navier-Stokes equations,
- $f_4$ - the pressure Poisson equation which is the integrability condition for $\{f_1, f_2, f_3\}$,
- $(u, v)$ - the velocity field,
- $p$ - the pressure,
- $\mathrm{Re}$ - the Reynolds number.

# Divergence form

The involutive Navier-Stokes system admits two-dimensional conservation law form

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} + \frac{\partial \mathbf{R}}{\partial y} = 0\,.$$

In terms of $\{f_1, f_2, f_3, f_4\}$ this form reads

**Conservation law form**

$$\begin{cases} f_1: \; \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0\,, \\ f_2: \; \frac{\partial}{\partial t} u + \frac{\partial}{\partial x}\left(u^2 + p - \frac{1}{\mathrm{Re}} u_x\right) + \frac{\partial}{\partial y}\left(vu - \frac{1}{\mathrm{Re}} u_y\right) = 0\,, \\ f_3: \; \frac{\partial}{\partial t} v + \frac{\partial}{\partial x}\left(uv - \frac{1}{\mathrm{Re}} v_x\right) + \frac{\partial}{\partial y}\left(v^2 + p - \frac{1}{\mathrm{Re}} v_y\right) = 0\,, \\ f_4: \; \frac{\partial}{\partial x}\left(uu_x + vu_y + p_x\right) + \frac{\partial}{\partial y}\left(vv_y + uv_x + p_y\right) = 0\,. \end{cases}$$

# Divergence form

The involutive Navier-Stokes system admits two-dimensional conservation law form

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} + \frac{\partial \mathbf{R}}{\partial y} = 0\,.$$

In terms of $\{f_1, f_2, f_3, f_4\}$ this form reads

**Conservation law form**

$$\left\{\begin{array}{l} f_1 : \; \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0\,, \\ f_2 : \; \frac{\partial}{\partial t} u + \frac{\partial}{\partial x}\left(u^2 + p - \frac{1}{\mathrm{Re}} u_x\right) + \frac{\partial}{\partial y}\left(vu - \frac{1}{\mathrm{Re}} u_y\right) = 0\,, \\ f_3 : \; \frac{\partial}{\partial t} v + \frac{\partial}{\partial x}\left(uv - \frac{1}{\mathrm{Re}} v_x\right) + \frac{\partial}{\partial y}\left(v^2 + p - \frac{1}{\mathrm{Re}} v_y\right) = 0\,, \\ f_4 : \; \frac{\partial}{\partial x}\left(uu_x + vu_y + p_x\right) + \frac{\partial}{\partial y}\left(vv_y + uv_x + p_y\right) = 0\,. \end{array}\right.$$

# Divergence form

The involutive Navier-Stokes system admits two-dimensional conservation law form

$$\frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \mathbf{Q}}{\partial x} + \frac{\partial \mathbf{R}}{\partial y} = 0\,.$$

In terms of $\{f_1, f_2, f_3, f_4\}$ this form reads

## Conservation law form

$$\left\{\begin{array}{l} f_1 : \ \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0\,, \\ f_2 : \ \frac{\partial}{\partial t} u + \frac{\partial}{\partial x}\left(u^2 + p - \frac{1}{\mathrm{Re}} u_x\right) + \frac{\partial}{\partial y}\left(vu - \frac{1}{\mathrm{Re}} u_y\right) = 0\,, \\ f_3 : \ \frac{\partial}{\partial t} v + \frac{\partial}{\partial x}\left(uv - \frac{1}{\mathrm{Re}} v_x\right) + \frac{\partial}{\partial y}\left(v^2 + p - \frac{1}{\mathrm{Re}} v_y\right) = 0\,, \\ f_4 : \ \frac{\partial}{\partial x}\left(uu_x + vu_y + p_x\right) + \frac{\partial}{\partial y}\left(vv_y + uv_x + p_y\right) = 0\,. \end{array}\right.$$

# Contents

# Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0\ (1 \leq i \leq 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p]\,,$$

where $\mathbb{K} := \mathbb{Q}(\mathrm{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0,\ h > 0,\ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\}\,|_{x=jh, y=kh, t=\tau n}\,.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t),\ \sigma_y \circ \phi = \phi(x, y + h, t),\ \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$.

# Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0\ (1 \le i \le 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p],$$

where $\mathbb{K} := \mathbb{Q}(\mathrm{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0,\ h > 0,\ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \mid_{x=jh, y=kh, t=\tau n}.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t),\ \sigma_y \circ \phi = \phi(x, y + h, t),\ \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$.

# Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \le i \le 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p]\,,$$

where $\mathbb{K} := \mathbb{Q}(\mathrm{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0,\ h > 0,\ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \,|_{x=jh, y=kh, t=\tau n}\,.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t),\ \sigma_y \circ \phi = \phi(x, y + h, t),\ \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$.

# Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \le i \le 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p]\,,$$

where $\mathbb{K} := \mathbb{Q}(\mathrm{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0,\ h > 0,\ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \mid_{x=jh, y=kh, t=\tau n}\,.$$

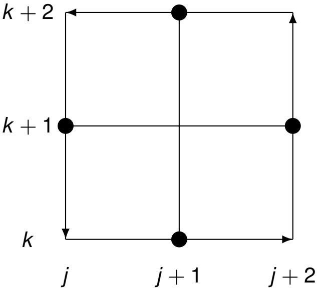
We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t),\ \sigma_y \circ \phi = \phi(x, y + h, t),\ \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$.

# Integration contour

To discretize NSS on the grid choose the integration contour $\Gamma$ in the $(x, y)$ plane



# The Navie-Stokes system in integral form

## Integral conservation law form

$$
\begin{cases}
\oint\limits_{\Gamma} -v dx + u dy = 0\,, \\
\int\limits_{x_j}^{x_{j+2}} \int\limits_{y_k}^{y_{k+2}} u dx dy \Big|_{t_n}^{t_{n+1}} - \int\limits_{t_n}^{t_{n+1}} \left( \oint\limits_{\Gamma} \left(vu - \frac{1}{\mathrm{Re}} u_y\right) dx - \left(u^2 + p - \frac{1}{\mathrm{Re}} u_x\right) dy \right) dt = 0\,, \\
\int\limits_{x_j}^{x_{j+2}} \int\limits_{y_k}^{y_{k+2}} v dx dy \Big|_{t_n}^{t_{n+1}} - \int\limits_{t_n}^{t_{n+1}} \left( \oint\limits_{\Gamma} \left(v^2 + p - \frac{1}{\mathrm{Re}} v_y\right) dx - \left(uv - \frac{1}{\mathrm{Re}} v_x\right) dy \right) dt = 0\,, \\
\oint\limits_{\Gamma} -\left((v^2)_y + (uv)_x + p_y\right) dx + \left((u^2)_x + (vu)_y + p_x\right) dy = 0\,.
\end{cases}
$$

# Additional relations

Now we add integral relations between dependent variables and derivatives

## Exact integral relations

$$
\left\{
\begin{array}{ll}
\int_{x_j}^{x_{j+1}} (u^2)_x dx = u(x_{j+1},y)^2 - u(x_j,y)^2\,, & \int_{y_k}^{y_{k+1}} (v^2)_y dy = v(x,y_{k+1})^2 - v(x,y_k)^2\,, \\
\int_{x_j}^{x_{j+1}} (uv)_x dx = u(x_{j+1},y)v(x_{j+1},y) - u(x_j,y)v(x_j,y)\,, & \\
\int_{y_k}^{y_{k+1}} (uv)_y dy = u(x,y_{k+1})v(x,y_{k+1}) - u(x,y_k)v(x,y_k)\,, & \\
\int_{x_j}^{x_{j+1}} u_x dx = u(x_{j+1},y) - u(x_j,y)\,, & \int_{y_k}^{y_{k+1}} u_y dy = u(x,y_{k+1}) - u(x,y_k)\,, \\
\int_{x_j}^{x_{j+1}} v_x dx = v(x_{j+1},y) - u(x_j,y)\,, & \int_{y_k}^{y_{k+1}} v_y dy = v(x,y_{k+1}) - u(x,y_k)\,, \\
\int_{x_j}^{x_{j+1}} p_x dx = p(x_{j+1},y) - u(x_j,y)\,, & \int_{y_k}^{y_{k+1}} p_y dy = p(x,y_{k+1}) - u(x,y_k)\,.
\end{array}
\right.
$$

# Finite difference approximation 1

By using the midpoint integration approximation for the integrals over $x$ and $y$ and the top-left corner approximation for integration over $t$. Then elimination of partial derivatives from the obtained difference system gives the following FDA with a $5 \times 5$ stencil (G.,Blinkov'09)

$$
\text{FDA 1} = \begin{cases}
{e_1}_{j,k}^n := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
{e_2}_{j,k}^n := \frac{u_{jk}^{n+1} - u_{jk}^n}{\tau} + \frac{{u_{j+1,k}^n}^2 - {u_{j-1,k}^n}^2}{2h} + \frac{v_{j,k+1}^n u_{j,k+1}^n - v_{j,k-1}^n u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{u_{j+2,k}^n - 2u_{jk}^n + u_{j-2,k}^n}{4h^2} + \frac{u_{j,k+2}^n - 2u_{jk}^n + u_{j,k-2}^n}{4h^2} \right) = 0, \\
{e_3}_{j,k}^n := \frac{v_{jk}^{n+1} - v_{jk}^n}{\tau} + \frac{u_{j+1,k}^n v_{j+1,k}^n - u_{j-1,k}^n v_{j-1,k}^n}{2h} \frac{{v_{j,k+1}^n}^2 - {v_{j,k-1}^n}^2}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{v_{j+2,k}^n - 2v_{jk}^n + v_{j-2,k}^n}{4h^2} + \frac{v_{j,k+2}^n - 2v_{jk}^n + v_{j,k-2}^n}{4h^2} \right) = 0, \\
{e_4}_{j,k}^n := \frac{{u_{j+2,k}^n}^2 - 2{u_{j,k}^n}^2 + {u_{j-2,k}^n}^2}{4h^2} + \frac{{v_{j,k+2}^n}^2 - 2{v_{j,k}^n}^2 + {v_{j,k-2}^n}^2}{4h^2} \\
\qquad + 2 \frac{u_{j+1,k+1}^n v_{j+1,k+1}^n - u_{j+1,k-1}^n v_{j+1,k-1}^n - u_{j-1,k+1}^n v_{j-1,k+1}^n + u_{j-1,k-1}^n v_{j-1,k-1}^n}{4h^2} \\
\qquad + \frac{p_{j+2,k}^n - 2p_{jk}^n + p_{j-2,k}^n}{4h^2} + \frac{p_{j,k+2}^n - 2p_{jk}^n + p_{j,k-2}^n}{4h^2} = 0\,.
\end{cases}
$$

# Finite difference approximation 1

By using the midpoint integration approximation for the integrals over $x$ and $y$ and the top-left corner approximation for integration over $t$. Then elimination of partial derivatives from the obtained difference system gives the following FDA with a $5 \times 5$ stencil (G.,Blinkov'09)

$$
\text{FDA 1} = \begin{cases}
{e_1}^n_{j,k} := \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} = 0, \\
{e_2}^n_{j,k} := \frac{u^{n+1}_{jk} - u^n_{jk}}{\tau} + \frac{{u^n_{j+1,k}}^2 - {u^n_{j-1,k}}^2}{2h} + \frac{v^n_{j,k+1} u^n_{j,k+1} - v^n_{j,k-1} u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k} - p^n_{j-1,k}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{u^n_{j+2,k} - 2u^n_{jk} + u^n_{j-2,k}}{4h^2} + \frac{u^n_{j,k+2} - 2u^n_{jk} + u^n_{j,k-2}}{4h^2} \right) = 0, \\
{e_3}^n_{j,k} := \frac{v^{n+1}_{jk} - v^n_{jk}}{\tau} + \frac{u^n_{j+1,k} v^n_{j+1,k} - u^n_{j-1,k} v^n_{j-1,k}}{2h} \frac{{v^n_{j,k+1}}^2 - {v^n_{j,k-1}}^2}{2h} + \frac{p^n_{j,k+1} - p^n_{j,k-1}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{v^n_{j+2,k} - 2v^n_{jk} + v^n_{j-2,k}}{4h^2} + \frac{v^n_{j,k+2} - 2v^n_{jk} + v^n_{j,k-2}}{4h^2} \right) = 0, \\
{e_4}^n_{j,k} := \frac{{u^n_{j+2,k}}^2 - 2{u^n_{j,k}}^2 + {u^n_{j-2,k}}^2}{4h^2} + \frac{{v^n_{j,k+2}}^2 - 2{v^n_{j,k}}^2 + {v^n_{j,k-2}}^2}{4h^2} \\
\qquad + 2 \frac{u^n_{j+1,k+1} v^n_{j+1,k+1} - u^n_{j+1,k-1} v^n_{j+1,k-1} - u^n_{j-1,k+1} v^n_{j-1,k+1} + u^n_{j-1,k-1} v^n_{j-1,k-1}}{4h^2} \\
\qquad + \frac{p^n_{j+2,k} - 2p^n_{jk} + p^n_{j-2,k}}{4h^2} + \frac{p^n_{j,k+2} - 2p^n_{jk} + p^n_{j,k-2}}{4h^2} = 0\,.
\end{cases}
$$

# Finite difference approximation 2

If one applies the trapezoidal approximation to the integral relations for $u_x, u_y, v_x, v_y, u^2)_x, (v^2)_y$ and $p$ instead of the midpoint approximation, then it produces FDA with a $3 \times 3$ stencil (G.,Blinkov'09)

$$
\text{FDA 2} = \begin{cases}
{e_1}_{j,k}^{n} := \frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} = 0, \\
{e_2}_{j,k}^{n} := \frac{u^{n+1}_{jk}-u^n_{jk}}{\tau} + u^n_{jk}\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + v^n_{jk}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k}-p^n_{j-1,k}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k}-2u^n_{jk}+u^n_{j-1,k}}{h^2} + \frac{u^n_{j,k+1}-2u^n_{jk}+u^n_{j,k-1}}{h^2}\right) = 0, \\
{e_3}_{j,k}^{n} := \frac{v^{n+1}_{jk}-v^n_{jk}}{\tau} + u^n_{jk}\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h} + v^n_{jk}\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} + \frac{p^n_{j,k+1}-p^n_{j,k-1}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k}-2v^n_{jk}+v^n_{j-1,k}}{h^2} + \frac{v^n_{j,k+1}-2v^n_{jk}+v^n_{j,k-1}}{h^2}\right) = 0, \\
{e_4}_{j,k}^{n} := \left(\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h}\right)^2 + 2\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \left(\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h}\right)^2 \\
\qquad + \frac{p^n_{j+1,k}-2p^n_{jk}+p^n_{j-1,k}}{h^2} + \frac{p^n_{j,k+1}-2p^n_{jk}+p^n_{j,k-1}}{h^2} = 0
\end{cases}
$$

# Finite difference approximation 2

If one applies the trapezoidal approximation to the integral relations for $u_x, u_y, v_x, v_y, u^2)_x, (v^2)_y$ and $p$ instead of the midpoint approximation, then it produces FDA with a $3 \times 3$ stencil (G.,Blinkov'09)

$$
\text{FDA 2} = \begin{cases}
{e_1}^n_{j,k} := \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} = 0, \\
{e_2}^n_{j,k} := \frac{u^{n+1}_{jk} - u^n_{jk}}{\tau} + u^n_{jk} \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} + v^n_{jk} \frac{u^n_{j,k+1} - u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k} - p^n_{j-1,k}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{u^n_{j+1,k} - 2u^n_{jk} + u^n_{j-1,k}}{h^2} + \frac{u^n_{j,k+1} - 2u^n_{jk} + u^n_{j,k-1}}{h^2} \right) = 0, \\
{e_3}^n_{j,k} := \frac{v^{n+1}_{jk} - v^n_{jk}}{\tau} + u^n_{jk} \frac{v^n_{j+1,k} - v^n_{j-1,k}}{2h} + v^n_{jk} \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} + \frac{p^n_{j,k+1} - p^n_{j,k-1}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}} \left( \frac{v^n_{j+1,k} - 2v^n_{jk} + v^n_{j-1,k}}{h^2} + \frac{v^n_{j,k+1} - 2v^n_{jk} + v^n_{j,k-1}}{h^2} \right) = 0, \\
{e_4}^n_{j,k} := \left( \frac{u^n_{j+1,k} - u^n_{j-1,k}}{2h} \right)^2 + 2 \frac{v^n_{j+1,k} - v^n_{j-1,k}}{2h} \frac{u^n_{j,k+1} - u^n_{j,k-1}}{2h} + \left( \frac{v^n_{j,k+1} - v^n_{j,k-1}}{2h} \right)^2 \\
\qquad + \frac{p^n_{j+1,k} - 2p^n_{jk} + p^n_{j-1,k}}{h^2} + \frac{p^n_{j,k+1} - 2p^n_{jk} + p^n_{j,k-1}}{h^2} = 0
\end{cases}
$$

# Finite difference approximation 3

The third approximation with $3\times 3$ stencil is obtained from NSS by the conventional discretization what consists of replacing the temporal derivatives with the forward differences and the spatial derivatives with the central differences.

$$
\text{FDA } 3 = \left\{
\begin{aligned}
& {e_1}^n_{j,k} := \frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} = 0,\\
& {e_2}^n_{j,k} := \frac{u^{n+1}_{jk}-u^n_{jk}}{\tau} + u^n_{jk}\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + v^n_{jk}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k}-p^n_{j-1,k}}{2h}\\
& \qquad - \frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k}-2u^n_{jk}+u^n_{j-1,k}}{h^2} + \frac{u^n_{j,k+1}-2u^n_{jk}+u^n_{j,k-1}}{h^2}\right) = 0,\\
& {e_3}^n_{j,k} := \frac{v^{n+1}_{jk}-v^n_{jk}}{\tau} + u^n_{jk}\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h} + v^n_{jk}\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} + \frac{p^n_{j,k+1}-p^n_{j,k-1}}{2h}\\
& \qquad - \frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k}-2v^n_{jk}+v^n_{j-1,k}}{h^2} + \frac{v^n_{j,k+1}-2v^n_{jk}+v^n_{j,k-1}}{h^2}\right) = 0,\\
& {e_4}^n_{j,k} := \left(\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h}\right)^2 + 2\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \left(\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h}\right)^2\\
& \qquad + \frac{p^n_{j+1,k}-2p^n_{jk}+p^n_{j-1,k}}{h^2} + \frac{p^n_{j,k+1}-2p^n_{jk}+p^n_{j,k-1}}{h^2} = 0
\end{aligned}
\right.
$$

# Finite difference approximation 3

The third approximation with $3 \times 3$ stencil is obtained from NSS by the conventional discretization what consists of replacing the temporal derivatives with the forward differences and the spatial derivatives with the central differences.

$$
\text{FDA } 3 = \begin{cases}
{e_1}^n_{j,k} := \frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + \frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} = 0, \\
{e_2}^n_{j,k} := \frac{u^{n+1}_{jk}-u^n_{jk}}{\tau} + u^n_{jk}\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h} + v^n_{jk}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \frac{p^n_{j+1,k}-p^n_{j-1,k}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}}\left(\frac{u^n_{j+1,k}-2u^n_{jk}+u^n_{j-1,k}}{h^2} + \frac{u^n_{j,k+1}-2u^n_{jk}+u^n_{j,k-1}}{h^2}\right) = 0, \\
{e_3}^n_{j,k} := \frac{v^{n+1}_{jk}-v^n_{jk}}{\tau} + u^n_{jk}\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h} + v^n_{jk}\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h} + \frac{p^n_{j,k+1}-p^n_{j,k-1}}{2h} \\
\qquad - \frac{1}{\mathrm{Re}}\left(\frac{v^n_{j+1,k}-2v^n_{jk}+v^n_{j-1,k}}{h^2} + \frac{v^n_{j,k+1}-2v^n_{jk}+v^n_{j,k-1}}{h^2}\right) = 0, \\
{e_4}^n_{j,k} := \left(\frac{u^n_{j+1,k}-u^n_{j-1,k}}{2h}\right)^2 + 2\frac{v^n_{j+1,k}-v^n_{j-1,k}}{2h}\frac{u^n_{j,k+1}-u^n_{j,k-1}}{2h} + \left(\frac{v^n_{j,k+1}-v^n_{j,k-1}}{2h}\right)^2 \\
\qquad + \frac{p^n_{j+1,k}-2p^n_{jk}+p^n_{j-1,k}}{h^2} + \frac{p^n_{j,k+1}-2p^n_{jk}+p^n_{j,k-1}}{h^2} = 0
\end{cases}
$$

# Contents

# Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1}(\sigma_y \circ \tilde{f})^{k_2}(\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!]\,.$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0\,, \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

**Differential and difference consequences**

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

# Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1} (\sigma_y \circ \tilde{f})^{k_2} (\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!] \,.$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0 \,, \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

### Differential and difference consequences

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

# Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1} (\sigma_y \circ \tilde{f})^{k_2} (\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!] \,.$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0 \,, \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

**Differential and difference consequences**

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

# Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1}(\sigma_y \circ \tilde{f})^{k_2}(\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!] \,.$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0\,, \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

## Differential and difference consequences

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

# Differential and difference consequences

A perfect difference ideal $[\![\tilde{F}]\!]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1}(\sigma_y \circ \tilde{f})^{k_2}(\sigma_t \circ \tilde{f})^{k_3} \in [\![\tilde{F}]\!] \Longrightarrow \tilde{f} \in [\![\tilde{F}]\!]\,.$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset R$ (NSS) generates radical differential ideal $[\![F]\!]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0\,, \quad \tilde{F} := \{\tilde{f}_1, \ldots \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

## Differential and difference consequences

A differential (resp. difference) polynomial $f \in R$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if $f \in [\![F]\!]$ (resp. $\tilde{f} \in [\![\tilde{F}]\!]$).

# Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \rhd f$ if $f$ does not contain the grid spacings $h, \tau$ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h,\tau) = 0$ where $O(h,\tau)$ denotes expression which vanishes when $h$ and $\tau$ go to zero.

**Definition**

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with $F$ if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F})\ (\exists f \in F)\ [\tilde{f} \rhd f].$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
2. A w-consistent FDA may not be good in view of inheritance of properties of the underlying differential equation(s) at the discrete level.

# Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \rhd f$ if $f$ does not contain the grid spacings $h, \tau$ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h,\tau) = 0$ where $O(h,\tau)$ denotes expression which vanishes when $h$ and $\tau$ go to zero.

**Definition**

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with $F$ if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F})\ (\exists f \in F)\ [\,\tilde{f} \rhd f\,]\,.$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
2. A w-consistent FDA may not be good in view of inheritance of properties of the underlying differential equation(s) at the discrete level.

# Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \rhd f$ if $f$ does not contain the grid spacings $h, \tau$ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h, \tau) = 0$ where $O(h, \tau)$ denotes expression which vanishes when $h$ and $\tau$ go to zero.

**Definition**

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with $F$ if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F})\ (\exists f \in F)\ [\tilde{f} \rhd f].$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
2. A w-consistent FDA may not be good in view of inheritance of properties of the underlying differential equation(s) at the discrete level.

# Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \rhd f$ if $f$ does not contain the grid spacings $h, \tau$ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h, \tau) = 0$ where $O(h, \tau)$ denotes expression which vanishes when $h$ and $\tau$ go to zero.

**Definition**

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with $F$ if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F}) \; (\exists f \in F) \; [\, \tilde{f} \rhd f \,] \,.$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
2. A w-consistent FDA may not be good in view of inheritance of properties of the underlying differential equation(s) at the discrete level.

# Strong consistency

**Definition**

An FDA to PDE(s) is strongly consistent or s-consistent if

$$(\forall \tilde{f} \in [\![\tilde{F}]\!]) \, (\exists f \in [F]) \, [\, \tilde{f} \rhd f \,] \, .$$

The algorithmic approach (G'12) to verification of s-consistency is based on the following statement.

**Theorem**

A difference approximation $\tilde{F} \subset \mathcal{R}$ to $F \subset R$ is s-consistent iff a (reduced) standard basis $G$ of the difference ideal $[\tilde{F}]$ satisfies

$$(\forall g \in G) \, (\exists f \in [F]) \, [\, g \rhd f \,] \, .$$

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[\![F]\!]$ by performing the involutive (Janet) reduction.

# Strong consistency

**Definition**

An FDA to PDE(s) is strongly consistent or s-consistent if

$$(\forall \tilde{f} \in [\![\tilde{F}]\!]) \, (\exists f \in [F]) \, [\, \tilde{f} \rhd f \,] \,.$$

The algorithmic approach (G'12) to verification of s-consistency is based on the following statement.

**Theorem**

A difference approximation $\tilde{F} \subset \mathcal{R}$ to $F \subset R$ is s-consistent iff a (reduced) standard basis $G$ of the difference ideal $[\tilde{F}]$ satisfies

$$(\forall g \in G) \, (\exists f \in [F]) \, [\, g \rhd f \,] \,.$$

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[\![F]\!]$ by performing the involutive (Janet) reduction.

# Strong consistency

**Definition**

An FDA to PDE(s) is strongly consistent or s-consistent if

$$(\forall \tilde{f} \in [\![\tilde{F}]\!])\,(\exists f \in [F])\,[\,\tilde{f} \rhd f\,]\,.$$

The algorithmic approach (G'12) to verification of s-consistency is based on the following statement.

**Theorem**

A difference approximation $\tilde{F} \subset \mathcal{R}$ to $F \subset R$ is s-consistent iff a (reduced) standard basis $G$ of the difference ideal $[\tilde{F}]$ satisfies

$$(\forall g \in G)\,(\exists f \in [F])\,[\,g \rhd f\,]\,.$$

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[\![F]\!]$ by performing the involutive (Janet) reduction.

# Strong consistency

**Definition**

An FDA to PDE(s) is strongly consistent or s-consistent if

$$(\forall \tilde{f} \in [\![\tilde{F}]\!])\,(\exists f \in [F])\,[\,\tilde{f} \rhd f\,]\,.$$

The algorithmic approach (G'12) to verification of s-consistency is based on the following statement.

**Theorem**

A difference approximation $\tilde{F} \subset \mathcal{R}$ to $F \subset R$ is s-consistent iff a (reduced) standard basis $G$ of the difference ideal $[\tilde{F}]$ satisfies

$$(\forall g \in G)\,(\exists f \in [F])\,[\,g \rhd f\,]\,.$$

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[\![F]\!]$ by performing the involutive (Janet) reduction.

# S-consistency analysis of FDA 1,2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

$$\tilde{F} := \{ {e_1}_{j,k}^n, {e_2}_{j,k}^n, {e_3}_{j,k}^n, {e_4}_{j,k}^n \}$$

about the grid point $\{hj, hk, n\tau\}$ when the grid spacings $h$ and $\tau$ go to zero.

**Proposition [Amodio,Blinkov,G.,La Scala'13]**

Among weakly consistent FDAs 1,2, and 3 only FDA 1 is strongly consistent.

**Corollary**

A standard basis $G$ of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

$$(\forall g \in G)\ (\exists f \in [F])\ [\, g \triangleright f \,].$$

# S-consistency analysis of FDA 1,2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

$$\tilde{F} := \{ {e_1}^n_{j,k}, {e_2}^n_{j,k}, {e_3}^n_{j,k}, {e_4}^n_{j,k} \}$$

about the grid point $\{hj, hk, n\tau\}$ when the grid spacings $h$ and $\tau$ go to zero.

**Proposition [Amodio,Blinkov,G.,La Scala'13]**

Among weakly consistent FDAs 1,2, and 3 only FDA 1 is strongly consistent.

**Corollary**

A standard basis $G$ of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

$$(\forall g \in G)\ (\exists f \in [F])\ [\, g \triangleright f \,].$$

# S-consistency analysis of FDA 1,2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

$$\tilde{F} := \{ {e_1}_{j,k}^{n}, {e_2}_{j,k}^{n}, {e_3}_{j,k}^{n}, {e_4}_{j,k}^{n} \}$$

about the grid point $\{hj, hk, n\tau\}$ when the grid spacings $h$ and $\tau$ go to zero.

**Proposition [Amodio,Blinkov,G.,La Scala'13]**

Among weakly consistent FDAs 1,2, and 3 only FDA 1 is strongly consistent.

**Corollary**

A standard basis $G$ of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

$$(\forall g \in G)\ (\exists f \in [F])\ [\, g \rhd f \,].$$

# S-consistency analysis of FDA 1,2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

$$\tilde{F} := \{e_1{}^n_{j,k}, e_2{}^n_{j,k}, e_3{}^n_{j,k}, e_4{}^n_{j,k}\}$$

about the grid point $\{hj, hk, n\tau\}$ when the grid spacings $h$ and $\tau$ go to zero.

**Proposition [Amodio,Blinkov,G.,La Scala'13]**

Among weakly consistent FDAs 1,2, and 3 only FDA 1 is strongly consistent.

**Corollary**

A standard basis $G$ of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

$$(\forall g \in G)\,(\exists f \in [F])\,[\,g \triangleright f\,].$$

# Contents

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$
\begin{aligned}
u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\
v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\
p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,.
\end{aligned}
$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$
\begin{aligned}
u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\
v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\
p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,.
\end{aligned}
$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$
\begin{aligned}
u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y) \,, \\
v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y) \,, \\
p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4 \,.
\end{aligned}
$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$\begin{aligned} u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\ v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\ p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,. \end{aligned}$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$\begin{aligned} u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\ v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\ p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,. \end{aligned}$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Numerical Experiments

Suppose (Amodio,Blinkov,G.,La Scala'13) that the NSS is defined for $t \geq 0$ in the square domain $\Omega = [0, \pi] \times [0, \pi]$ and provide initial conditions for $t = 0$ and boundary conditions for $t > 0$ and $(x, y) \in \partial\Omega$ according to the exact solution (Pearson'64)

$$\begin{aligned} u &:= -e^{-2t/\mathrm{Re}} \cos(x) \sin(y)\,, \\ v &:= e^{-2t/\mathrm{Re}} \sin(x) \cos(y)\,, \\ p &:= -e^{-4t/\mathrm{Re}} (\cos(2x) + \cos(2y))/4\,. \end{aligned}$$

Let $[0, \pi] \times [0, \pi]$ be discretized in the $(x, y)$-directions by means of the $(m+2)^2$ equispaced points $x_j = jh$ and $y_k = kh$, for $j, k = 0, \ldots m+1$, and $h = \pi/(m+1)$.

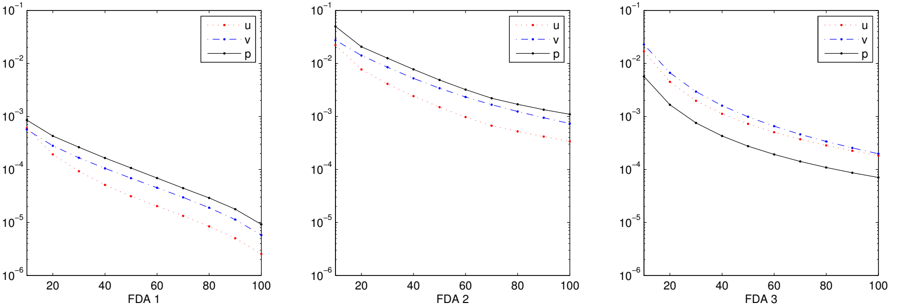
Then, starting from IC, the 2nd and the 3rd equations in every FDA give explicit formulae to compute $u_{jk}^{n+1}$ and $v_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 4th equation can be used to derive a $m^2 \times m^2$ linear system that computes the unknowns $p_{jk}^{n+1}$ for $j, k = 1, \ldots, m$. The 1st equation is unnecessary and may be used to validate the obtained solution. This procedure is iterated for $n = 0, 1, \ldots, N$ being $t_f = N\tau$ the end point of the time interval.

# Relative error for $\mathrm{Re} = 10^5$

We computed error by means of the formula

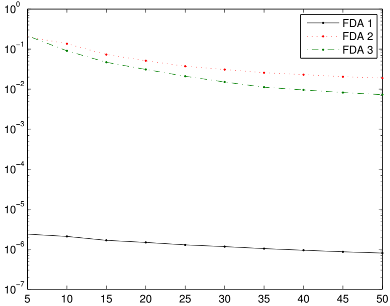
$$e_g = \max_{j,k} \frac{|g^N_{j,k} - g(x_j, y_k, t_f)|}{1 + |g(x_j, y_k, t_f)|}.$$

where $g \in \{u, v, p\}$ and $g(x, y, t)$ belongs to the exact solution.



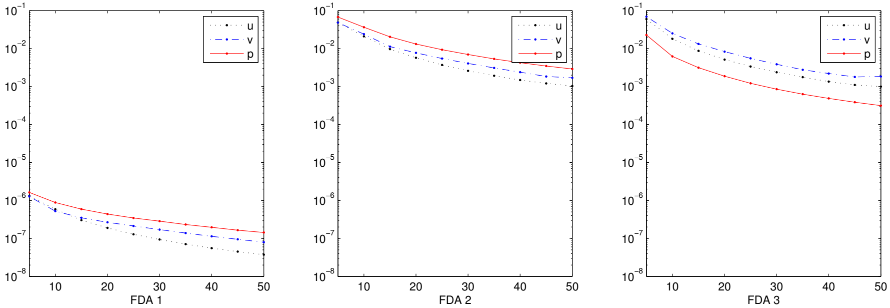

Relative error for $N = 10$, $t_f = N\tau = 1$, $\mathrm{Re} = 10^5$ and varying $m$ from 5 to 50

# Computed value of $u_x + v_y$



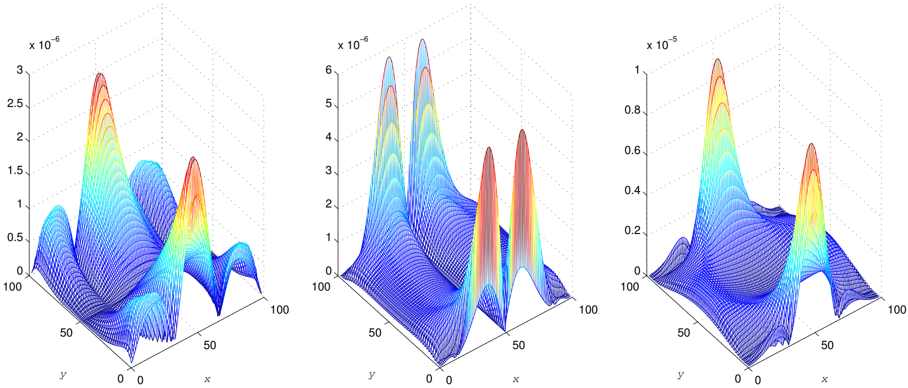

Computed value of $f_1$ in NSS for FDA 1, FDA 2 and FDA 3 with $N = 10$, $t_f = 1$, $Re = 10^5$ and varying $m$ from 5 to 50

# Relative error for Re = $10^2$




Computed errors in $u, v$ and $p$ for FDA 1 (left), FDA 2 (middle) and FDA 3 (right): $N = 40$, $t_f = 1$, $Re = 10^2$ and varying $m$ from 10 to 100

# Relative error in $u$, $v$ and $p$ with FDA 1 for $\mathrm{Re} = 10^2$




Computed error with FDA 1 ($u$, $v$ and $p$, respectively): $N = 40$, $t_f = 1$, $Re = 10^2$ and $m = 100$

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# Conclusions

## Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.
- The structure of FDA depends on the numerical methods used to approximate integrals.
- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.
- This result is at variance with universally accepted opinion that discretization with a more compact stencil is numerically favoured.
- Our computer experimentation revealed much better numerical behavior of the s-consistent approximation in comparison with the considered s-inconsistent ones.

# References I


C.E. Pearson (1964).
A computational method for time dependent two dimensional incompressible viscous flow problems.
Report No. SRRC-RR-64-17, Sperry-Rand Research Center, Sudbury, Mass.

V.P. Gerdt, Yu. A. Blinkov, V.V. Mozzhilkin (2006).
Gröbner Bases and Generation of Difference Schemes for Partial Differential Equation.
Symmetry, Integrability and Geometry: Methods and Applications. Vol. 2. P. 26. arXiv:math.RA/0605334

V.P. Gerdt, Yu. A. Blinkov (2009).
Involution and Difference Schemes for the Navier–Stokes Equations.
Proceedings of CASC 2009 (September 13-17, Kobe, Japan), V.P.Gerdt, E.W.Mayr, E.V.Vorozhtsov (Eds.), LNCS, vol. 5743, Springer-Verlag, Berlin, pp. 94–105.

V.P. Gerdt (2012).
Consistency Analysis of Finite Difference Approximations to PDE Systems.
Proceedings of MMCP 2011 (July 3-8, 2011, Stará Lesná, High Tatra Mountains, Slovakia), G.Adam, J.Buša, M.Hnatič (Eds.), LNCS, vol. 7125, Springer-Verlag, Berlin, pp. 28–42. arXiv:math.AP/1107.4269

# References II


Yu.A. Blinkov, S.V. Ivanov, L.I. Mogilevich (2012).
Mathematical and Computer Modeling of Non-linear Deformation Waves in the Cover with Viscous Liquid Inside.
Herald of Peoples' Friendship University of Russia: Series Mathematics, Information Sciences, Physics, No.3, 2012, 52–60 (in Russian).

V. P. Gerdt, D. Robertz (2012).
Computation of Difference Gröbner Bases.
Computer Science Journal of Moldova. Vol. 20, No.2, 2012, 203–226.
arXiv:1206.3463 [cs.SC]

P. Amodio, Yu. Blinkov, V. Gerdt, R. La Scala (2013).
On Consistency of Finite Difference Approximations to the Navier-Stokes Equations.
Proceedins of CASC 2013 (September 9-13, 2013, Berlin, Germany), V.P.Gerdt, W.Koepff, E.W.Mayr, E.V.Vorozhtsov (Eds.), LNCS, vol. 8136, Springer-Verlag, Berlin, pp. 46–60. arXiv:math.NA/1307.0914 [math.NA]