Computer algebra aided numerical solving nonlinear PDEs

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In the finite difference method (FDM) differential equations (PDE(s)) are replaced with their finite difference approximation (FDA) on a grid with spacings \( h := \{h_1, \ldots, h_n\} \).

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\text{PDE(s)} \implies \text{FDA}
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The initial conditions (ICs) and/or boundary conditions (BCs) are also discretized. Then, together with FDA it gives a finite difference scheme (FDS).
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Solving PDEs in Practice

PDE(s) + IC(s) or/and BC(s)  
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Discretization (FDM, FEM, FVM)  
⇓
Algebraic (difference) equations  
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Numerical solving  
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Approximate solution

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FDA requirements

Convergence of an approximate solution to a solution to PDE(s) at $|h| \longrightarrow 0$.

Challenge: find FDA whose solutions converge to solutions to PDE(s).

Such FDA must inherit at the discrete level all algebraic properties of PDE(s) such as conservation laws, symmetries, maximum principle, etc.).

For polynomially nonlinear PDE(s) s(strong)-consistency of FDA (Gerdt’12).

S-consistency

Definition. FDA is s-consistent with PDE(s) if any difference consequence of FDA in the limit $|h| \rightarrow 0$ is reduced to a differential consequence of PDE(s).

Theorem. For polynomial nonlinear PDE(s) its FDA is s-consistent iff all elements in a canonical form of FDA (Gröbner basis) are reduced to differential consequences of PDEs.
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Family of KdV-like equations

We consider a 5-parameter family of the 3rd order quasilinear PDEs

\[ \Phi := \{ u_t + (f_1 u + f_2 u^2 + f_3 u^3)_x + u_{xxx} + s_2 u_{xx} + su = 0 \mid f_1, f_2, f_3, s, s_2 \in \mathbb{R} \} . \]

Motivation

1. Korteweg-de Vries (KdV) and modified KdV (MKdV) equations are contained in \( \Phi \)

\[ u_t + u_{xxx} + 6uu_x = 0 \in \Phi, \quad u_t + u_{xxx} + 6u^2u_x = 0 \in \Phi. \]

They possess infinitely many conservation laws and symmetries.

2. Equations in \( \Phi \) admit a wide class of exact solutions.

3. Equations in \( \Phi \) describe propagation of nonlinear deformation waves in elastic cylinder shells containing viscous incompressible liquid (Blinkov, Ivanov, Mogilevich’2012). The sign of \( s \) characterizes the shell material: nonorganic \((s < 0)\), living organisms \((s > 0)\), rubber \((s = 0)\).
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Generation of FDA

We use our algorithmic approach (Gerdt, Blinkov, Mozghilkin’06) based on FVM combined with numerical integration and difference elimination.

1. Convert into integral form (Green’s theorem)

\[ \oint_{\partial \Omega} - (F + u_{xx} + s_2 u_x) \, dt + u \, dx + s \iint_{\Omega} u \, dt \, dx = 0, \quad F := f_1 u + f_2 u^2 + f_3 u^3. \]

Ω is arbitrary region in the plane (t, x) bounded by ∂Ω.

2. Choose of a “control volume” Ω

3. Add the integral relations

\[ \int_{x_j}^{x_{j+1}} u_x \, dx = u(t, x_{j+1}) - u(t, x_j), \quad \int_{x_j}^{x_{j+1}} u_{xx} \, dx = u_x(t, x_{j+1}) - u_x(t, x_j). \]
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Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply

- the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$, and for integration of $u_x$ in the additional relation
- the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$\begin{align*}
- \left( (F_j^n + F_j^{n+1} - F_{j+2}^n - F_{j+2}^{n+1}) + (u_{xx}^n_j + u_{xx}^{n+1}_j - u_{xx}^n_{j+2} - u_{xx}^{n+1}_{j+2}) + \\
&\quad s_2 \left( u_{xj}^n + u_{xj}^{n+1} - u_{xj}^n_{j+2} - u_{xj}^{n+1}_{j+2} \right) \right) \cdot \frac{\tau}{2} + \\
&\quad + (u_{j+1}^n - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1}_j + u_{j+1}^n) \cdot h\tau = 0,
\end{align*}$$

$$\begin{align*}
(u_{xj+1}^n + u_{xj}^n) \cdot \frac{h}{2} = u_{j+1}^n - u_{j}^n, \quad u_{xxj+1}^n \cdot 2h = u_{xj+2}^n - u_{xj}^n.
\end{align*}$$
Discretization

Set \( t_{n+1} - t_n = \tau \), \( x_{j+1} - x_j = h \) and apply

- the trapezoidal rule for integration over \( t \), for integration of \( u \) and \( u_{xx} \) over \( x \) and for integration of \( u_x \) in the additional relation
- the midpoint rule for the other integrations

In the standard notations for a grid function \( \phi^n_j := \phi(t_n, x_j) \) this gives

\[
- \left( \left( F^n_j + F^{n+1}_j - F^n_{j+2} - F^{n+1}_{j+2} \right) + \left( u_{xx}^n_j + u_{xx}^{n+1}_j - u_{xx}^n_{j+2} - u_{xx}^{n+1}_{j+2} \right) + s_2 \left( u_{xj}^n + u_{xj}^{n+1} - u_{xj+2}^n - u_{xj+2}^{n+1} \right) \right) \cdot \frac{\tau}{2} + \\
+ (u^{n+1}_{j+1} - u^n_{j+1}) \cdot 2h + s(u^{n+1}_{j+1} + u^n_{j+1}) \cdot h\tau = 0, \\
(u^n_{xj+1} + u^n_{xj}) \cdot \frac{h}{2} = u^n_{j+1} - u^n_j, \quad u_{xx}^n_{j+1} \cdot 2h = u^n_{xj+2} - u^n_{xj}.
\]
Set \( t_{n+1} - t_n = \tau \), \( x_{j+1} - x_j = h \) and apply

- the trapezoidal rule for integration over \( t \), for integration of \( u \) and \( u_{xx} \) over \( x \) and for integration of \( u_x \) in the additional relation
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\quad s_2 \left( u_{x}^n_j + u_{x}^{n+1}_j - u_{x}^n_{j+2} - u_{x}^{n+1}_{j+2} \right) \cdot \frac{\tau}{2} + \\
\quad + (u_{j+1}^{n+1} - u_{j+1}^n) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h\tau = 0, \\
(u_{x}^n_{j+1} + u_{x}^n_j) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad u_{xx}^n_{j+1} \cdot 2h = u_{x}^n_{j+2} - u_{x}^n_j.
\]
Discretization

Set \( t_{n+1} - t_n = \tau, \ x_{j+1} - x_j = h \) and apply
- the trapezoidal rule for integration over \( t \), for integration of \( u \) and \( u_{xx} \) over \( x \) and for integration of \( u_x \) in the additional relation
- the midpoint rule for the other integrations

In the standard notations for a grid function \( \phi_j^n := \phi(t_n, x_j) \) this gives

\[
- \left( \left( F_j^n + F_{j+1}^n - F_{j+2}^n - F_{j+2}^{n+1} \right) + \left( u_{xx}^n + u_{xx}^{n+1} - u_{xx}^{n+2} - u_{xx}^{n+2} \right) + s_2 \left( u_x^n + u_{x+1}^{n+1} - u_{x+2}^n - u_{x+2}^{n+1} \right) \right) \cdot \frac{\tau}{2} + \\
+ (u_{j+1}^{n+1} - u_j^n) \cdot 2h + s(u_{j+1}^{n+1} + u_j^n) \cdot h\tau = 0,
\]

\[
(u_{xj+1}^n + u_{xj}^n) \cdot \frac{h}{2} = u_{j+1}^n - u_j^n, \quad u_{xxj+1}^n \cdot 2h = u_{xj+2}^n - u_{xj}^n.
\]
Discretization

Set $t_{n+1} - t_n = \tau$, $x_{j+1} - x_j = h$ and apply

- the trapezoidal rule for integration over $t$, for integration of $u$ and $u_{xx}$ over $x$ and for integration of $u_x$ in the additional relation
- the midpoint rule for the other integrations

In the standard notations for a grid function $\phi_j^n := \phi(t_n, x_j)$ this gives

$$
- \left( (F^n_j + F^{n+1}_j - F^n_{j+2} - F^{n+1}_{j+2}) + (u_{xx}^n_j + u_{xx}^{n+1}_j - u_{xx}^n_{j+2} - u_{xx}^{n+1}_{j+2}) + s_2 \left( u_x^n_j + u_x^{n+1}_j - u_x^n_{j+2} - u_x^{n+1}_{j+2} \right) \right) \cdot \frac{\tau}{2} + \\
+ \left( u_{j+1}^{n+1} - u_{j+1}^n \right) \cdot 2h + s(u_{j+1}^{n+1} + u_{j+1}^n) \cdot h \tau = 0, \\
(\frac{u_{xj+1}^n + u_{xj}^n}{2} - 2) = u_{j+1}^n - u_j^n, \quad u_{xx}^n_{j+1} \cdot 2h = u_{xj+2}^n - u_{xj}^n.
$$
Discretization

Set \( t_{n+1} - t_n = \tau \), \( x_{j+1} - x_j = h \) and apply

- the trapezoidal rule for integration over \( t \), for integration of \( u \) and \( u_{xx} \) over \( x \) and for integration of \( u_x \) in the additional relation
- the midpoint rule for the other integrations

In the standard notations for a grid function \( \phi^n_j := \phi(t_n, x_j) \) this gives

\[
\begin{align*}
&- \left( \left( F^n_j + F^{n+1}_j - F^n_{j+2} - F^{n+1}_{j+2} \right) + \left( u_{xx}^n_j + u_{xx}^{n+1}_j - u_{xx}^n_{j+2} - u_{xx}^{n+1}_{j+2} \right) + \\
&\quad s_2 \left( u_{x}^n_j + u_{x}^{n+1}_j - u_{x}^n_{j+2} - u_{x}^{n+1}_{j+2} \right) \right) \cdot \frac{\tau}{2} + \\
&\quad + (u'_{j+1}^{n+1} - u'_{j+1}^n) \cdot 2h + s(u'^{n+1}_{j+1} + u'^n_{j+1}) \cdot h\tau = 0, \quad (u_{x}^n_{j+1} + u_{x}^n_j) \cdot \frac{h}{2} = u^n_{j+1} - u^n_j, \quad u_{xx}^n_{j+1} \cdot 2h = u^n_{xj+2} - u^n_{xj}. 
\end{align*}
\]
Elimination of $u_x$ and $u_{xx}$ by computing a difference Gröbner basis for an elimination monomial ordering extending the ranking $u_{xx} \succ u_x \succ u \succ F$. The input for the Maple package LDA (Gerdt, Robertz’12)

```maple
> restart;
> libname:=libname, "/usr/local/lib/LDA";
> L:=-((F(n,j)+F(n+1,j)-F(n,j+2)-F(n+1,j+2)) +
  (uxx(n,j)+uxx(n+1,j)-uxx(n,j+2)-uxx(n+1,j+2)) +
  s2(ux(n,j)+ux(n+1,j)-ux(n,j+2)-ux(n+1,j+2)) )tau/2 +
  (u(n+1,j+1)-u(n,j+1))2h,
> s(ux(n,j+1)+ux(n,j))h/2-(u(n,j+1)-u(n,j)),
> 2uxx(n,j+1)h-(ux(n,j+2)-ux(n,j)));
> JanetBasis(L, [n,j], [uxx,ux,u,F],2):
> collect(%[1,1]/(4*tau*h^3),[s,s2,tau,h]);
```
Difference elimination II

\[
\begin{align*}
&su(n + 1, j + 2) + u(n, j + 2) + s2 \frac{1}{2h^2}(-2u(n + 1, j + 2) + u(n, j + 3) + \\
&+u(n, j + 1) + u(n + 1, j + 1) - 2u(n, j + 2) + u(n + 1, j + 3)) + \\
&+\frac{F(n + 1, j + 3) - F(n + 1, j + 1) + F(n, j + 3) - F(n, j + 1)}{4h} + \\
&+ \frac{1}{4h^3}(-u(n, j) - 2u(n, j + 3) + 2u(n + 1, j + 1) - u(n + 1, j) + \\
&+2u(n, j + 1) + u(n + 1, j + 4) + u(n, j + 4) - 2u(n + 1, j + 3)) + \\
&+ \frac{u(n + 1, j + 2) - u(n, j + 2)}{\tau}
\end{align*}
\]
Strong consistency

**FDA**

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{\tau} + \frac{(F_{j+1}^{n+1} - F_{j-1}^{n+1}) + (F_{j+1}^{n} - F_{j-1}^{n})}{4h} + \\
\left(\frac{u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}}{4h^3}\right) + \\
\left(\frac{u_{j+2}^{n} - 2u_{j+1}^{n} + 2u_{j-1}^{n} - u_{j-2}^{n}}{4h^3}\right)
\]

\[
+ s_2 \frac{(u_{j+1}^{n+1} - 2u_{j+1}^{n+1} + u_{j-1}^{n+1}) + (u_{j+1}^{n} - 2u_{j+1}^{n} + u_{j-1}^{n})}{2h^2} + s_2 \frac{u_{j+1} + u_{j}}{2} = 0
\]

**S-consistency**

If one chooses an admissible difference monomial ordering such that \(u_{j+2}^{n+1}\) is the leading monomial in the above FDA, then its left-hand side is a Gröbner basis. In the limit \(\tau, h \rightarrow 0\) it is reduced to the original PDE.
KdV-like PDEs

Finite Difference Approximation

Strong consistency

**FDA**

\[
\begin{align*}
    u_{j+1}^{n+1} - u_j^n &\quad \frac{\tau}{\tau} + \frac{(F_{j+1}^{n+1} - F_{j-1}^{n+1}) + (F_{j+1}^n - F_{j-1}^n)}{4h} + \\
    (u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}) &\quad 4h^3 + \\
    \frac{(u_{j+1}^{n+1} - 2u_{j+1}^{n+1} + u_{j-1}^{n+1}) + (u_{j+1}^n - 2u_{j+1}^n + u_{j-1}^n)}{2h^2} + \\
    s_2 &\quad \frac{u_{j+1}^{n+1} + u_j^n}{2} = 0
\end{align*}
\]

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If one chooses an admissible difference monomial ordering such that \( u_{j+2}^{n+1} \) is the leading monomial in the above FDA, then its left-hand side is a Gröbner basis. In the limit \( \tau, h \longrightarrow 0 \) it is reduced to the original PDE.
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   - Involutive Navier-Stokes System
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Exact solutions

Let \( s = 0, \ U, n_0, n_1, d_0, d_1, k, \omega, \in \mathbb{R} \) and solution be the form

\[
    u = U \frac{n_0 + n_1 \exp(\zeta) + \exp(2\zeta)}{d_0 + d_1 \exp(\zeta) + \exp(2\zeta)}, \quad \zeta = kx - \omega t
\]

Then the method of indefinite coefficients gives the following multi-parametric solution

\[
    f_1 = \frac{\omega}{k} + 2k^2 + \frac{6k^2 d_0^2 (2n_0 - n_1^2)}{d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2}, \\
    f_2 = - \frac{6(d_0 + n_0 - n_1 d_1) d_0^2 k^2}{U(d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)}, \\
    f_3 = \frac{2k^2 d_0^2 (2d_0 - d_1^2)}{U^2(d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)}, \\
    s_2 = - \frac{3kd_1(d_0 - n_0)(d_0 d_1 - 2n_1 d_0 + n_0 d_1)}{d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2}.
\]
Exact solutions

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$$f_3 = \frac{2k^2 d_0^2 (2d_0 - d_1^2)}{U^2(d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2)},$$

$$s_2 = -\frac{3kd_1 (d_0 - n_0)(d_0 d_1 - 2n_1 d_0 + n_0 d_1)}{d_0^2 (n_1 - d_1)^2 + (n_1 d_0 - n_0 d_1)^2}.$$
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\]

\[
    f_2 = -\frac{6(d_0 + n_0 - n_1d_1)d_0^2k^2}{U(d_0^2(n_1 - d_1)^2 + (n_1d_0 - n_0d_1)^2)},
\]

\[
    f_3 = \frac{2k^2d_0^2(2d_0 - d_1^2)}{U^2(d_0^2(n_1 - d_1)^2 + (n_1d_0 - n_0d_1)^2)},
\]

\[
    s_2 = -\frac{3kd_1(d_0 - n_0)(d_0d_1 - 2n_1d_0 + n_0d_1)}{d_0^2(n_1 - d_1)^2 + (n_1d_0 - n_0d_1)^2}.
\]
Exact solutions with $u \neq \text{const}$

There are 4 types of such solutions:

1. \( \{d_0 = 0, \ n_0 = -d_1^2 + n_1 d_1 \} \Rightarrow \{f_1 = (2k^3 + \omega)/k, f_2 = 0, f_3 = 0, s_2 = 3k\} \)
   
   In this case the equation is linear and its solution is given by
   
   \[ u = U \frac{n_1 - d_1 + \exp(\zeta)}{\exp(\zeta)}, \]

2. \( d_0 = d_1^2/6, \ n_0 = d_1^2/6, \)

3. \( d_1 = \frac{n_1(n_0 + d_0) \pm (d_0 - n_0)\sqrt{n_1^2 - 4n_0}}{2n_0}, \)

4. \( d_0 = -d_1^2, \ n_0 = d_1 \left(\frac{1 \pm \sqrt{5}}{2} n_1 - \frac{3 \pm \sqrt{5}}{2} d_1\right). \)

This solution is blowup.
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   \]

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   \]

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4. $d_0 = -d_1^2, \ n_0 = d_1 \left(\frac{1 + \sqrt{5}}{2}n_1 - \frac{3 + \sqrt{5}}{2}d_1\right).$

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Exact solutions with $u \neq \text{const}$

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**Linearization**

The above FDA contains nonlinear terms for the grid function on the next time layer. To treat them in construction of a numerical solution we used the following linearization

\[ v_{k+1}^3 = v_{k+1}^3 - v_k^3 + v_k^3 = (v_{k+1} - v_k)(v_{k+1}^2 + v_{k+1}v_k + v_k^2) + v_k^3 \approx \\
\approx v_{k+1} \cdot 3v_k^2 - 2v_k^3 , \]

\[ v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx \\
\approx v_{k+1} \cdot 2v_k - v_k^2 . \]

We implemented numerical procedure for construction of a solution in Python with the use of package SciPy. In doing so, we fixed \( \tau := h/2 \).
The above FDA contains nonlinear terms for the grid function on the next time layer. To treat them in construction of a numerical solution we used the following linearization

\[
\begin{align*}
\nu_{k+1}^{3} & = \nu_{k+1}^{3} - \nu_{k}^{3} + \nu_{k}^{3} = (\nu_{k+1} - \nu_{k})(\nu_{k+1}^{2} + \nu_{k+1} \nu_{k} + \nu_{k}^{2}) + \nu_{k}^{3} \approx \\
& \approx \nu_{k+1} \cdot 3 \nu_{k}^{2} - 2 \nu_{k}^{3}, \\
\nu_{k+1}^{2} & = \nu_{k+1}^{2} - \nu_{k}^{2} + \nu_{k}^{2} = (\nu_{k+1} - \nu_{k})(\nu_{k+1} + \nu_{k}) + \nu_{k}^{2} \approx \\
& \approx \nu_{k+1} \cdot 2 \nu_{k} - \nu_{k}^{2}.
\end{align*}
\]

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Linearization

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\[
v_{k+1}^3 = v_{k+1}^3 - v_k^3 + v_k^3 = (v_{k+1} - v_k)(v_{k+1}^2 + v_{k+1}v_k + v_k^2) + v_k^3 \approx v_{k+1} \cdot 3v_k^2 - 2v_k^3,
\]

\[
v_{k+1}^2 = v_{k+1}^2 - v_k^2 + v_k^2 = (v_{k+1} - v_k)(v_{k+1} + v_k) + v_k^2 \approx v_{k+1} \cdot 2v_k - v_k^2.
\]

We implemented numerical procedure for construction of a solution in Python with the use of package SciPy. In doing so, we fixed \( \tau := h/2 \).
Exact solution of type 2

\[ U = 1.00, \quad n_0 = 1.50, \quad n_1 = -3.00, \quad d_0 = 1.50, \quad d_1 = 3.00, \quad f_1 = 0.00, \quad f_2 = -0.04, \quad f_3 = -0.01, \quad s_2 = -0.00, \quad s = 0, \quad t = 0.00 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 60.12 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 120.24 \]
Exact solution of type 2

$U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 180.36$
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 240.48 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 300.60 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 360.72 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 420.84 \]
Exact solution of type 2

$U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 480.96$
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 541.08 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 601.20 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 661.32 \]
Exact solution of type 2

\[ U = 1.00, n_0 = 1.50, n_1 = -3.00, d_0 = 1.50, d_1 = 3.00, f_1 = 0.00, f_2 = -0.04, f_3 = -0.01, s_2 = -0.00, s = 0, t = 721.44 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 0.00 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 6.01 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 12.02 \]
Exact solution of type 3

$U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 18.04$
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 24.05 \]
Exact solution of type 3

\[ U = 1.00, \quad n_0 = 2.00, \quad n_1 = 3.00, \quad d_0 = 3.00, \quad d_1 = 4.00, \quad f_1 = 0.00, \quad f_2 = 1.51, \quad f_3 = -0.72, \quad s_2 = -0.48, \quad s = 0, \quad t = 30.06 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 36.07 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 42.08 \]
Exact solution of type 3

\[ U = 1.00, \quad n_0 = 2.00, \quad n_1 = 3.00, \quad d_0 = 3.00, \quad d_1 = 4.00, \quad f_1 = 0.00, \quad f_2 = 1.51, \quad f_3 = -0.72, \quad s_2 = -0.48, \quad s = 0, \quad t = 48.10 \]
Exact solution of type 3

\[ U = 1.00, \, n_0 = 2.00, \, n_1 = 3.00, \, d_0 = 3.00, \, d_1 = 4.00, \, f_1 = 0.00, \, f_2 = 1.51, \, f_3 = -0.72, \, s_2 = -0.48, \, s = 0, \, t = 54.11 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 60.12 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 66.13 \]
Exact solution of type 3

\[ U = 1.00, n_0 = 2.00, n_1 = 3.00, d_0 = 3.00, d_1 = 4.00, f_1 = 0.00, f_2 = 1.51, f_3 = -0.72, s_2 = -0.48, s = 0, t = 72.14 \]
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Involutional PDE system of the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity can be written in the following form \( \text{(G.,Blinkov’09)} \) obtained by the method suggested in \( \text{(G.,Blinkov, Mozzhilkin’06)} \)

\[
F := \begin{cases} 
    f_1 := u_x + v_y = 0, \\
    f_2 := u_t + uu_x + v u_y + p_x - \frac{1}{\text{Re}} (u_{xx} + u_{yy}) = 0, \\
    f_3 := v_t + uv_x + v v_y + p_y - \frac{1}{\text{Re}} (v_{xx} + v_{yy}) = 0, \\
    f_4 := u_x^2 + 2 v_x u_y + v_y^2 + p_{xx} + p_{yy} = 0.
\end{cases}
\]

Here

\( f_1 \) - the continuity equation,
\( f_2, f_3 \) - the proper Navier-Stokes equations,
\( f_4 \) - the pressure Poisson equation which is the integrability condition for \( \{f_1, f_2, f_3\} \),
\( (u, v) \) - the velocity field,
\( p \) - the pressure,
\( \text{Re} \) - the Reynolds number.
Involutive PDE system of the Navier-Stokes equations for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity can be written in the following form (G., Blinkov’09) obtained by the method suggested in (G., Blinkov, Mozzhilkin’06)

\[
F := \begin{cases} 
  f_1 := u_x + v_y = 0, \\
  f_2 := u_t + uu_x + vu_y + p_x - \frac{1}{Re}(u_{xx} + u_{yy}) = 0, \\
  f_3 := v_t + uv_x + vv_y + p_y - \frac{1}{Re}(v_{xx} + v_{yy}) = 0, \\
  f_4 := u_x^2 + 2u_x u_y + v_y^2 + p_{xx} + p_{yy} = 0.
\end{cases}
\]

Here

- \( f_1 \) - the continuity equation,
- \( f_2, f_3 \) - the proper Navier-Stokes equations,
- \( f_4 \) - the pressure Poisson equation which is the integrability condition for \( \{f_1, f_2, f_3\} \),
- \((u, v)\) - the velocity field,
- \( p \) - the pressure,
- \( \text{Re} \) - the Reynolds number.
The involutive Navier-Stokes system admits two-dimensional conservation law form

\[
\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial R}{\partial y} = 0.
\]

In terms of \(\{f_1, f_2, f_3, f_4\}\) this form reads

Conservation law form

\[
\begin{align*}
 f_1 & : \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0, \\
 f_2 & : \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left( u^2 + p - \frac{1}{Re} u_x \right) + \frac{\partial}{\partial y} \left( vu - \frac{1}{Re} u_y \right) = 0, \\
 f_3 & : \frac{\partial}{\partial t} v + \frac{\partial}{\partial x} \left( uv - \frac{1}{Re} v_x \right) + \frac{\partial}{\partial y} \left( v^2 + p - \frac{1}{Re} v_y \right) = 0, \\
 f_4 & : \frac{\partial}{\partial x} \left( uu_x + vu_y + p_x \right) + \frac{\partial}{\partial y} \left( vv_y + uv_x + p_y \right) = 0.
\end{align*}
\]
The involutive Navier-Stokes system admits two-dimensional conservation law form

\[
\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial R}{\partial y} = 0.
\]

In terms of \( \{f_1, f_2, f_3, f_4\} \) this form reads

\[
\begin{align*}
f_1 & : \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0, \\
f_2 & : \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} (u^2 + p - \frac{1}{Re} u_x) + \frac{\partial}{\partial y} (v u - \frac{1}{Re} u_y) = 0, \\
f_3 & : \frac{\partial}{\partial t} v + \frac{\partial}{\partial x} (u v - \frac{1}{Re} v_x) + \frac{\partial}{\partial y} (v^2 + p - \frac{1}{Re} v_y) = 0, \\
f_4 & : \frac{\partial}{\partial x} (u u_x + v u_y + p_x) + \frac{\partial}{\partial y} (v v_y + u v_x + p_y) = 0.
\end{align*}
\]
The involutive Navier-Stokes system admits two-dimensional conservation law form

\[ \frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x} + \frac{\partial R}{\partial y} = 0 . \]

In terms of \( \{f_1, f_2, f_3, f_4\} \) this form reads

### Conservation law form

\[
\begin{align*}
    f_1 & : \quad \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0 , \\
    f_2 & : \quad \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \left( u^2 + p - \frac{1}{\text{Re}} u_x \right) + \frac{\partial}{\partial y} \left( vu - \frac{1}{\text{Re}} u_y \right) = 0 , \\
    f_3 & : \quad \frac{\partial}{\partial t} v + \frac{\partial}{\partial x} \left( uv - \frac{1}{\text{Re}} v_x \right) + \frac{\partial}{\partial y} \left( v^2 + p - \frac{1}{\text{Re}} v_y \right) = 0 , \\
    f_4 & : \quad \frac{\partial}{\partial x} \left( uu_x + vu_y + p_x \right) + \frac{\partial}{\partial y} \left( vv_y + uv_x + p_y \right) = 0 .
\end{align*}
\]
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The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \leq i \leq 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p],$$

where $\mathbb{K} := \mathbb{Q}(\text{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0, \ h > 0, \ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$${u^j_n}_k, {v^j_n}_k, {p^j_n}_k := \{u, v, p\} \mid x=jh, y=kh, t=n\tau.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t), \quad \sigma_y \circ \phi = \phi(x, y + h, t), \quad \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$. 
Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \leq i \leq 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p],$$

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In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u_{j,k}^n, v_{j,k}^n, p_{j,k}^n\} := \{u, v, p\} \ |_{x=jh, y=kh, t=\tau n}.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t), \quad \sigma_y \circ \phi = \phi(x, y + h, t), \quad \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

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Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \leq i \leq 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p],$$

where $\mathbb{K} := \mathbb{Q} (\text{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0, \ h > 0, \ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \mid x=jh, y=kh, t=\tau n.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x+h, y, t), \quad \sigma_y \circ \phi = \phi(x, y+h, t), \quad \sigma_t \circ \phi = \phi(x, y, t+\tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$. 
Computational grid

The l.h.s. of the Navier–Stokes system (NSS) can be considered as elements in the differential polynomial ring $R$

$$f_i = 0 \ (1 \leq i \leq 4), \quad F := \{f_1, f_2, f_3, f_4\} \subset R := \mathbb{K}[u, v, p],$$

where $\mathbb{K} := \mathbb{Q}(\text{Re})$ is the differential field of constants.

We use an orthogonal and uniform computational grid as the set of points

$$(jh, kh, n\tau) \in \mathbb{R}^3, \quad \tau > 0, \ h > 0, \ (j, k, n) \in \mathbb{Z}^3.$$

In a grid node $(jh, kh, n\tau)$ a solution to NSS is approximated by the triple of grid functions

$$\{u^n_{j,k}, v^n_{j,k}, p^n_{j,k}\} := \{u, v, p\} \mid x = jh, y = kh, t = \tau n.$$

We introduce differences $\{\sigma_x, \sigma_y, \sigma_t\}$ acting on a grid function $\phi(x, y, t)$ as

$$\sigma_x \circ \phi = \phi(x + h, y, t), \quad \sigma_y \circ \phi = \phi(x, y + h, t), \quad \sigma_t \circ \phi = \phi(x, y, t + \tau)$$

and denote by $\mathcal{R}$ the ring of difference polynomials over $\mathbb{K}$. 

Integration contour

To discretize NSS on the grid choose the integration contour \( \Gamma \) in the \((x, y)\) plane
The Navie-Stokes system in integral form

Integral conservation law form

\[
\begin{align*}
\oint_{\Gamma} - vdx + udy &= 0, \\
\frac{\partial}{\partial t} \left[ \int_{x_j}^{x_{j+2}} \int_{y_k}^{y_{k+2}} u \, dx \, dy \right]_{t_n}^{t_{n+1}} &- \int_{t_n}^{t_{n+1}} \left( \oint_{\Gamma} \left( vu - \frac{1}{Re} u_y \right) \, dx - \left( u^2 + p - \frac{1}{Re} u_x \right) \, dy \right) \, dt = 0, \\
\frac{\partial}{\partial t} \left[ \int_{x_j}^{x_{j+2}} \int_{y_k}^{y_{k+2}} v \, dx \, dy \right]_{t_n}^{t_{n+1}} &- \int_{t_n}^{t_{n+1}} \left( \oint_{\Gamma} \left( v^2 + p - \frac{1}{Re} v_y \right) \, dx - \left( uv - \frac{1}{Re} v_x \right) \, dy \right) \, dt = 0, \\
\oint_{\Gamma} \left( (v^2)_y + (uv)_x + p_y \right) \, dx + \left( (u^2)_x + (vu)_y + p_x \right) \, dy &= 0.
\end{align*}
\]
Additional relations

Now we add integral relations between dependent variables and derivatives

**Exact integral relations**

\[
\begin{align*}
\int_{x_j}^{x_{j+1}} (u^2) \, dx &= u(x_{j+1}, y)^2 - u(x_j, y)^2, & \int_{y_k}^{y_{k+1}} (v^2) \, dy &= v(x, y_{k+1})^2 - v(x, y_k)^2, \\
\int_{x_j}^{x_{j+1}} (uv) \, dx &= u(x_{j+1}, y)v(x_{j+1}, y) - u(x_j, y)v(x_j, y), & \int_{y_k}^{y_{k+1}} (uv) \, dy &= u(x, y_{k+1})v(x, y_{k+1}) - u(x, y_k)v(x, y_k), \\
\int_{x_j}^{x_{j+1}} u_x \, dx &= u(x_{j+1}, y) - u(x_j, y), & \int_{y_k}^{y_{k+1}} u_y \, dy &= u(x, y_{k+1}) - u(x, y_k), \\
\int_{x_j}^{x_{j+1}} v_x \, dx &= v(x_{j+1}, y) - u(x_j, y), & \int_{y_k}^{y_{k+1}} v_y \, dy &= v(x, y_{k+1}) - u(x, y_k), \\
\int_{x_j}^{x_{j+1}} p_x \, dx &= p(x_{j+1}, y) - u(x_j, y), & \int_{y_k}^{y_{k+1}} p_y \, dy &= p(x, y_{k+1}) - u(x, y_k).
\end{align*}
\]
Finite difference approximation

By using the **midpoint integration approximation** for the integrals over $x$ and $y$ and the **top-left corner approximation** for integration over $t$. Then elimination of partial derivatives from the obtained difference system gives the following FDA with a $5 \times 5$ stencil (G., Blinkov’09)

\[
\begin{aligned}
\varepsilon_{1_{j,k}}^n := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\varepsilon_{2_{j,k}}^n := \frac{u_{j+1,k}^{n+1} - u_{j,k}^n}{\tau} + \frac{u_{j+1,k}^n 2 - u_{j-1,k}^n 2}{2h} + \frac{v_{j,k+1}^n u_{j,k+1}^n - v_{j,k-1}^n u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
- \frac{1}{Re} \left( \frac{u_{j+2,k}^n - 2u_{j,k}^n + u_{j-2,k}^n}{4h^2} + \frac{u_{j,k+2}^n - 2u_{j,k}^n + u_{j,k-2}^n}{4h^2} \right) = 0, \\
\varepsilon_{3_{j,k}}^n := \frac{v_{j,k+1}^n - v_{j,k}^n}{\tau} + \frac{u_{j,k+1}^n v_{j+1,k}^n - v_{j-1,k}^n v_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n v_{j,k+1}^n - v_{j,k-1}^n v_{j,k-1}^n}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
- \frac{1}{Re} \left( \frac{v_{j+2,k}^n - 2v_{j,k}^n + v_{j-2,k}^n}{4h^2} + \frac{v_{j,k+2}^n - 2v_{j,k}^n + v_{j,k-2}^n}{4h^2} \right) = 0, \\
\varepsilon_{4_{j,k}}^n := \frac{u_{j+2,k}^n 2 - 2u_{j,k}^n 2 + u_{j-2,k}^n 2}{4h^2} + \frac{v_{j,k+2}^n 2 - 2v_{j,k}^n 2 + v_{j,k-2}^n 2}{4h^2} \\
+ \frac{2u_{j+1,k}^n v_{j+1,k}^n + u_{j-1,k}^n v_{j-1,k}^n}{4h^2} + \frac{u_{j+1,k}^n v_{j+1,k}^n + u_{j-1,k}^n v_{j-1,k}^n}{4h^2} + \frac{p_{j+1,k}^n - 2p_{j,k}^n + p_{j-1,k}^n}{4h^2} + \frac{p_{j,k+2}^n - 2p_{j,k}^n + p_{j,k-2}^n}{4h^2} = 0.
\end{aligned}
\]
Finite difference approximation

By using the midpoint integration approximation for the integrals over $x$ and $y$ and the top-left corner approximation for integration over $t$. Then elimination of partial derivatives from the obtained difference system gives the following FDA with a $5 \times 5$ stencil (G.,Blinkov’09)

$$
\begin{align*}
\mathbf{e}_{1,j,k}^n & := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\mathbf{e}_{2,j,k}^n & := \frac{u_{jk}^{n+1} - u_{jk}^n}{\tau} + \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{u_{j,k+2}^n - 2u_{j,k+1}^n + u_{j,k}^n}{4h^2} + \frac{u_{j,k+2}^n - 2u_{j,k}^n + u_{j,k-2}^n}{4h^2} = 0, \\
\mathbf{e}_{3,j,k}^n & := \frac{v_{jk}^{n+1} - v_{jk}^n}{\tau} + \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{v_{j+2,k}^n - 2v_{j+1,k}^n + v_{j,k}^n}{4h^2} + \frac{v_{j+2,k}^n - 2v_{j,k}^n + v_{j,k-2}^n}{4h^2} = 0, \\
\mathbf{e}_{4,j,k}^n & := \frac{u_{j+2,k}^n - 2u_{j+1,k}^n + u_{j-2,k}^n}{4h^2} + \frac{v_{j+2,k}^n - 2v_{j+1,k}^n + v_{j,k}^n}{4h^2} + \frac{u_{j+1,k+1}^n - u_{j+1,k-1}^n}{4h^2} + \frac{v_{j+1,k+1}^n - v_{j+1,k-1}^n}{4h^2} + \frac{p_{j+2,k}^n - 2p_{j+1,k}^n + p_{j-2,k}^n}{4h^2} + \frac{p_{j+2,k}^n - 2p_{j,k}^n + p_{j-2,k}^n}{4h^2} = 0.
\end{align*}
$$

\[ \text{FDA 1} = \]
If one applies the trapezoidal approximation to the integral relations for $u_x, u_y, v_x, v_y, u^2_x, (v^2)_y$ and $p$ instead of the midpoint approximation, then it produces FDA with a $3 \times 3$ stencil (G., Blinkov’09)

$$
\begin{aligned}
\mathbf{e}_1^n_{j,k} &:= \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\mathbf{e}_2^n_{j,k} &:= \frac{u_{j,k+1}^{n+1} - u_{j,k-1}^{n-1}}{\tau} + \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
&- \frac{1}{\text{Re}} \left( \frac{u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{j,k}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_3^n_{j,k} &:= \frac{v_{j,k+1}^{n+1} - v_{j,k-1}^{n-1}}{\tau} + \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
&- \frac{1}{\text{Re}} \left( \frac{v_{j+1,k}^n - 2v_{j,k}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{j,k}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_4^n_{j,k} &:= \left( \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + \frac{2v_{j+1,k}^n - 2v_{j-1,k}^n}{2h} + \frac{2u_{j,k+1}^n - 2u_{j,k-1}^n}{2h} + \left( \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} \right)^2 \\
&+ \frac{p_{j+1,k}^n - 2p_{j,k}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j,k+1}^n - 2p_{j,k}^n + p_{j,k-1}^n}{h^2} = 0
\end{aligned}
$$
Finite difference approximation 2

If one applies the trapezoidal approximation to the integral relations for $u_x, u_y, v_x, v_y, u^2, (v^2)_y$ and $p$ instead of the midpoint approximation, then it produces FDA with a $3 \times 3$ stencil (G., Blinkov’09)

\[
\begin{aligned}
\mathbf{e}_1^{n}_{j,k} &:= \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\mathbf{e}_2^{n}_{j,k} &:= \frac{u_{j,k+1}^{n+1} - u_{j,k}^{n}}{\tau} + u_{jk}^n \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + v_{jk}^n \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
&\quad - \frac{1}{\text{Re}} \left( \frac{u_{j+1,k}^n - 2u_{jk}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{jk}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_3^{n}_{j,k} &:= \frac{v_{j,k+1}^{n+1} - v_{j,k}^{n}}{\tau} + u_{jk}^n \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + v_{jk}^n \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
&\quad - \frac{1}{\text{Re}} \left( \frac{v_{j+1,k}^n - 2v_{jk}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{jk}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_4^{n}_{j,k} &:= \left( \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + \left( \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} \right)^2 + \frac{p_{j+1,k}^n - 2p_{jk}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j+1,k}^n - 2p_{jk}^n + p_{j-1,k}^n}{h^2} = 0.
\end{aligned}
\]
The third approximation with $3 \times 3$ stencil is obtained from NSS by the conventional discretization what consists of replacing the temporal derivatives with the forward differences and the spatial derivatives with the central differences.

\[
\begin{align*}
\mathcal{E}_{1,j,k}^n & := \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j+1,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\mathcal{E}_{2,j,k}^n & := \frac{u_{j,k}^{n+1} - u_{j,k}^n}{\tau} + u_{j,k}^n \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + v_{j,k}^n \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
& \quad - \frac{1}{\text{Re}} \left( \frac{u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{j,k}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
\mathcal{E}_{3,j,k}^n & := \frac{v_{j,k}^{n+1} - v_{j,k}^n}{\tau} + u_{j,k}^n \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + v_{j,k}^n \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
& \quad - \frac{1}{\text{Re}} \left( \frac{v_{j+1,k}^n - 2v_{j,k}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{j,k}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
\mathcal{E}_{4,j,k}^n & := \left( \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + 2 \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \left( \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} \right)^2 \\
& \quad + \frac{p_{j+1,k}^n - 2p_{j,k}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j,k+1}^n - 2p_{j,k}^n + p_{j,k-1}^n}{h^2} = 0
\end{align*}
\]
Finite difference approximation 3

The third approximation with $3 \times 3$ stencil is obtained from NSS by the conventional discretization what consists of replacing the temporal derivatives with the forward differences and the spatial derivatives with the central differences.

$$
\begin{aligned}
\mathbf{e}_{1,j,k}^n &:= \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} = 0, \\
\mathbf{e}_{2,j,k}^n &:= \frac{u_{j,k+1}^{n+1} - u_{j,k}^n}{\tau} + \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j+1,k}^n - p_{j-1,k}^n}{2h} \\
&- \frac{1}{\text{Re}} \left( \frac{u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n}{h^2} + \frac{u_{j,k+1}^n - 2u_{j,k}^n + u_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_{3,j,k}^n &:= \frac{v_{j,k+1}^{n+1} - v_{j,k}^n}{\tau} + \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} + \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} + \frac{p_{j,k+1}^n - p_{j,k-1}^n}{2h} \\
&- \frac{1}{\text{Re}} \left( \frac{v_{j+1,k}^n - 2v_{j,k}^n + v_{j-1,k}^n}{h^2} + \frac{v_{j,k+1}^n - 2v_{j,k}^n + v_{j,k-1}^n}{h^2} \right) = 0, \\
\mathbf{e}_{4,j,k}^n &:= \left( \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2h} \right)^2 + 2 \frac{v_{j+1,k}^n - v_{j-1,k}^n}{2h} \frac{u_{j,k+1}^n - u_{j,k-1}^n}{2h} + \left( \frac{v_{j,k+1}^n - v_{j,k-1}^n}{2h} \right)^2 \\
&+ \frac{p_{j+1,k}^n - 2p_{j,k}^n + p_{j-1,k}^n}{h^2} + \frac{p_{j,k+1}^n - 2p_{j,k}^n + p_{j,k-1}^n}{h^2} = 0
\end{aligned}
$$
Contents

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2 KdV-like PDEs
   • 5-parameter Family of PDEs
   • Finite Difference Approximation
   • Exact Solutions
   • Numerical Experiments

3 Navier-Stokes Equations
   • Involutive Navier-Stokes System
   • Finite Difference Approximation
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   • Numerical Experiments

4 Conclusions

5 References
Differential and difference consequences

A perfect difference ideal $[\tilde{F}]$ generated by $\tilde{F} \subset \mathcal{R}$ is the smallest difference ideal containing $\tilde{F}$ and such that for any $\tilde{f} \in \mathcal{R}$ and $k_1, k_2, k_3 \in \mathbb{N}_{\geq 0}$

$$(\sigma_x \circ \tilde{f})^{k_1} (\sigma_y \circ \tilde{f})^{k_2} (\sigma_t \circ \tilde{f})^{k_3} \in [\tilde{F}] \implies \tilde{f} \in [\tilde{F}].$$

In difference algebra, perfect ideals play the same role as radical ideals in commutative and differential algebra.

Set $F \subset \mathcal{R}$ (NSS) generates radical differential ideal $[F]$.

Let a finite set of difference polynomials

$$\tilde{f}_1 = \cdots = \tilde{f}_p = 0, \quad \tilde{F} := \{\tilde{f}_1, \ldots, \tilde{f}_p\} \subset \mathcal{R}$$

be a FDA to $F$. Generally, $p$ needs not to be equal 4.

Differential and difference consequences

A differential (resp. difference) polynomial $f \in \mathcal{R}$ (resp. $\tilde{f} \in \mathcal{R}$) is differential-algebraic (resp. difference-algebraic) consequence of $F$ (resp. $\tilde{F}$) if

$f \in [F]$ (resp. $\tilde{f} \in [\tilde{F}]$).
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\[
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\]

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Conventional (weak) consistency of FDA

We shall say that a difference equation $\tilde{f} = 0$ implies (in the continuous limit) the differential equation $f = 0$ and write $\tilde{f} \triangleright f$ if $f$ does not contain the grid spacings $h, \tau$ and the Taylor expansion about a grid point $(u^n_{j,k}, v^n_{j,k}, p^n_{j,k})$ transforms equation $\tilde{f} = 0$ into $f + O(h, \tau) = 0$ where $O(h, \tau)$ denotes expression which vanishes when $h$ and $\tau$ go to zero.

Definition

The difference approximation $\tilde{F}$ is (weakly or w-)consistent with $F$ if $p = 4$ and

$$(\forall \tilde{f} \in \tilde{F}) (\exists f \in F) [\tilde{f} \triangleright f].$$

The requirement of w-consistency which has been universally accepted in the literature, is not satisfactory by the following two reasons:

1. The cardinality of FDA to a system of differential equations may be different from that in the system.
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# Strong consistency

## Definition

An FDA to PDE(s) is **strongly consistent** or s-consistent if

\[
( \forall \tilde{f} \in [\tilde{F}] ) ( \exists f \in [F] ) [ \tilde{f} \triangleright f ].
\]

The algorithmic approach (G’12) to verification of s-consistency is based on the following statement.

## Theorem

A difference approximation \( \tilde{F} \subset \mathcal{R} \) to \( F \subset \mathcal{R} \) is s-consistent iff a (reduced) standard basis \( G \) of the difference ideal \([\tilde{F}]\) satisfies

\[
( \forall g \in G ) ( \exists f \in [F] ) [ g \triangleright f ].
\]

Given a differential polynomial \( f \in \mathcal{R} \), one can algorithmically check its membership in \([F]\) by performing the involutive (Janet) reduction.
**Strong consistency**

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An FDA to PDE(s) is **strongly consistent** or **s-consistent** if

\[
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Strong consistency

**Definition**

An FDA to PDE(s) is **strongly consistent** or s-consistent if

\[(\forall \tilde{f} \in [\tilde{F}]) \ (\exists f \in [F]) \ [\tilde{f} \triangleright f].\]

The algorithmic approach (G’12) to verification of s-consistency is based on the following statement.

**Theorem**

A difference approximation $\tilde{F} \subset R$ to $F \subset R$ is s-consistent iff a (reduced) standard basis $G$ of the difference ideal $[\tilde{F}]$ satisfies

\[(\forall g \in G) \ (\exists f \in [F]) \ [g \triangleright f].\]

Given a differential polynomial $f \in R$, one can algorithmically check its membership in $[F]$ by performing the involutive (Janet) reduction.

Gerdt & Blinkov (JINR & SSU) CAIM 2013 34 / 44
Strong consistency

Definition

An FDA to PDE(s) is strongly consistent or s-consistent if

\[(\forall \tilde{f} \in [\tilde{F}]) \ (\exists f \in [F]) \ [\tilde{f} \triangleright f].\]

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A difference approximation \(\tilde{F} \subset \mathcal{R}\) to \(F \subset \mathcal{R}\) is s-consistent iff a (reduced) standard basis \(G\) of the difference ideal \([\tilde{F}]\) satisfies

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Given a differential polynomial \(f \in \mathcal{R}\), one can algorithmically check its membership in \([F]\) by performing the involutive (Janet) reduction.
S-consistency analysis of FDA 1, 2 and 3

All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

\[ \tilde{F} := \{ e_{1,j,k}^n, e_{2,j,k}^n, e_{3,j,k}^n, e_{4,j,k}^n \} \]

about the grid point \( \{ h_j, h_k, n\tau \} \) when the grid spacings \( h \) and \( \tau \) go to zero.

**Proposition [Amodio, Blinkov, G., La Scala’13]**

Among weakly consistent FDAs 1, 2, and 3 only FDA 1 is strongly consistent.

**Corollary**

A standard basis \( G \) of the difference ideal generated by the set of polynomials in FDA 1 satisfies the condition

\[ (\forall g \in G) \ (\exists f \in [F]) \ [g \triangleright f]. \]
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All three FDAs are w-consistent. This can be easily verified by the Taylor expansion of the finite differences in the set

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\begin{align*}
    u &:= -e^{-2t/Re} \cos(x) \sin(y), \\
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Let \([0, \pi] \times [0, \pi]\) be discretized in the \((x, y)\)-directions by means of the \((m+2)^2\) equispaced points \(x_j = jh\) and \(y_k = kh\), for \(j, k = 0, \ldots, m+1\), and \(h = \pi/(m+1)\).

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Relative error for $\text{Re} = 10^5$

We computed error by means of the formula

$$e_g = \max_{j,k} \frac{|g_{j,k}^N - g(x_j, y_k, t_f)|}{1 + |g(x_j, y_k, t_f)|}.$$ 

where $g \in \{u, v, p\}$ and $g(x, y, t)$ belongs to the exact solution.

Relative error for $N = 10$, $t_f = N\tau = 1$, $\text{Re} = 10^5$ and varying $m$ from 5 to 50
Computed value of $u_x + v_y$

Computed value of $f_1$ in NSS for FDA 1, FDA 2 and FDA 3 with $N = 10$, $t_f = 1$, $Re = 10^5$ and varying $m$ from 5 to 50
Compared errors in \( u, v \) and \( p \) for FDA 1 (left), FDA 2 (middle) and FDA 3 (right): \( N = 40, \ t_f = 1, \ \text{Re} = 10^2 \) and varying \( m \) from 10 to 100
Relative error in $u$, $v$ and $p$ with FDA 1 for $Re = 10^2$

Computed error with FDA 1 ($u$, $v$ and $p$, respectively): $N = 40$, $t_f = 1$, $Re = 10^2$ and $m = 100$
Conclusions

Main results obtained

- We applied the finite volume method, numerical integration and difference elimination to obtain FDA to the KdV-like PDEs and to the NSS for unsteady two-dimensional motion of incompressible viscous liquid of constant viscosity.

- The structure of FDA depends on the numerical methods used to approximate integrals.

- By using algorithmic methods of differential and difference algebra we shown that the FDA for the KdV-like PDEs is s-consistent whereas for NSS one of the approximations which is characterized by a $5 \times 5$ stencil is s-consistent whereas the other two with a $3 \times 3$ stencil are not.

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