Symbolic Algorithms for Studying and Solving Pseudo-Linear Systems

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Introduction

Pseudo-linear Systems

- Introduced by Jacobson 1937.
- Let K be a field of characteristic 0.
- Let ϕ a automorphism over K and δ be a ϕ-derivation, i.e:

$$\delta(ab) = \phi(a)\delta(b) + \delta(a)b \quad \forall a, b \in K.$$

- If $\phi = \mathrm{id}_K$, then δ is the standard derivation.
- If $\phi \neq \mathrm{id}_K$ then $\delta = \gamma(\mathrm{id}_K - \phi)$ for some $\gamma \in K$.
- The subfield $\mathcal{C}_K \subset K$ containing all elements c in K satisfying $\phi(c) = c$ and $\delta(c) = 0$ is called the field of constants of K

Pseudo-linear Systems

- A pseudo-linear system PLS of size n over K is a system of the form

$$\delta(\mathbf{y}) = M\,\phi(\mathbf{y}), \quad \text{where} \quad M \in \mathcal{M}_n(K).$$

- Pseudo-linear systems is a unified class for expressing common types of linear functional systems (differential, (q-)difference, etc..).

$$\mathbf{y}' = A\,\mathbf{y} \quad \longrightarrow \quad \delta(\mathbf{y}) = M\,\phi(\mathbf{y}), \quad \phi = \mathrm{id}_K, \quad \delta = \frac{d}{dx}, \quad M = A.$$

$$\underbrace{\phi(\mathbf{y}) = A\,\mathbf{y}}_{\phi\text{-system}} \quad \longrightarrow \quad \delta(\mathbf{y}) = M\,\phi(\mathbf{y}), \quad \delta = \mathrm{id}_K - \phi, \quad M = A^{-1} - I.$$

Singularities

- A point x_0 in $\overline{\mathcal{C}_K}$ is said to be ϕ-fixed, if $x - x_0$ and $\phi^j(x - x_0)$ divide each other for some $j \in \mathbb{Z}^*$.
- In the differential case $\phi = \mathrm{id}_K$, all the points in $\overline{\mathcal{C}_K}$ are ϕ-fixed.
- In the difference case $\phi : x \mapsto x + r$ with $r \in \mathcal{C}_K^*$, there are no ϕ-fixed points.
- In the q-difference case $\phi : x \mapsto q\, x$ where $q \in \mathcal{C}_K^*$ is not a root of unity, the point 0 is the only ϕ-fixed point.
- The singularities of a pseudo-linear system are all ϕ-fixed points, and the point at infinity.

Objectives / Motivation

- Objective 1: Develop a recursive algorithm for computing rational solutions of a partial pseudo-linear system PPLS:

$$\begin{cases} \delta_1(\mathbf{y}) - M_1\, \phi_1(\mathbf{y}) = 0, \\ \quad \vdots \\ \delta_m(\mathbf{y}) - M_m\, \phi_m(\mathbf{y}) = 0. \end{cases}$$

- This requires, in particular, and algorithm to compute rational solutions of a sole PLS: $\delta(\mathbf{y}) - M\, \phi(\mathbf{y}) = 0$.

- Objective 2: Develop such an algorithm.

- This also requires, in particular, some information at a singularity. These information can be computed using simple forms.

- Objective 3: Develop an algorithm to compute a simple form of a pseudo-linear system.

Simple forms of pseudo-linear systems

Simple forms

- Simple forms was introduced by Barkatou 1999 as an alternative to super-irreducibles forms.

- They give useful information:
  - Local data: nature of a singularity, indicial equation, formal solutions,...
  - Closed-form solutions: polynomial, rational, hypergeometric, ...

- Methods for computing simple forms of pseudo-linear systems have been developed before. They require applying super-reduction algorithms first.

- Direct algorithms to compute simple forms appeared in the differential and difference case [Barkatou, Cluzeau, El Bacha 2011, 2018].

Simple forms

- Let C be a field of characteristic 0 and $K = C((t))$ be the field of Laurent series in t, equipped with the t-adic valuation ν.
- Consider a pseudo-linear system:

$$A\,\delta(\mathbf{y}) + B\,\phi(\mathbf{y}) = 0,$$

where $A,\ B \in \mathbb{M}_n(C[[t]])$.

- We assume that ϕ preserves the valuation: $\nu(\phi(a)) = \nu(a) \quad \forall a \in K$.
- $A = A_0 + t\,A_1 + t^2\,A_2 + \ldots, \qquad B = B_0 + t\,B_1 + t^2\,B_2 + \ldots$
- Leading pencil: $L_\lambda = A_0\,\lambda + B_0 \in \mathbb{M}_n(C[\lambda])$.

Definition

We say that a pseudo-linear system is simple if $\det(L_\lambda) \neq 0$.

Equivalent pseudo-linear systems

- To any pseudo-linear system $A\,\delta(\mathbf{y}) + B\,\phi(\mathbf{y}) = 0$, we associate the operator
$$L = A\,\delta + B\,\phi.$$
- The system can be written $L(y) = 0$.
- Two operators $L = A\,\delta + B\,\phi$ and $L' = A'\,\delta + B'\,\phi$ are said to be equivalent if $\exists\ S, T \in \mathrm{GL}_n(K)$ such that $L' = S\,L\,T$, that is:
$$A' = S\,A\,T, \qquad B' = S\,A\,\delta(T) + S\,B\,\phi(T).$$
- Two pseudo-linear systems $L(y) = 0$ and $L'(y) = 0$ are equivalent if the operators L and L' are equivalent.

Computing simple forms

- The method is a generalization of the ideas in the differential and difference case [Barkatou, Cluzeau, El Bacha 2011, 2018].
- Two key points: ϕ preserves the valuation, and the notion of equivalence.
- Remark that non-simple pseudo-linear system $\implies$ $\nu(\det(A)) > 0$.
- Principle of the algorithm: Compute iteratively equivalent pseudo-linear systems

$$A^{(i)}\,\delta(y) + B^{(i)}\,\phi(y) = 0, \; i \geq 1$$

satisfying $\nu(\det(A^{(i+1)})) < \nu(\det(A^{(i)}))$.

- We are sure we reach a simple system.

- $A_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad r = \mathrm{rank}(A_0), \quad B_0 = \begin{pmatrix} B_0^{11} & B_0^{12} \\ B_0^{21} & B_0^{22} \end{pmatrix}.$

- The matrix L_λ is written as

$$L_\lambda = \begin{pmatrix} I_r\lambda + B_0^{11} & B_0^{12} \\ B_0^{21} & B_0^{22} \end{pmatrix}.$$

- The rows of $(B_0^{21} \quad B_0^{22})$ are called the λ-free rows.

- Two cases arise:

  Case 1. $\mathrm{rank}(B_0^{21} \quad B_0^{22}) < n - r \implies$ we can decrease $\nu(\det(A))$.

  Case 2. $\mathrm{rank}(B_0^{21} \quad B_0^{22}) = n - r \implies$ we can reduce to Case 1 without modifying $\nu(\det(A))$.

Case 1: λ-free rows are linearly dependent

- If we have $\mathrm{rank}(B_0^{21} \quad B_0^{22}) < n - r$, then we can always construct an invertible matrix $S = D\,C \in \mathrm{GL}_n(C[[t^{-1}]])$ such that the equivalent operator $\widehat{L} = S\,L = \widehat{A}\,\delta + \widehat{B}\,\phi$ satisfies

$$\nu(\det(\widehat{A})) < \nu(\det(A)).$$

- D is an "appropriate" diagonal matrix.
- C is an "appropriate" constant matrix.

Case 1: λ-free rows are linearly dependent

Example

Let $K = C((x))$, $\phi(x) = q\,x$, $q \in C^*$ is not a root of unity, and $\delta = \phi - \mathrm{id}_K$. Consider the q-difference operator:

$$L = \underbrace{\begin{bmatrix} 1+x & \frac{x}{q^2} \\ 0 & x \end{bmatrix}}_{A} \delta + \begin{bmatrix} (1+x(q+1))(q-1) & \frac{x(q^2-1)}{q^2} \\ (xq+1)\,xq & \frac{(-q+1+xq)x}{q} \end{bmatrix} \phi,$$

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad L_\lambda = \begin{bmatrix} \lambda + q - 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\nu(\det(A)) = 1.$$

Case 1: λ-free rows are linearly dependent

- $C = I_2$ and $D = \mathrm{diag}(1, x^{-1})$.
- Multiplication of L on the left by $S = D\,C$

$$\Longrightarrow \widehat{L} = \underbrace{\begin{bmatrix} x+1 & \frac{x}{q^2} \\ 0 & 1 \end{bmatrix}}_{\widehat{A}} \delta + \begin{bmatrix} (q-1)\,(1+x\,(q+1)) & \frac{x\left(q^2-1\right)}{q^2} \\ (qx+1)\,q & \frac{1+(x-1)q}{q} \end{bmatrix} \phi$$

- $\nu(\det(\widehat{A})) = 0 < \nu(\det(A)) = 1.$
- The leading pencil of $\widehat{L}$:

$$\widehat{L}_\lambda = \begin{bmatrix} \lambda+q-1 & 0 \\ q & \frac{1+(\lambda-1)q}{q} \end{bmatrix} \quad \text{and} \quad \det(\widehat{L}_\lambda) \neq 0$$

$\Longrightarrow$ Simple system

Case 2: λ-free rows are linearly independent

- If we have $\mathrm{rank}(B_0^{21} \quad B_0^{22}) = n - r$, then we can always construct two matrices $S = D\,C \in \mathrm{GL}_n(C[[t^{-1}]])$ and $T \in \mathrm{GL}_n(C[[t]])$ such that the λ-free rows of the equivalent operator $\widehat{L} = SLT$ are linearly dependent.

- Moreover if we note $\widehat{L} = \widehat{A}\,\delta + \widehat{B}\,\phi$, then we have

$$\nu(\det(\widehat{A})) = \nu(\det(A)).$$

- D is an "appropriate" diagonal matrix.

- C is an "appropriate" constant matrix.

- $T = D^{-1}$.

Case 2: λ-free rows are linearly independent

Example

Let $K = C((x))$, $\phi(x) = q\,x$, $q \in C^*$ is not a root of unity, and $\delta = \phi - \mathrm{id}_K$. Consider the q-difference operator:

$$L = \underbrace{\begin{bmatrix} x & 0 \\ 0 & x+1 \end{bmatrix}}_{A} \delta + \begin{bmatrix} -x & q\,x+1 \\ -\frac{x}{q} & \frac{(q^3-q^2+q)x+1}{q^2} \end{bmatrix} \phi$$

- The leading matrix pencil of the operator is:

$$L_\lambda = \begin{bmatrix} \lambda + q^{-2} & 0 \\ 1 & 0 \end{bmatrix}.$$

- The operator L is not simple and we have:

$$\nu(\det(A)) = 1.$$

Case 2: λ-free rows are linearly independent

- Multiply on the left by $S = D\,C$ and on the right by $T = D^{-1}$ where:

$$C = \begin{bmatrix} 1 & -\frac{1}{q^2} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \mathrm{diag}(x^{-1}, 1)$$

$$\Longrightarrow \widehat{L} = \underbrace{\begin{bmatrix} 1+x & \frac{x}{q^2} \\ 0 & x \end{bmatrix}}_{\widehat{A}} \delta + \begin{bmatrix} q-1+x\,q^2-x & \frac{x\left(q^2-1\right)}{q^2} \\ (x\,q+1)\,x\,q & \frac{(-q+1+x\,q)x}{q} \end{bmatrix} \phi$$

- The leading pencil is:

$$\widehat{L}_\lambda = \begin{bmatrix} \lambda+q-1 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Moreover $\nu(\det(\widehat{A})) = \nu(\det(A)) = 1$.

Rational Solutions of First Order Pseudo-Linear Systems

Strategy

Let $K = C(x)$. Consider a PLS $\delta(\mathbf{y}) = M\,\phi(\mathbf{y})$ where $M \in \mathbb{M}_n(K)$.

1. Compute first a universal denominator $u \in C[x]$: a polynomial which is a multiple of the denominator of any rational solution.

2. Perform the change of variable $\mathbf{y} = \mathbf{z}/u$.

3. Compute polynomial solutions of the resulting system

$$\delta(\mathbf{z}) = \phi\left(u^{-1}\right)\left(u\,M + \delta(u)\right)\,\phi(\mathbf{z}),$$

which is of the same type as the original one.

Polynomial solutions

Let $K = C(x)$. Consider a PLS $\delta(\mathbf{y}) = M\,\phi(\mathbf{y})$ where $M \in \mathbb{M}_n(K)$.

Strategy to compute polynomial solutions:

1. Compute a degree bound of the polynomial. This can be done using simple forms.

2. Once a degree bound is obtained, compute the different monomials $\alpha_i\, x^{\mu_i}$ ($\alpha_i \in C^n$, $\mu_i \in \mathbb{N}$) of the polynomial. This can be done using the monomial-by-monomial approach of Barkatou, Broughton & Pflügel 2010.

We focus on the computation of a universal denominator.

ϕ-systems

- Let $K = C(x)$. For a PLS $\delta(\mathbf{y}) = M\,\phi(\mathbf{y})$, we have two cases:

  1. $\phi = \mathrm{id}_K$: corresponds to differential systems. An algorithm is already developed: Barkatou 1999 using simple forms.

  2. $\phi \neq \mathrm{id}_K$: corresponds to ϕ-systems

     $$\phi(\mathbf{y}) = N\,\mathbf{y},$$

     where $N \in \mathrm{GL}_n(K)$ and $\phi : x \mapsto q\,x + r$. Here $r \in C$ and $q \in C^*$ is not a root of unity, but if $r \neq 0$ then q is allowed to be equal to 1.

- We focus on ϕ-systems, which includes pure difference (Abramov & Barkatou 1998) and q-difference (Abramov 2002) systems.

Review: the q-difference case

- We refer to the work of Abramov 2002.

- In the pure q-difference case $\phi : x \mapsto q\,x$, a universal denominator is written as:
$$x^{\alpha}\,U(x),$$
where $\alpha \in \mathbb{N}$ and $U(x)$ is not divisible by x.

- x^{α} is called the fixed part. It corresponds to the only ϕ-fixed singularity 0. It can be computed using EG-eliminations (Abramov 1999) or simple forms.

- $U(x)$ is the non fixed part. It can be computed using resultant and gcd computations.

General formula for a universal denominator

- Similarly, for general ϕ-systems with $\phi : x \mapsto q\,x + r$, a universal denominator is written as:

$$\left(x - \frac{r}{1-q}\right)^{\alpha} U(x),$$

where $\alpha \in \mathbb{N}$ and $U(x)$ is not divisible by $x - \frac{r}{1-q}$.

- $\left(x - \frac{r}{1-q}\right)^{\alpha}$ is the fixed part. It corresponds to the only ϕ-fixed singularity $x_0 = \frac{r}{1-q}$. It can be computed using simple forms.

- $U(x)$ is the non fixed part. It can be computed using resultant and gcd computations.

- However, we extended the ideas of Man and Write 1994 to have a more efficient method for computing $U(x)$.

Example

Let $\phi : x \mapsto 3\,x+2$. Consider the ϕ-system

$$\phi(\mathbf{y}) = \begin{bmatrix} \dfrac{3\,x+2}{9\,x} & 0 \\ \dfrac{2\,(x+1)^3\,(13\,x+2)}{3\,(3\,x+2)\,(3\,x+1)\,x} & \dfrac{(x-1)\,(x-2)}{3\,(3\,x+2)\,(3\,x+1)} \end{bmatrix} \mathbf{y}.$$

- The only ϕ-fixed singularity is $x_0 = -1$ and thus we write a universal denominator under the form

$$(x+1)^{\alpha}\, U(x),$$

where $U(x)$ is not divisible by $x+1$.

Example

- Let $t = x + 1$ and write the system as a PLS $A\,\delta(\mathbf{y}) + B\,\phi(y)$ over $K = C((t))$, where $\delta = \mathrm{id}_K - \phi$.
- Compute a simple form.
- The indicial polynomial then reads $\varphi(\lambda) = \left(-1 + 9\,3^{\lambda}\right)\left(-1 + 3^{\lambda}\right)$.
- Compute $\alpha = \max\{\, \lambda \in \mathbb{N} \quad ; \quad \varphi(-\lambda) = 0 \,\} \longrightarrow \alpha = 2$.
- $U(x)$ can be computed as $U(x) = x\,(x-1)\,(x-2)^2$.

Rational solutions of Partial Pseudo-Linear Systems

Motivation example

Compute rational solutions of the partial system:

$$\left\{\mathbf{y}(x,k+1)=A(x,k)\mathbf{y}(x,k),\quad \frac{\partial \mathbf{y}}{\partial x}(x,k)=B(x,k)\mathbf{y}(x,k)\right\}$$

$$A=\begin{bmatrix} \frac{kx+k+x}{2k+2} & \frac{x}{2(k+1)k} & \frac{(x-1)k+x}{2k+2} \\ 0 & \frac{k+1}{k} & 0 \\ \frac{(x-1)k+x}{2k+2} & \frac{-x}{2(k+1)k} & \frac{kx+k+x}{2k+2} \end{bmatrix}, \quad B=\begin{bmatrix} \frac{k+1}{2x} & \frac{-1}{2k} & \frac{k-1}{2x} \\ 0 & -\frac{1}{x} & 0 \\ \frac{k-1}{2x} & \frac{1}{2k} & \frac{k+1}{2x} \end{bmatrix}.$$

- **Step 1:** Compute rational solutions of the difference system $\mathbf{y}(x,k+1)=A(x,k)\mathbf{y}(x,k)$ considering x as a parameter:

$$\longrightarrow \quad W_1=\begin{bmatrix} \frac{x}{2} & -\frac{1}{k} \\ k & 0 \\ -\frac{x}{2} & \frac{1}{k} \end{bmatrix}$$

- **Step 2:** Let $P = (W_1 \quad v) \in \mathrm{GL}_n(K)$ with some $v \in K^3$. Perform the change of variable $\mathbf{y} = P\,\mathbf{z}$. We are then reduced to solving the linear system $W_1\,N = -\left(\frac{\partial W_1}{\partial x} - B\,W_1\right)$:

$$\longrightarrow \quad N = \begin{bmatrix} -\frac{1}{x} & 0 \\ 0 & \frac{1}{x} \end{bmatrix} \quad \text{independent of } k$$

- **Step 3:** Compute rational solutions of the differential system $\frac{d}{d\,x}\mathbf{z}(x) = N(x)\,\mathbf{z}(x)$ in one variable x:

$$\longrightarrow \quad W_2 = \begin{bmatrix} 0 & \frac{1}{x} \\ x & 0 \end{bmatrix}$$

- Rational solutions $\longrightarrow \quad W_1\,W_2 \quad = \begin{bmatrix} -\frac{x}{k} & \frac{1}{2} \\ 0 & \frac{k}{x} \\ \frac{x}{k} & -\frac{1}{2} \end{bmatrix}$

Problem

- Let $K = C(x_1, \ldots, x_m)$. Compute all rational solutions of partial pseudo-linear system PPLS

$$\begin{cases} L_1(\mathbf{y}) := \delta_1(\mathbf{y}) - M_1\,\phi_1(\mathbf{y}) = 0, \\ \quad\vdots \\ L_m(\mathbf{y}) := \delta_m(\mathbf{y}) - M_m\,\phi_m(\mathbf{y}) = 0, \end{cases}$$

  where $M_i \in \mathcal{M}_n(K)$.

- The integrability conditions are $L_i \circ L_j = L_j \circ L_i$.
- For all $j \neq i$, x_j is a constant with respect to ϕ_i and δ_i.
- Each system $L_i(\mathbf{y}) = 0$ is considered as a system with one variable x_i, and the other variables are parameters.

General algorithm

1. Let $W \in \mathbb{M}_{n\times s}(K)$ be a matrix whose columns form a basis of the rational solutions of $L_1(y) = 0$.
2. if $m = 1$, **RETURN** W.
3. For $i = 2, \ldots, m$, Compute the matrices $N_i \in \mathbb{M}_s(C(x_2, \ldots, x_m))$, solutions of the $m - 1$ matrix linear systems $W\, N_i = -L_i(W)$.
4. **Return** W multiplied by the result of applying the algorithm to the system
$$\begin{cases} L_2(\mathbf{y}) := \delta_2(\mathbf{y}) - N_2\, \phi_2(\mathbf{y}) = 0, \\ \quad \vdots \\ L_m(\mathbf{y}) := \delta_m(\mathbf{y}) - N_m\, \phi_m(\mathbf{y}) = 0, \end{cases}$$
which is free of the variable x_1.

Example

Let $K = C(x_1, x_2, x_3)$. Consider the PPLS

$$\left\{ \begin{array}{l} L_1(\mathbf{y}) := \delta_1(\mathbf{y}) - M_1\,\phi_1(\mathbf{y}), \\ L_2(\mathbf{y}) := \delta_2(\mathbf{y}) - M_2\,\phi_2(\mathbf{y}), \\ L_3(\mathbf{y}) := \delta_3(\mathbf{y}) - M_3\,\phi_3(\mathbf{y}), \end{array} \right.$$

with

$$\phi_1 : (x_1, x_2, x_3) \mapsto (x_1 - 1, x_2, x_3), \quad \delta_1 = \mathrm{id}_K - \phi_1,$$

$$\phi_2 : (x_1, x_2, x_3) \mapsto (x_1, x_2/q, x_3), \quad \delta_2 = \mathrm{id}_K - \phi_2,$$

$$\phi_3 = \mathrm{id}, \quad \delta_3 = \partial/\partial x_3,$$

$$M_1 = \begin{bmatrix} \frac{x_1}{x_1-1} & -\frac{q x_3 (x_3+x_1-1)}{{x_2}^2 (x_1-1)} \\ 0 & \frac{x_3+x_1-1}{x_3+x_1} \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & -\frac{x_3 (x_3+x_1)(q-1) q^2}{{x_2}^2} \\ 0 & q \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & \frac{q (x_3+x_1)}{{x_2}^2} \\ 0 & -(x_3+x_1)^{-1} \end{bmatrix}.$$

Example

- Compute a basis of rational solutions of $L_1(\mathbf{y}) = 0$ in one variable x_1:

$$\longrightarrow \quad W_1 = \begin{bmatrix} \frac{x_1 - x_3}{x_2^4} & \frac{1}{x_2^4} \\ \frac{-1}{q\, x_2^2\, (x_3 + x_1)} & \frac{1}{q\, x_2^2\, x_3 (x_3 + x_1)} \end{bmatrix}.$$

- Solving the linear systems $W_1\, N_2 = -L_2(W_1)$ and $W_1\, N_3 = -L_3(W_1)$:

$$\longrightarrow \quad N_2 = \begin{bmatrix} -q^4 + 1 & 0 \\ -q^3\,(q-1)\,x_3 & -q^3 + 1 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

- We are reduced to solving the PPLS:

$$\begin{cases} \widetilde{L_2}(\mathbf{y}) := \delta_2(\mathbf{y}) - N_2\, \phi_2(\mathbf{y}) = 0, \\ \widetilde{L_3}(\mathbf{y}) := \delta_3(\mathbf{y}) - N_3\, \phi_3(\mathbf{y}) = 0. \end{cases}$$

which is free of the variable x_1.

Example

- Compute a basis of rational solutions of $\widetilde{L_2}(\mathbf{y}) = 0$ in one variable x_2:

$$\longrightarrow \quad W_2 = \begin{bmatrix} x_2^4 & 0 \\ x_2^4\, x_3 & x_2^3 \end{bmatrix}.$$

- Solving the linear system $W_2\, \widehat{N_3} = -\widetilde{L_3}(W_2)$:

$$\longrightarrow \quad \widehat{N_3} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

- We are then reduced to solving the PLS:

$$\widehat{L_3}(\mathbf{y}) := \delta_3(\mathbf{y}) - \widehat{N_3}\, \phi_3(\mathbf{y}) = 0$$

which is free of the variables x_1 and x_2.

Example

- Compute a basis of rational solutions of $\widehat{L_3}(\mathbf{y}) = 0$:

$$\longrightarrow \quad W_3 = \begin{bmatrix} 0 & 1 \\ x_3 & 0 \end{bmatrix}.$$

- Finally, a basis of rational solutions of the original system is spanned by the columns of

$$W_1\, W_2\, W_3 = \begin{bmatrix} \frac{x_3}{x_2} & x_1 \\ \frac{x_2}{(x_3 + x_1)\, q} & 0 \end{bmatrix}.$$

Comparison of strategies

- The systems in the PPLS can be considered in arbitrary order. Are some orders better than others from the computational point of view?
- We give some timings of one of our experiments carried on several PPLS composed of one differential and one difference system (with two variables x and k).
- The matrices of the systems are generated from a randomly chosen fundamental matrix of rational solutions but whose denominator U is fixed as a product of some of the following three polynomials:

$$U_1(x,k) = (x+k)(x-k)^2\left(-k^2+x\right)\left(-k^3+x^2+3\right),$$

$$U_2(x,k) =$$

$$77\,k^8x^6 - 51\,k^2x^{12} + 31\,k^5x^8 + 10\,k^4x^9 + 68\,x^{13} + 91\,x^{12} - 81\,k^{10} + 40\,k^4x^6 - 47\,k^2x^5 - 49\,kx,$$

$$U_3(x,k) =$$

$$k\left(6\,k^{10}x + 5\,kx^9 + 6\,k^2x^7 + 3\,k^7 + 2\,k^6x - 4\,x^7 + 4\,k^4x^2 + k^4x - 3\,x^4 - 5\,k\right).$$

$$U_1(x,k) = (x+k)(x-k)^2(-k^2+x)(-k^3+x^2+3),$$

$$U_2(x,k) =$$

$$77\,k^8x^6 - 51\,k^2x^{12} + 31\,k^5x^8 + 10\,k^4x^9 + 68\,x^{13} + 91\,x^{12} - 81\,k^{10} + 40\,k^4x^6 - 47\,k^2x^5 - 49\,kx,$$

$$U_3(x,k) =$$

$$k\,(6\,k^{10}x + 5\,kx^9 + 6\,k^2x^7 + 3\,k^7 + 2\,k^6x - 4\,x^7 + 4\,k^4x^2 + k^4x - 3\,x^4 - 5\,k).$$

Strategy 1: we start with the differential system.
Strategy 2: we start with the difference system.

	$U = U_1$			$U = U_1U_2$			$U = U_1U_2U_3$		
	$n=3$	$n=6$	$n=9$	$n=3$	$n=6$	$n=9$	$n=3$	$n=6$	$n=9$
St 1	0.4	2.2	9.0	22.9	187.5	574.8	249.9	912.9	1703
St 2	0.3	2.8	16.4	0.3	2.1	12.2	0.9	3.3	15.1

Conclusion: We always treat the ϕ-systems first and consider the differential systems at the end of the iterative process.

The PseduoLinearSystems package

The PseduoLinearSystems package

- All algorithms developed are implemented in our Maple package PseduoLinearSystems.

- Our package is dedicated to the study of general pseudo-linear systems, unlike existing packages such as ISOLDE and LinearFunctionalSystems which are dedicated to the individual differential and (q-)difference systems.

- It uses simple forms to compute necessary local data (not super-reduction, not EG-eliminations).

- The PseudoLinearSystems package is freely available at `http://www.unilim.fr/pages_perso/ali.el-hajj/PseudoLinearSystems.html`.

Important procedures

- Simple forms of pseudo-linear systems.
- Local data: indicial polynomials, super-irreducible forms, k-simple forms, ...
- Rational solutions of first order PLS (including differential and (q-)difference systems).
- Exponential solutions of differential systems.
- Hypergeometric solutions of difference systems.
- Rational solutions of partial pseudo-linear systems.
- Hypergeometric solutions of PPLS composed of differential and/or difference systems.
- Eigenring of (partial) pseudo-linear systems.

References

S. Abramov. (1999) EG-eliminations. Journal of Difference Equations and Applications, 5(4-5), 393-433, 1999.

Abramov S. (2002) A direct algorithm to compute rational solutions of first order linear q-difference systems. Discrete Mathematics 246, p. 3-12.

S. Abramov, A. Gheffar. (2010) Factorization of Polynomials and GCD Computations for Finding Universal Denominators. Computer Algebra in Scientific Computing, 12th International Workshop, CASC 2010.

Barkatou M. (1999).Rational solutions of matrix difference equations: the problem of equivalence and factorization. In Proc. ISSAC'99, 277-282, 1999.

Barkatou M. (1999) On rational solutions of systems of linear differential equations. J. of Symb. Comp., 28, 547-567.

Barkatou M., Broughton G., Pflügel E. (2010) A monomial-by-monomial method for computing regular solutions of systems of pseudo-linear equations. Math. in Comp. Sci., 4(2-3), p. 267-288.

M. A. Barkatou, T. Cluzeau, A. El Hajj. (2019) Simple forms and rational solutions of pseudo-linear systems. In Proc. ISSAC'19, 26-33, 2019.

M. Bronstein, M. Petkovsek (1996). An introduction to pseudo-linear algebra. Theor. Comput. Sci., 157, 3-33, 1996.

Y. K. Man, F. J. Write. (1994) Fast polynomial dispersion computation and its application to indefnite summation. In Proc. ISSAC'94, 175-180, 1994.

Thank you !