Consistency Analysis of Finite Difference Approximations to Systems of Partial Differential Equations

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MSU, 28-09-2011, arXiv:math.AP/1107.4269

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Introduction

The numerical solution of partial differential equations is a fundamental task in science and engineering.

Solving PDEs in practice:

PDEs (+ IC or/and BC)

$\Downarrow$

Discretizing (FDM, FEM, FVM)

$\Downarrow$

Algebraic (difference) equations

$\Downarrow$

Numerical solving

$\Downarrow$

Approximate solution

The main research problem here is to find good discretization which inherit fundamental properties of the PDEs such as topology, conservation, symmetries and maximum principle.

Basic goal of difference methods

1. Replace the given PDE(s) in n independent variables by certain finite-difference approximation(s) (FDA) defined on a mesh (grid) in the prescribed domain.
2. Ensure that the solution of FDA converges to solution of PDE(s) as the increments in the independent variables (mesh sizes) go to zero.

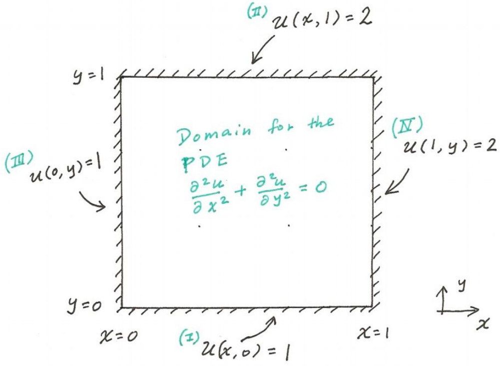
Q.: can computer algebra (CA) help to achieve this goal?
A.: yes. In particular, for a wide number of problems,

- to generate FDA automatically (G., Blinkov, Mozzhilkin'06)
- to investigate such important properties of FDA as consistency (to be analyzed in this talk) and stability (Ganzha, Vorozhtsov'96, G., Blinkov'07, Levandovskyy, Martin'11)

Simple Example of Approximation

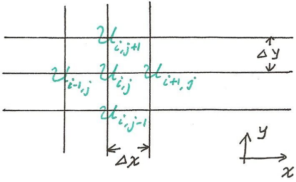
Consider Laplace equation

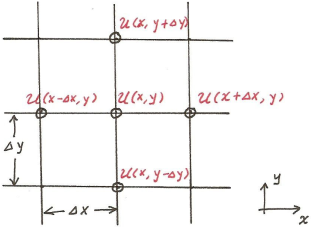
$$\begin{cases} u_{xx} + u_{yy} = 0, & \text{domain: } x \in [0,1],\ y \in [0,1] \\ \text{BC: } u(x,0) = u(0,y) = 1, & u(x,1) = u(1,y) = 2 \end{cases}$$



Mesh

We consider orthogonal and uniform mesh with sizes Δx and Δy (increments in independent variables)





Then the discrete version of Laplace equation at the mesh point (i, j) which can easily be algorithmically generated (G.,Blinkov, Mozzhilkin'06) is

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 0$$

Standard Discretizations of Derivatives

Let the set of independent variables be $\mathbf{x} := \{x_1, \ldots, x_n\}$ and the set of dependent variables be $\mathbf{u} := \{u^{(1)}, \ldots, u^{(m)}\}$. Then on the orthogonal mesh with the set of mesh steps $\mathbf{h} := \{h_1, \ldots, h_n\}$ $(h_i > 0)$

$$u^{(i)}_{x_j} = \frac{u^{(i)}_{k_1,\ldots,k_j+1,\ldots,k_n} - u^{(i)}_{k_1,\ldots,k_j,\ldots,k_n}}{h_j} + O(h_j),\ \text{forward difference}$$

$$u^{(i)}_{x_j} = \frac{u^{(i)}_{k_1,\ldots,k_j,\ldots,k_n} - u^{(i)}_{k_1,\ldots,k_j-1,\ldots,k_n}}{h_j} + O(h_j),\ \text{backward difference}$$

$$u^{(i)}_{x_j} = \frac{u^{(i)}_{k_1,\ldots,k_j+1,\ldots,k_n} - u^{(i)}_{k_1,\ldots,k_j-1,\ldots,k_n}}{2h_j} + O(h_j^2),\ \text{centered difference}$$

Here approximation of $\mathbf{u}(\mathbf{x})$ in the grid node is given by the grid function

$$u^{(i)}_{k_1,\ldots,k_n} := u^i(k_1 h_1, \ldots, k_n h_n), \quad (k_1, \ldots, k_n) \in \mathbb{Z}^n$$

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Consistency

Definition. Given a PDE $f = 0$ and its FDA $\tilde{f} = 0$, the FDA is said to be consistent with the PDE if for sufficiently differentiable $\mathbf{u}(\mathbf{x})$

$$f(\mathbf{u}) - \tilde{f}(\mathbf{u}) \to 0 \ \text{ as } \ |\mathbf{h}| \to 0$$

the convergence being pointwise at each point $(\mathbf{x})$.

Example. $f(u) := u_x + \nu u_y \ (\nu = \mathrm{const})$. FDA for the uniform grid $(h_1 = h_2 = h)$ by using forward differences is

$$\tilde{f}(u) := \frac{u_{i+1,j} - u_{i,j}}{h} + \nu \frac{u_{i,j+1} - u_{i,j}}{h}$$

The Taylor expansion about the grid point $(x = ih, y = jh)$ shows that FDA is consistent

$$u_{i+1,j} = u_{i,j} + hu_x + \frac{h^2}{2}u_{xx} + O(h^3),\ \ u_{i,j+1} = u_{i,j} + hu_y + \frac{h^2}{2}u_{yy} + O(h^3)$$

$$f(u) - \tilde{f}(u) = -\frac{h}{2}(u_{xx} + \nu u_{yy}) + O(h^2) \xrightarrow[h \to 0]{} 0\,.$$

Nonuniform Grids

For nonuniform grids, in some cases, one has to restrict the manner in which $|\mathbf{h}| \to 0$. Consider again equation $f(u) := u_x + \nu u_y = 0$ and its FDA in the Lax-Friedrichs form

$$\tilde{f} = \frac{2u_{i+1,j+1} - u_{i,j+2} - u_{i,j}}{2h_1} + \nu \frac{u_{i,j+2} - u_{i,j}}{2h_2}$$

The Taylor expansion of $\tilde{f}$ about the point $x = h_1 i, y = h_2(j+1)$ reads

$$\begin{aligned}\tilde{f} &= u_x + \nu u_y + \tfrac{h_1}{2} u_{xx} - \tfrac{h_2^2}{2h_1} u_{yy} + \nu \tfrac{h_2^2}{6} u_{yyy} + \tfrac{1}{6} \nu u_{xxx} h_1^2 \\ &+ \tfrac{1}{6} \nu u_{xxx} h_1^2 - \tfrac{h_2^4}{24h_1} u_{xxxx} + \nu \tfrac{h_2^4}{120} u_{xxxxx} + O(h_1^3 + \tfrac{h_2^6}{h_1} + h_2^6).\end{aligned}$$

It shows that the consistency holds only if $h_1 \to 0$ and $h_2^2/h_1 \to 0$

Consistency of FDAs to Systems of PDEs

Definition. A FDA to a system of PDEs is called w(eakly)-consistent if there is a passage $|\mathbf{h}| \to 0$ such that every difference equation in the FDA is consistent with its counterpart in the PDE system.

Remark. For a uniform grid $h_1 = \cdots = h_n = h$ the w-consistency of a FDA to a PDE system admits the straightforward algorithmic verification by means of Taylor expansion in h of the difference operators.

Definition (informal). A discretization of a system of PDEs will be called s(trongly)-consistent if there exists a passage $|\mathbf{h}| \to 0$ such that any (difference) consequence is reduced to a differential consequence of the PDEs.

Remark. It is clear that

$$\text{s-consistency} \Longrightarrow \text{w-consistency}$$

Q.: Is the converse true? We shall show that the converse is false.

Stability

Even for a single PDE consistency of its discrete version is not sufficient for convergency. Another important issue is (numerical) stability.

Definition (informal). A FDS (FDA + discretization of BC or/end IC) of PDE(s) is called stable if the error caused by a small perturbation in the numerical solution of the difference equations remain bounded.

Remark. Stability of difference schemes, for linear (parabolic or hyperbolic) PDEs, can be analyzed by several methods:

- the von Neumann or Fourier method,
- modified equation (differential approximation),
- CFL (Courant-Friedrichs-Lewy) stability (necessary) condition.

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Lax-Richtmyer Equivalence Theorem

In many cases, for a single linear PDE consistency and stability of its FDA are equivalent to convergence

$$\text{consistency} + \text{stability} \Longleftrightarrow \text{convergence}$$

The strict statement is given by the fundamental Lax-Richtmyer equivalence theorem (Lax, Richtmyer'56):

Theorem. A consistent FDS for a linear PDE for which the initial value problem (IVP) is well-posed is convergent iff it is stable.

Definition. (Hadamard'1902) A IVP is well-posed if its solution

1. exists
2. is unique
3. depends smoothly on the initial (Cauchy) data

Importance of Completion to Involution

Remark. The proof of Lax-Richtmyer equivalence theorem is heavily based on advanced analysis. In spite of numerous attempts, it was extended to a very restricted class of single nonlinear equations of evolution type.

Generally, given a system of PDEs, for well-posedness of a Cauchy (IVP) problem it is necessary to complete the system to involution (Cauchy, Kavalevskaya'1875, Finikov'48, G.'09) .

Besides, we shall show that completion of a differential system to involution provides an algorithmic verification of consistency of a FDA to a system of PDEs.

Hence, there are two interrelated and practically important reasons to complete systems of PDEs to involution.

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Integrability Conditions

Let $\mathcal{R}_q$ be a system of PDEs of order q.

Definition. An integrability condition for $\mathcal{R}_q$ is an equation of order $\leq q$ which is differential and not pure algebraic consequence of $\mathcal{R}_q$.

Example. (Seiler'94)

$$\mathcal{R}_1 : \begin{cases} u_z + y\, u_x = 0 \\ u_y = 0 \end{cases} \Longrightarrow \begin{cases} u_{yz} + y\, u_{xy} + u_x = 0 \\ u_{xy} = u_{yz} = 0 \end{cases} \Longrightarrow \boxed{u_x = 0}$$

$$\Longrightarrow \mathcal{R}_1 : \{u_x = u_y = u_z = 0$$

Formal Integrability and Involutivity

Definition. A formally integrable system has all the integrability conditions incorporated in it.

Definition. An involutive system is a formally integrable one with the complete (or involutive) set of the leading derivatives (symbol of $\mathcal{R}_q$).

Remark. The last condition means that any prolongation (i.e. differentiation w.r.t. an independent variable) of an element in the symbol is equal to finitely many prolongations of (generally another) element in the symbol w.r.t. a subset of the variables called multiplicative (or differentially admissible) for this element.

Definition. Given a system of PDEs, its transformation into an involutive form is called completion.

Remark. An involutive system is a differential Gröbner basis which (in the Gröbner sense) is generally redundant.

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Differential Systems

Definition. Let $S^{=}$ and $S^{\neq}$ be finite sets of differential polynomials such that $S^{=} \neq \emptyset$ and contains equations

$$(\,\forall s \in S^{=}\,)\ \ [\,s = 0\,]$$

whereas $S^{\neq}$ contains inequations

$$(\,\forall s \in S^{\neq}\,)\ \ [\,s \neq 0\,]$$

Then the pair $\left(S^{=}, S^{\neq}\right)$ of sets $S^{=}$ and $S^{\neq}$ is differential system.

Let $\mathfrak{Sol}(S^{=}/S^{\neq})$ denote the set of common solutions of differential equations $\{\, s = 0 \mid s \in S^{=}\}$ that do not annihilate elements $s \in S^{\neq}$.

Decomposition into Involutive Subsystems

Theorem. (Thomas'37,62) Any differential system $(S^{=}, S^{\neq})$ can be decomposed into a finite set of involutive subsystems $\left(S_i^{=}, S_i^{\neq}\right)$ with disjoint set of solutions

$$(S^{=}/S^{\neq}) \Longrightarrow \bigcup_i (S_i^{=}/S_i^{\neq}), \qquad \mathfrak{Sol}(S^{=}/S^{\neq}) = \biguplus_i \mathfrak{Sol}(S_i^{=}/S_i^{\neq})$$

The decomposition for Janet division is done fully algorithmically and have been implemented as a Maple package (Bächler, G., Lange-Hegermann, Robertz'10).

Given such a decomposition, one can algorithmically verify is a differential equation is algebraic consequence of the system $(S^{=}, S^{\neq})$

$$(\forall a \in \mathfrak{Sol}(S^{=}/S^{\neq}))\ [\,f(a) = 0\,] \Longleftrightarrow (\,\forall\, i\,)\,[\,\mathrm{dprem}_{\mathcal{J}}(f, S_i^{=}) = 0\,]$$

where $\mathrm{dprem}_{\mathcal{J}}(f, P)$ denotes differential Janet pseudo-reminder of f modulo P which is computed in the package.

Example of Thomas Decomposition

$$\left(\begin{array}{l} (u_y + v)u_x + 4v\,u_y - 2v^2 \\ (u_y + 2v)u_x + 5v\,u_y - 2v^2 \end{array},\ \emptyset \right)$$

$$\Downarrow$$

involutive subsystems

$$\left(\begin{array}{l} (u_y + v)u_x + 4v\,u_y - 2v^2 \\ u_y^2 - 3u_y + 2v^2 \\ v_x + v_y \end{array},\ v \right) \bigcup \left(\begin{array}{l} u_x \\ v \end{array},\ u_y \right) \bigcup \left(\begin{array}{l} u_y \\ v \end{array},\ \emptyset \right)$$

$$\Downarrow$$

Cauchy conditions

$$\left\{ \begin{array}{l} u(x_o, y_o) = C \\ v(x_o, y) = \phi(y) \neq 0 \end{array} \right\} \ \{\, u(x_0, y) = \psi(y),\ \psi_y' \neq 0 \,\} \ \{\, u(x, y_o) = \xi(x) \,\}$$

Remark. For linear PDEs the decomposition algorithm performs its completion to involution and for the Janet division returns the Janet basis (JB) form of the input system.

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Some Notions and Notations

$\mathcal{K} := \mathbb{Q}(\mathbf{x})$: field of rational functions over rationals ($\mathbb{Q}$) in the (independent) variables $\mathbf{x} := \{x_1, \ldots, x_n\}$

$\sigma := \{\sigma_1, \ldots, \sigma_n\}$: set of differences acting on functions $\phi \in \mathcal{K}$ as the right-shift operators

$$\sigma_i \circ \phi(x_1, \ldots, x_n) = \phi(x_1, \ldots, x_i + h_i, \ldots, x_n) \quad (\, h_i > 0 \,)$$

$\Theta := \{\sigma_1^{i_1} \circ \cdots \circ \sigma_n^{i_n}\}$: monoid (free commutative semigroup) generated by σ

$\mathcal{K}[\mathbf{u}]$: differential / difference polynomial ring (σ-ring) over $\mathcal{K}$ with the indeterminates (dependent variables) $\mathbf{u} := \{u^{(1)}, \ldots, u^{(m)}\}$.

$f(\mathbf{u}) \in \mathcal{K}[\mathbf{u}]$: differential polynomial, i.e. polynomial in dependent variables and their derivatives with coefficients from $\mathcal{K}$

$\tilde{f}(\mathbf{u}) \in \mathcal{K}[\mathbf{u}]$: difference polynomial, i.e. polynomial in $\{\theta \circ u^\alpha \mid \theta \in \Theta\}$ with coefficients from $\mathcal{K}$

$$\text{PDE} \Longrightarrow \text{FDA} : \begin{cases} \mathbf{x} \Longrightarrow \{k_1 h_1, \ldots, k_n h_n\} \\ \mathbf{u} \Longrightarrow \mathbf{u}_{k_1,\ldots,k_n} = \mathbf{u}(k_1 h_1, \ldots, k_n h_n) \end{cases} \qquad \{k_1, \ldots, k_n\} \in \mathbb{Z}^n$$

Difference Rankings and Monomial Orders

Definition. A total ordering $\prec$ on $\{\,\theta \circ u^\alpha \mid \theta \in \Theta,\ 1 \le \alpha \le m\,\}$ is ranking if $\forall \sigma_i, \theta, \theta_1, \theta_2, \alpha, \beta$

$$(a)\ \sigma_i \circ \theta \circ u^\alpha \succ \theta \circ u^\alpha \quad (b)\ \theta_1 \circ u^\alpha \succ \theta_2 \circ u^\beta \iff \theta \circ \theta_1 \circ u^\alpha \succ \theta \circ \theta_2 \circ u^\beta$$

The set of difference monomials in $\mathcal{K}[\mathbf{u}]$ is defined as

$$\mathcal{M} := \{\, (\theta_1 \circ u^{(1)})^{i_1} \cdots (\theta_m \circ u^{(m)})^{i_m} \mid \theta_j \in \Theta,\ i_j \in \mathbb{N}_{\ge 0},\ 1 \le j \le m \,\}$$

Definition. A total ordering $\succ$ on $\mathcal{M}$ which is extension of ranking such that

$$(\forall t \in \mathcal{M} \setminus \{1\})\,[t \succ 1] \wedge (\forall \theta \in \Theta)\,(\forall t, v, w \in \mathcal{M})\,[\, v \succ w \iff t \cdot \theta \circ v \succ t \cdot \theta \circ w \,]$$

is admissible.

Remark. Given $\succ$, every difference polynomial $\tilde{f}$ has the leading monomial $\mathrm{lm}(\tilde{f})$ w.r.t. $\succ$. In so doing, all $\tilde{f}$ are assumed to be normalized (monic), i.e. with the unit coefficient at $\mathrm{lm}(\tilde{f})$.

Difference Standard Bases

Definition. A set $\mathcal{I} \subset \mathcal{K}[\mathbf{u}]$ is difference polynomial ideal or (σ-ideal) if

$$(\forall a, b \in \mathcal{I})\ \ (\forall c \in \mathcal{K}[\mathbf{u}])\ \ (\forall \theta \in \Theta)\, [\, a + b \in \mathcal{I},\ a \cdot c \in \mathcal{I},\ \theta \circ a \in \mathcal{I}\,]$$

If $\tilde{F} \subset \mathcal{K}[\mathbf{u}] \setminus \mathcal{K}$, then the minimal σ-ideal containing $\tilde{F}$ is denoted by $[\tilde{F}]$ and $\tilde{F}$ is a generated set for $[\tilde{F}]$:

$$[\tilde{F}] = \{\, \sum_{\tilde{f} \in \tilde{F}} a_{\tilde{f}} \cdot \theta_{\tilde{f}} \circ \tilde{f} \mid a_{\tilde{f}} \in \mathcal{K}[\mathbf{u}],\ \theta_{\tilde{f}} \in \Theta \,\}$$

Definition. If for $v, w \in \mathcal{M}$ the equality $w = t \cdot \theta \circ v$ holds with $\theta \in \Theta$ and $t \in \mathcal{M}$ we shall say that v divides w and write $v \mid w$.

Definition. Given a $\sigma-$ideal $\mathcal{I}$ and an admissible monomial ordering $\succ$, a subset $\tilde{G} \subset \mathcal{I}$ is its (difference) standard basis (SB) if $[\tilde{G}] = \mathcal{I}$ and

$$(\,\forall \tilde{f} \in \mathcal{I}\,)(\,\exists \tilde{g} \in \tilde{G}\,)\ \ [\,\mathrm{lm}(\tilde{g}) \mid \mathrm{lm}(\tilde{f})\,]$$

If SB is finite it is called Gröbner basis (GB).

Algorithm: ***StandardBasis*** $(\tilde{F}, \succ)$

Input: $\tilde{F} \in \tilde{\mathcal{R}} \setminus \{0\}$, a finite set of nonzero polynomials;
$\succ$, a monomial ordering

Output: G, an interreduced standard basis of $[F]$

$\tilde{G} := \tilde{F}$
do
    $\tilde{H} := \tilde{G}$
    for all $S-$polynomials $\tilde{s}$ associated with elements in $\tilde{H}$ **do**
        $\tilde{g} := NF(\tilde{s}, \tilde{H})$
        if $\tilde{g} \neq 0$ **then**
            $\tilde{G} := \tilde{G} \cup \{\tilde{g}\}$
        fi
    od
od while $\tilde{G} \neq \tilde{H}$
$\tilde{G} :=$ ***Interreduce*** $(\tilde{G})$
return $\tilde{G}$

Example of Difference Gröbner Basis

$$F := \{\tilde{g}_1\}, \quad \tilde{g}_1 := u(x)\cdot u(x+2) - x\cdot u(x+1), \quad \sigma\circ u(x) = u(x+1)$$

and $\succ$ be pure lexicographic. Then having nonzero normal form $S-$polynomials and their normal forms are

$$s_1 := u(x+4)\cdot\tilde{g}_1 - u(x)\cdot\sigma^2\circ\tilde{g}_1, \quad \tilde{G} := \{\tilde{g}_1\}$$

$$\tilde{g}_2 := NF(s_1,\tilde{G}) = u(x+1)\cdot u(x+4) - \frac{x+2}{x}\cdot u(x)$$

$$s_2 := u(x+4)\cdot\sigma\circ\tilde{g}_1 - u(x+3)\cdot\tilde{g}_2, \quad \tilde{G} := \tilde{G}\cup\{\tilde{g}_2\}$$

$$\tilde{g}_3 := NF(s_2,\tilde{G}) = u(x)\cdot u(x+3)^2 - x\cdot(x+1)\cdot u(x+3)$$

$$s_3 := \sigma\circ\cdot\tilde{g}_3 - u(x+4)\cdot\tilde{g}_2, \quad \tilde{G} := \tilde{G}\cup\{\tilde{g}_3\}$$

$$\tilde{g}_4 := NF(s_3,\tilde{G}) = u(x)\cdot u(x+3)\cdot u(x+4) - x\cdot(x+1)\cdot u(x+4)$$

$$s_4 := u(x+5)\cdot\tilde{g}_3 - \sigma\circ\tilde{g}_4, \quad \tilde{G} := \tilde{G}\cup\{\tilde{g}_4\}$$

$$\tilde{g}_5 := NF(s_4,\tilde{G}) = u(x+5) - \frac{x+3}{x\cdot(x+1)}u(x)\cdot u(x+4)$$

All $S-$polynomials associated with elements in $\tilde{G} := \{\tilde{g}_1,\tilde{g}_2,\tilde{g}_3,\tilde{g}_4,\tilde{g}_5\}$ are reduced to zero modulo $\tilde{G}$. $\tilde{G}$ is an interreduced standard basis of $[F]$.

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Algebraic Consequences of Equation Systems

Given a finite set

$$F := \{f_1, \dots, f_k\} \subset \mathcal{K}[\mathbf{u}]$$

of differential polynomials, $[\![F]\!]$ will denote the set of their algebraic differential consequences, that is, the set of all differential polynomials which vanish on common solutions to the PDE system $\{\, f = 0 \mid f \in F \,\}$.

Similarly, if

$$\tilde{F} := \{\tilde{f}_1, \dots, \tilde{f}_k\} \subset \mathcal{K}[\mathbf{u}]$$

is a set of difference polynomials then $[\![\tilde{F}]\!] \in \mathcal{K}[\mathbf{u}]$ will denote the set of their algebraic difference consequences, that is, the set of difference polynomials which vanish on common solutions to the difference system $\{\, \tilde{f} = 0 \mid \tilde{f} \in \tilde{F} \,\}$.

Remark. $[\![\tilde{F}]\!]$ is a perfect difference ideal generated by $\tilde{F}$, and

$$[\tilde{F}] \subseteq [\![\tilde{F}]\!]$$

Definition of S-consistency

Definition. (G., Robertz'10) We shall say that a difference equation $\tilde{f}(\mathbf{u}) = 0$ implies the differential equation $f(\mathbf{u}) = 0$ and write $\tilde{f} \rhd f$ when there is a limit $|\mathbf{h}| \to 0$ such that the Taylor expansion about a grid point yields

$$\tilde{f}(\mathbf{u}) \xrightarrow[h\to 0]{} f(\mathbf{u})|\mathbf{h}|^k + O(|\mathbf{h}|^{k+1}),\ k \in \mathbb{Z}_{\geq 0}\,.$$

In this terminology, consistency of $\tilde{f}$ with f means $\tilde{f} \rhd f$.

Definition. Given a PDE system F and its FDA $\tilde{F}$, we shall say that $\tilde{F}$ is strongly consistent or s-consistent with F if there is a limit $|\mathbf{h}| \to 0$ such that

$$(\,\forall \tilde{f} \in [\![\tilde{F}]\!]\,)\,(\,\exists f \in [\![F]\!]\,)\ \ [\ \tilde{f} \rhd f\ ]$$

Main Theorem

Theorem 1. (G., Robertz'10) A FDA $\{\,\tilde f = 0 \mid \tilde f \in \tilde F\,\}$ to a linear PDE system $\{\,f = 0 \mid f \in F\,\}$ is s-consistent iff there exists a limit $|\mathbf{h}| \to 0$ such that a Gröbner basis (or involutive basis) $\tilde G$ of $[\tilde F]$ satisfies

$$(\forall \tilde g \in \tilde G)\,(\exists g \in [\![F]\!])\ \ [\,\tilde g \rhd g\,]$$

Remark. For linear systems Gröbner basis always exists and the above theorem provides the algorithmic check of s-consistency.

Theorem 2. A FDA $\{\,\tilde f = 0 \mid \tilde f \in \tilde F\,\}$ to a PDE system $\{\,f = 0 \mid f \in F\,\}$ is s-consistent if and only if there is a limit $|\mathbf{h}| \to 0$ such that a standard basis $\tilde G$ satisfies

$$(\forall \tilde g \in \tilde G)\,(\exists g \in [\![F]\!])\ \ [\,\tilde g \rhd g\,]$$

Remark. The proof of Theorem 2 is a generalization of that of Theorem 1 to nonlinear systems.

Main Theorem

Theorem 1. (G., Robertz'10) A FDA $\{\, \tilde{f} = 0 \mid \tilde{f} \in \tilde{F} \,\}$ to a linear PDE system $\{\, f = 0 \mid f \in F \,\}$ is s-consistent iff there exists a limit $|\mathbf{h}| \to 0$ such that a Gröbner basis (or involutive basis) $\tilde{G}$ of $[\tilde{F}]$ satisfies

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$$(\forall \tilde{g} \in \tilde{G})\,(\exists g \in [\![F]\!])\ \ [\,\tilde{g} \rhd g\,]$$

Remark. The proof of Theorem 2 is a generalization of that of Theorem 1 to nonlinear systems.

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Algorithm: ***ConsistencyCheck*** $(F, \tilde{F})$

choose differential ranking $\succ_1$ and difference ordering $\succ_2$
$\mathcal{T} :=$ ***DifferentialThomasDecomposition*** $(F, \succ_1)$
$\mathcal{P}_0 := \{\, P \mid \langle P, Q \rangle \in \mathcal{T} \,\}$
$\tilde{G} :=$ ***StandardBasis*** $(\tilde{F}, \succ_2)$ (* *may not terminate* *)
$C :=$ **true**
while $\tilde{G} \neq \emptyset$ and $C =$ **true do**
  choose $\tilde{g} \in \tilde{G}$; $\tilde{G} := \tilde{G} \setminus \{\tilde{g}\}$; $\mathcal{P} := \mathcal{P}_0$
  compute g such that $\tilde{g} \rhd g$
  while $\mathcal{P} \neq \emptyset$ and $C =$ **true do**
    choose $S \in \mathcal{P}$; $\mathcal{P} := \mathcal{P} \setminus \{S\}$; $d :=$ ***dprem***$_{\mathcal{J}}(g, S)$
    if $d \neq 0$ **then**
      $C :=$ **false**
    fi
  od
od
return C

Example 1

Consider PDE system

$$f_1 := u_{xz} + yu = 0, \quad f_2 := u_{yw} + zu = 0$$

For $\partial_x \succ \partial_y \succ \partial_z \succ \partial_w$ G(röbner)B=J(anet)B is

$$g_1 := yu_y - zu_z, \quad g_2 := u_x - u_w, \quad g_3 := u_{zw} + yu$$

If we use forward differences to discretize the system at the grid point $x = ih, y = jh, z = kh, w = lh$:

$$\tilde{f}_1 := (\Delta_1\Delta_3)(u) + jhu_{i,j,k,l}, \quad \tilde{f}_2 := (\Delta_2\Delta_4)(u) + khu_{i,j,k,l}.$$

then GB=JB w.r.t. degrevlex (with $\sigma_1 \succ \sigma_2 \succ \sigma_3 \succ \sigma_4$) for the difference ideal generated by $\tilde{f}_1$ and $\tilde{f}_2$ is

$$\Delta_1(u) - jh^2 u_{i,j,k,l}, \; u_{i,j+1,k,l}, \; u_{i,j,k+1,l}, \; \Delta_4(u) - kh^2 u_{i,j,k,l}.$$

It is easily verified that the FDA $\tilde{f}_1$, $\tilde{f}_2$ is s-inconsistent.

Example 2

Overdetermined PDE system

$$u_x + yu_z + u = 0, \quad u_y + xu_w = 0$$

For $\partial_x \succ \partial_y \succ \partial_z \succ \partial_w$ GB=JB and its FDA with forward differences and $x = ih, y = jh, z = kh, w = lh$ are

$$\begin{array}{lll} g_1 := u_x + yu_w + u, & g_2 := u_y + xu_w, & g_3 := u_z - u_w, \\ \tilde{g}_1 := \Delta_1(u) + jh\Delta_3(u) + u, & \tilde{g}_2 := \Delta_2(u) + ih\Delta_4(u), & \Delta_3(u) - \Delta_4(u) \end{array}$$

$\Downarrow$ difference JB

$$\Delta_1(u) + u, \quad \Delta_2(u), \quad \Delta_3(u), \quad \Delta_4(u) \qquad \text{s-inconsistent}$$

whereas

$$\mathrm{JB}\{\tilde{g}_1, \tilde{g}_2\} \rhd \{g_1, g_2, g_3, yu_z - yu_w, xu_z - xu_w\} \qquad \text{s-consistent}$$

Navier-Stokes Equations

For two-dimensional motion of incompressible viscous liquid of constant viscosity the equations are given by

$$\begin{cases} f^1 := u_x + v_y = 0\,, \\ f^2 := u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\triangle u = 0\,, \\ f^3 := v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\triangle v = 0\,. \end{cases}$$

Here

f^1 - the continuity equation

f^1, f^2 - the proper Navier-Stokes equations

(u, v) - the velocity field

p - the pressure

Re - the Reynolds number

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Completion to Involution

For the orderly ranking $\succ$ on the derivatives s.t.

$$\partial_x \succ \partial_y \succ \partial_t, \quad u \succ v \succ p$$

completion of the system to involution based on the Janet division reveals the only integrability condition – the Pressure Poisson equation

$$f^4 := u_x^2 + 2v_x u_y + v_y^2 + \triangle p = 0$$

which is the differential consequence of the Navier-Stokes equations:

$$f_x^2 + f_y^3 - f_t^1 - uf_x^1 - vf_y^1 + \frac{1}{\mathrm{Re}}\triangle f^1 = f^4$$

Involutive Navier-Stokes System

Thus, the involutive Navier-Stokes equation system is

$$F := \begin{cases} f^1 : \; u_x + v_y = 0\,, \\ f^2 : \; u_t + uu_x + vu_y + p_x - \frac{1}{\mathrm{Re}}\triangle u = 0\,, \\ f^3 : \; v_t + uv_x + vv_y + p_y - \frac{1}{\mathrm{Re}}\triangle v = 0\,, \\ f^4 : \; u_x^2 + 2v_x u_y + v_y^2 + \triangle p = 0\,. \end{cases}$$

Its Janet autoreduced form is given by

$$F_1 := \begin{cases} u_x + v_y = 0\,, \\ \frac{1}{\mathrm{Re}}(u_{yy} - v_{xy} - uv_y) - vu_y - u_t - p_x = 0\,, \\ \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy}) - uv_x - vv_y - v_t - p_y = 0\,, \\ 2v_x u_y + p_{xx} + p_{yy} + 2v_y^2 = 0\,. \end{cases}$$

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S-consistent Approximation

For the following system (G.,Blinkov'09) $\tilde{F} := \{\sigma_y \circ \tilde{f}_1, \sigma_y \circ \tilde{f}_2, \tilde{f}_3, \tilde{f}_4\}$ is a difference Gröbner basis for the lex order compatible with the orderly ranking s.t. $\sigma_t \succ \sigma_x \succ \sigma_y$ and $p \succ u \succ v$. In the continuous limit ($\tau \to 0,\ h \to 0$) it retains the involutive differential Navier-Stokes system $\Longrightarrow$ s-consistency.

$$
\left\{
\begin{array}{l}
\tilde{f}_1 := \dfrac{u^n_{j+1\,k}-u^n_{j-1\,k}}{2h} + \dfrac{v^n_{j\,k+1}-v^n_{j\,k-1}}{2h} = 0\,, \\[2ex]
\tilde{f}_2 := \dfrac{u^{n+1}_{j\,k}-u^n_{j\,k}}{\tau} + \dfrac{{u^2}^n_{j+1\,k}-{u^2}^n_{j-1\,k}}{2h} + \dfrac{{uv}^n_{j\,k+1}-{uv}^n_{j\,k-1}}{2h} + \\[2ex]
\qquad + \dfrac{p^n_{j+1\,k}-p^n_{j-1\,k}}{2h} - \dfrac{1}{Re}\left(\dfrac{u^n_{j+2\,k}-2u^n_{j\,k}+u^n_{j-2\,k}}{4h^2} + \dfrac{u^n_{j\,k+2}-2u^n_{j\,k}+u^n_{j\,k-2}}{4h^2}\right) = 0\,, \\[2ex]
\tilde{f}_3 := \dfrac{v^{n+1}_{j\,k}-v^n_{j\,k}}{\tau} + \dfrac{{uv}^n_{j+1\,k}-{uv}^n_{j-1\,k}}{2h} + \dfrac{{v^2}^n_{j\,k+1}-{v^2}^n_{j\,k-1}}{2h} + \\[2ex]
\qquad + \dfrac{p^n_{j\,k+1}-p^n_{j\,k-1}}{2h} - \dfrac{1}{Re}\left(\dfrac{v^n_{j+2\,k}-2v^n_{j\,k}+v^n_{j-2\,k}}{4h^2} + \dfrac{v^n_{j\,k+2}-2v^n_{j\,k}+v^n_{j\,k-2}}{4h^2}\right) = 0\,, \\[2ex]
\tilde{f}_4 := \dfrac{{u^2}^n_{j+2\,k}-2{u^2}^n_{j\,k}+{u^2}^n_{j-2\,k}}{4h^2} + 2\dfrac{{uv}^n_{j+1\,k+1}-{uv}^n_{j+1\,k-1}-{uv}^n_{j-1\,k+1}+{uv}^n_{j-1\,k-1}}{4h^2} + \\[2ex]
\qquad + \dfrac{{v^2}^n_{j\,k+2}-2{v^2}^n_{j\,k}+{v^2}^n_{j\,k-2}}{4h^2} + \left(\dfrac{p^n_{j+2\,k}-2p^n_{j\,k}+p^n_{j-2\,k}}{4h^2} + \dfrac{p^n_{j\,k+2}-2p^n_{j\,k}+p^n_{j\,k-2}}{4h^2}\right) = 0\,.
\end{array}
\right.
$$

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Another Approximation

A FDA with 3×3 stencil looks like numerically more attractive than the previous scheme whose stencil is 5×5.

An example of w-consistent FDA with 3×3 stencil was constructed in (G.,Blinkov'09).

$$
\left\{
\begin{aligned}
\tilde{e}_1 &:= \frac{u^n_{j+1\,k}-u^n_{j-1\,k}}{2h} + \frac{v^n_{j\,k+1}-v^n_{j\,k-1}}{2h} = 0\,,\\
\tilde{e}_2 &:= \frac{u^{n+1}_{j\,k}-u^n_{j\,k}}{\tau} + \frac{{u^2}^n_{j+1\,k}-{u^2}^n_{j-1\,k}}{2h} + \frac{uv^{\,n}_{j\,k+1}-uv^{\,n}_{j\,k-1}}{2h} + \\
&\quad + \frac{p^n_{j+1\,k}-p^n_{j-1\,k}}{2h} - \frac{1}{Re}\left(\frac{u^n_{j+1\,k}-2u^n_{j\,k}+u^n_{j-1\,k}}{h^2} + \frac{u^n_{j\,k+1}-2u^n_{j\,k}+u^n_{j\,k-1}}{h^2}\right) = 0\,,\\
\tilde{e}_3 &:= \frac{v^{n+1}_{j\,k}-v^n_{j\,k}}{\tau} + \frac{uv^{\,n}_{j+1\,k}-uv^{\,n}_{j-1\,k}}{2h} + \frac{{v^2}^n_{j\,k+1}-{v^2}^n_{j\,k-1}}{2h} + \\
&\quad + \frac{p^n_{j\,k+1}-p^n_{j\,k-1}}{2h} - \frac{1}{Re}\left(\frac{v^n_{j+1\,k}-2v^n_{j\,k}+v^n_{j-1\,k}}{h^2} + \frac{v^n_{j\,k+1}-2v^n_{j\,k}+v^n_{j\,k-1}}{h^2}\right) = 0\,,\\
\tilde{e}_4 &:= \frac{{u^2}^n_{j+1\,k}-2{u^2}^n_{j\,k}+{u^2}^n_{j-1\,k}}{h^2} + 2\,\frac{uv^{\,n}_{j+1\,k+1}-uv^{\,n}_{j+1\,k-1}-uv^{\,n}_{j-1\,k+1}+uv^{\,n}_{j-1\,k-1}}{4h^2} + \\
&\quad + \frac{{v^2}^n_{j\,k+1}-2{v^2}^n_{j\,k}+{v^2}^n_{j\,k-1}}{h^2} + \left(\frac{p^n_{j+1\,k}-2p^n_{j\,k}+p^n_{j-1\,k}}{h^2} + \frac{p^n_{j\,k+1}-2p^n_{j\,k}+p^n_{j\,k-1}}{h^2}\right) = 0\,.
\end{aligned}
\right.
$$

S-inconsistency

$\tilde{F}_1 := \{\sigma_y \circ \tilde{e}_1, \sigma_y \circ \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\}$, in contrast to $\tilde{F}$, is not Gröbner basis. The difference consequence $\tilde{q} := NF(s_{1,2}, \tilde{F}_1) \neq 0$ of where $s_{1,2}$ is the $S-$polynomial associated with $\sigma_y \circ \tilde{e}_1$ and $\sigma_y \circ \tilde{e}_2$. Furthermore,

$$\tilde{q} \rhd q := u_{xx}^2 + v_{yy}^2 + p_{xx} + p_{yy} \,,$$

and $q = 0$ is not a consequence of the Navier-Stokes equations.

- One way to check it is to compute $d :=\boldsymbol{dprem}_{\mathcal{J}}(q, F_1)$

  $$d = \frac{1}{\mathrm{Re}^2}\left(u_{yy}^2 + v_{yy}^2 - 2u_y v_x - 2v_y^2\right) + \frac{2}{\mathrm{Re}}\left(uv_y u_{yy} - vu_y u_{yy} - u_t u_{yy} - p_x u_{yy}\right) + 2\left(vu_t u_y - uu_t v_y + vu_y p_x - uv_y p_x - uvv_y u_y + u_t p_x\right) + u_t^2 + p_x^2 + v^2 u_y^2 + u^2 v_y^2 \,.$$

- Another way is to substitute the exact solution of the Navier-Stokes system (Kim, Moin'85)

  $$\begin{cases} u = -\exp(-2t)\cos(x)\sin(y) \\ v = \exp(-2t)\sin(x)\cos(y) \\ p = -\exp(-4t)(\cos(2x) + \cos(2y))/4 \end{cases}$$

  into q. The result is nonzero.

Hence, the FDA $\tilde{F}_1$ is s-inconsistent.

Implementation in Maple of Gröbner Bases (GB) / Janet Bases (JB) / Decomposition

Software	Differential systems	Difference systems	Comment
Groebner diffalg Rif	Linear GB GB/Decomposition GB/Decomposition	Linear GB – –	Built-in Built-in Built-in
Epsilon	ODE/Decomposition	–	Package
JanetOre	Linear JB	–	Package
Janet	Liner GB/JB	–	Package
LDA	–	Linear GB/JB	Maple
Differential-Thomas-Decomposition	JB/Decomposition	–	Package

Conclusions

- For FDA to systems of PDEs we introduced notion of s-consistency.
- Together with stability, consistency may provide convergence, if PDEs admit well-posedness of Cauchy problem.
- Completion of a PDE system to involution or (generally) its Thomas decomposition provides a tool for verification of well-posedness and s-consistency.
- S-consistency of a FDA to a linear PDE system admits full algorithmic verification and there is software for doing that.
- Construction of nonlinear difference Gröbner bases (GB) allows to verify s-consistency algorithmically.
- For FDA to nonlinear PDE systems computation of GB may be very hard or impossible (infinite SB). In this case it is useful to analyze intermediate polynomials in subalgorithm ***StandardBasis*** in the limit $|\mathbf{h}| \to 0$ as the necessary conditions of s-consistency.
- At present there are several software packages for constructing linear difference GB and none for nonlinear bases.