Stability Problems in Symbolic Integration

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Moscow Seminar in Computer Algebra

June 15, 2022 (on-line)

Dedicated to Professor Ziming Li on the occasion of his 60th birthday

Working with Ziming

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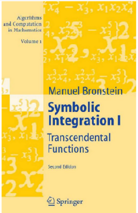
1. An Additive Decomposition in Logarithmic Towers and Beyond (with H. Du, J. Guo and E. Wong). In *Proce*
2. Apparent Singularities of D-finite Systems (with S. Chen, M. Kauers and Y. Zhang). In *Journal of Symbolic C*
3. Additive Decompositions in Primitive Extensions (with S. Chen and H. Du). In *Proceedings of the 2018 Inte*
4. A q-Analogue of the Modified Abramov-Petkovsek Reduction (with H. Du and H. Huang). Advances in Com
5. On the existence of telescopers for mixed hypergeometric terms (with S. Chen, F. Chyzak, R. Feng and G. Fu
6. A modified Abramov-Petkovsek reduction and creative telescoping for hypergeometric terms (with S. Chen,
7. Parallel telescoping and parameterized Picard-Vessiot theory (with S. Chen, R. Feng, and M.F. Singer). In *Pr*
8. Hermite reduction and creative telescoping for hyperexponential functions (with A. Bostan, S. Chen, F. Chyz
9. Transforming linear functional systems into fully integrable systems (with M. Wu). Journal of Symbolic Con
10. Fast computation of common left multiples of linear ordinary differential operators (with A. Bostan, F. Chyza
11. On the structure of compatible rational functions (with S. Chen, G. Fu and R. Feng). In *Proceedings of the 20*
12. Some remarks on Kahler differentials and ordinary differentials in nonlinear control theory (with G. Fu, M. H
13. Complexity of creative telescoping for bivariate rational functions (with A. Bostan, S. Chen and F. Chyzak).



Ziming, thanks for your supervising and collaborations!

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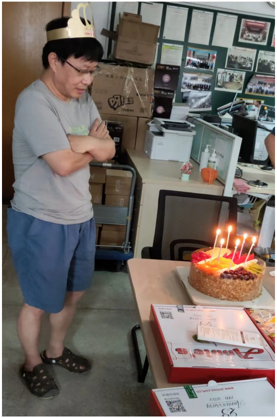

Ziming, Happy Birthday!

Integration Problems

Indefinite Integration. Given a function $f(x)$ in certain class $\mathfrak{C}$, decide whether there exists $g(x) \in \mathfrak{C}$ such that

$$f = \frac{dg}{dx} \triangleq g'.$$

Example. For $f = \log(x)$, we have $g = x\log(x) - x$.

Definite Integration. Given a function $f(x)$ that is continuous in the interval $I \subseteq \mathbb{R}$, compute the integral

$$\int_I f(x) dx.$$

Example. For $f = \log(x)$ and $I = [1,2]$, we have

$$\int_I f(x) dx = 2\log(2) - 1.$$

Fundamental Theorem of Calculus

Newton–Leibniz Theorem. Let $f(x)$ be a continuous function on $[a,b]$ and let $F(x)$ be defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for all } x \in [a,b].$$

Then $F(x)' = f(x)$ for all $x \in [a,b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(Newton–Leibniz formula)}$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_1^2 \log(x)\,dx = F(2) - F(1) = 2\log(2) - 1, \quad \text{where } F(x) = x\log(x) - x.$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_0^{+\infty} \exp(-x^2)\,dx = ?$$

What is Elementary Functions?

$$\mathfrak{E} := (\{\mathbb{C}, x\}, \quad \{+, -, \times, \div\}, \quad \{\exp(\cdot), \log(\cdot), \mathrm{RootOf}(\cdot)\}).$$

Definition. An elementary function is a function of x which is the composition of a finite number of

- binary operations: $+, -, \times, \div$;
- unary operations: exponential, logarithms, constants, solutions of polynomial equations.

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Example.

$$3x^2 + 3x + 1$$

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Example.

$$\frac{1}{3x^2+3x+1}$$

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Example.

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$$\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)$$

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Example.

$$\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1$$

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Definition. An elementary function is a function of x which is the composition of a finite number of

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Example.

$$\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)$$

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What is Elementary Functions?

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Definition. An elementary function is a function of x which is the composition of a finite number of

- binary operations: $+, -, \times, \div$;
- unary operations: exponential, logarithms, constants, solutions of polynomial equations.

Example.

$$\frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D: R \to R$ is called a derivation on R if

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

The pair (R, D) is called a differential ring. If R is a field, it is then called a differential field.

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Example.
Polynomial ring: $(\mathbb{C}[x], ')$

$$P = \sum_{i=0}^{n} p_i x^i \quad \rightsquigarrow \quad P' = \sum_{i=0}^{n} i p_i x^{i-1}.$$

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Example.
Rational-function field: $(\mathbb{C}(x), ')$

$$f = \frac{P}{Q} \quad \rightsquigarrow \quad f' = \frac{P'Q - PQ'}{Q^2}.$$

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Example.
Elementary-function field: algebraic case

$$(\mathbb{C}(x)(\alpha), ') \text{ with } \alpha \text{ algebraic over } \mathbb{C}(x)$$

$$r_d \alpha^d + r_{d-1} \alpha^{d-1} + \cdots + r_0 = 0 \quad \rightsquigarrow \quad \alpha'(x) = -\frac{r_d' \alpha^d + \cdots + r_0'}{d r_d \alpha^{d-1} + \cdots + r_1}$$

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Example.
Elementary-function field: exponential case

$$(\mathbb{C}(x)(\exp(x)), ')$$

$$f = \frac{1 + x + \exp(x)}{x^2 + \exp(x)} \quad \rightsquigarrow \quad f' = \frac{x(x\exp(x) - 3\exp(x) - x - 2)}{(x^2 + \exp(x))^2}.$$

Differential Algebra

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Example.
Elementary-function field: logarithmic case

$$(\mathbb{C}(x)(\log(x)), ')$$

$$f = \frac{1 + x + \log(x)}{x^2 + \log(x)} \quad \rightsquigarrow \quad f' = -\frac{2\log(x)x^2 + x^3 - \log(x)x + x^2 + x + 1}{(x^2 + \log(x))^2 x}.$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D: R \to R$ is called a derivation on R if

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Example.
Elementary-function field: general case

$$(\mathbb{C}(x)(t_1, t_2, t_3, \ldots, t_n), ')$$

$$t_1 = \sqrt{x^2+1}, \quad t_2 = \log(1+t_1^2), \quad t_3 = \exp\left(\frac{1+t_1}{t_1+t_2^2}\right), \ldots$$

Elementary Extensions

Differential Extension. (R^*, D^*) is called a differential extension of (R, D) if $R \subseteq R^*$ and $D^* |_R = D$.

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Elementary Extension. Let (E, D) be a differential extension of (F, D). An element $t \in E$ is elementary over F if one of the following conditions holds:

- t is algebraic over F;
- $D(t)/t = D(u)$ for some $u \in F$, i.e., $t = \exp(u)$;
- $D(t) = D(u)/u$ for some $u \in F$, i.e., $t = \log(u)$.

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Example. $(F, D) = (\mathbb{C}(x), ')$ and $(E, D) = (\mathbb{C}(x, \log(x)), ')$.

Elementary Functions

Definition. An function $f(x)$ is elementary if $\exists$ a differential extension $(E,')$ of $(\mathbb{C}(x),')$ s.t. $E = \mathbb{C}(x)(t_1,\ldots,t_n)$ and t_i is elementary over $\mathbb{C}(x)(t_1,\ldots,t_{i-1})$ for all $i = 2,\ldots,n$.

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Example.

$$f(x) = \frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Then $f(x)$ is elementary since $\exists$ a differential extension

$$E = \mathbb{C}(x)(t_1,t_2,t_3,t_4),$$

where

$$t_1 = \sqrt{\frac{1}{3x^2+3x+1}}, \quad t_2 = \exp(t_1), \quad t_3 = \log(t_2^2 + x^2 + 1), \quad t_4 = \sqrt{t_3}.$$

Symbolic Integration

Let (F, D) and (E, D) be two differential fields such that $F \subseteq E$.

Problem. Given $f \in F$, decide whether there exists $g \in E$ s.t. $f = D(g)$. If such g exists, we say f is integrable in E.

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Elementary Integration Problem. Given an elementary function $f(x)$ over $\mathbb{C}(x)$, decide whether $\int f(x)dx$ is elementary or not.

Example. The following integrals are not elementary over $\mathbb{C}(x)$:

$$\int \exp(x^2)dx, \quad \int \frac{1}{\log(x)}dx, \quad \int \frac{\sin(x)}{x}dx, \quad \int \frac{dx}{\sqrt{x(x-1)(x-2)}}, \ \cdots$$

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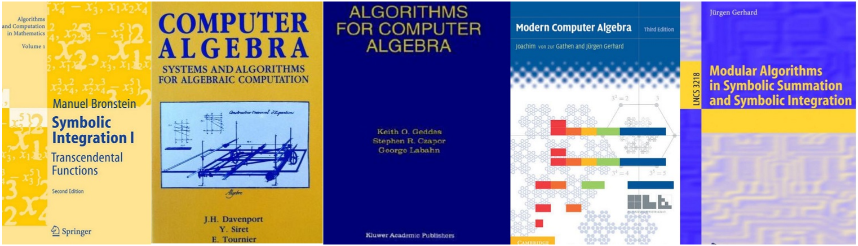
Selected books on Symbolic Integration:



Liouville's Theorem

Theorem (Liouville1835). Let $f(x)$ be elementary over $\mathbb{C}(x)$, i.e.,

$$f \in F = \mathbb{C}(x)(t_1, t_2, \ldots, t_n).$$

If $\int f(x)\,dx$ is elementary, then

$$\int f(x)\,dx = \underbrace{g_0}_{F\text{-part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

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where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Remark. With the above theorem, Liouville proved that the integrals

$$\int \exp(x^2)\,dx, \quad \int \frac{1}{\log(x)}\,dx, \quad \int \frac{\sin(x)}{x}\,dx, \ldots$$

are not elementary.

Two classical theorems

Liouville-Hardy Theorem. Let $f \in \mathbb{C}(x)$. Then $f \cdot \log(x)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = \frac{c}{x} + g' \quad \text{for some } c \in \mathbb{C} \text{ and } g \in \mathbb{C}(x).$$

Liouville's Theorem. Let $f, g \in \mathbb{C}(x)$. Then $f \cdot \exp(g)$ is elementary integrable over $\mathbb{C}(x)$ if and only if

$$f = h' + g'h \quad \text{for some } h \in \mathbb{C}(x).$$

Why $\exp(x^2)$ is not Elementary Integrable?

Let $t = \exp(x^2)$. We prove by contradiction.

Proof. If $\int t\,dx$ is elementary, Liouville's theorem implies that $\exists g_0, \ldots, g_n \in \mathbb{C}(x,t)$ and $c_0, \ldots, c_n \in \mathbb{C}$ s.t.

$$\int t\,dx = g_0 + \sum_{i=1}^{n} c_i \log(g_i) \quad \Leftrightarrow \quad t = g_0' + \sum_{i=1}^{n} c_i \frac{g_i'}{g_i}$$

$$\Downarrow$$

$$t = (ft)' \quad \text{for some } f \in \mathbb{C}(x) \quad \Leftrightarrow \quad 1 = f' + 2xf$$

Claim. The differential equation

$$y(x)' + 2x \cdot y(x) = 1$$

has no rational-function solution!

The irrationality of π

Suppose that $\pi/2 = a/b \in \mathbb{Q}$. Consider

$$I_n(x) = \int_{-1}^{1} (1-z^2)^n \cdot \cos(xz)\, dz \quad (n \in \mathbb{N})$$

Let $J_n(x) := x^{2n+1} I_n(x)$. Then

$$J_n(x) = 2n(2n-1)J_{n-1}(x) - 4n(n-1)x^2 J_{n-2}(x).$$

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$$J_n(x) = x^{2n+1} I_n(x) = n!\,(P_n(x)\sin(x) + Q_n(x)\cos(x)),$$

where $P_n, Q_n \in \mathbb{Z}[x]$ are of degree $\leq n$.

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where $P_n, Q_n \in \mathbb{Z}[x]$ are of degree $\leq n$. Taking $x = \pi/2$ yields

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) = P_n(\pi/2) b^{2n+1} \in \mathbb{N}.$$

But $0 < I_n(\pi/2) < 2$, which implies

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) \to 0 \quad (\text{as } n \to +\infty).$$

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But $0 < I_n(\pi/2) < 2$, which implies

$$\frac{a^{2n+1}}{n!} I_n(\pi/2) \to 0 \quad (\text{as } n \to +\infty).$$ a contradiction!

Stability in dynamical systems

A (discrete) dynamical system is a pair (X,ϕ) with X being any set and $\phi: X \to X$ a self-map on X.

- Subset of fixed points:

$$\text{Fix}(\phi,X) = \{x \in X \mid \phi(x) = x\}.$$

- Subset of periodic points:

$$\text{Per}(\phi,X) = \{x \in X \mid \phi^n(x) = x \text{ for some } n \in \mathbb{N}\setminus\{0\}\}.$$

- Subset of stable points:

$$\text{Stab}(\phi,X) = \{x \in X \mid \exists \{x_i\}_{i\geq 0} \text{ s.t. } x_0 = x \text{ and } \phi(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\}.$$

- Subset of attractive points:

$$\text{Attrac}(\phi,X) = \bigcap_{i\in\mathbb{N}} \phi^i(X).$$

Stability in dynamical systems

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- Subset of fixed points:

$$\mathsf{Fix}(\phi, X) = \{x \in X \mid \phi(x) = x\}.$$

- Subset of periodic points:

$$\mathsf{Per}(\phi, X) = \{x \in X \mid \phi^n(x) = x \text{ for some } n \in \mathbb{N} \setminus \{0\}\}.$$

- Subset of stable points:

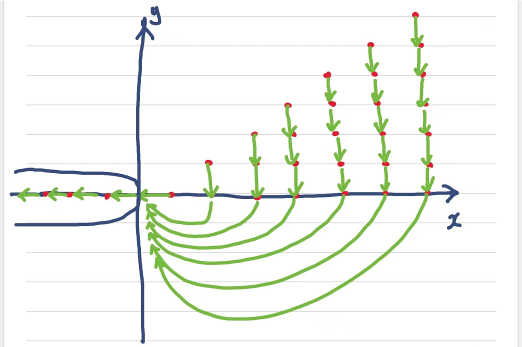
$$\mathsf{Stab}(\phi, X) = \{x \in X \mid \exists \{x_i\}_{i \geq 0} \text{ s.t. } x_0 = x \text{ and } \phi(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\}.$$

- Subset of attractive points:

$$\mathsf{Attrac}(\phi, X) = \bigcap_{i \in \mathbb{N}} \phi^i(X).$$

$$\mathsf{Fix}(\phi, X) \subseteq \mathsf{Per}(\phi, X) \subseteq \mathsf{Stab}(\phi, X) \subseteq \mathsf{Attrac}(\phi, X).$$

Godelle's example



Example. Let $X = \{(i,j) \in \mathbb{Z}^2 \mid 0 \leq j \leq \max\{i-1,0\}\}$ and $\phi : X \to X$ be such that

$$\phi((i,j)) = (i,j-1) \text{ if } j > 0 \text{ and } \phi((i,0)) = (\min i-1,0,0).$$

Then $\text{Stab}(\phi,X) = \emptyset$ and $\text{Attrac}(\phi,X) = \{(i,0) \mid i \leq 0\}$.

Stability in differential fields

Idea. Viewing a differential field (K, D) as a dynamical system.

$$D(f+g) = D(f) + D(g) \quad \text{and} \quad D(fg) = gD(f) + fD(g).$$

Definition. $C_K := \{c \in K \mid D(c) = 0\}$ is called the constant subfield of (K, D).

Remark. K is a C_K-vector space and $D : K \to K$ is C_K-linear.

Proposition. Let (K, D) be a differential field of char. zero. Then

$$\text{Stab}(D, K) = \text{Attrac}(D, K).$$

Stability Problem. Given $f \in K$, decide whether f is stable or not, i.e., for all $i \in \mathbb{N}$, $f = D^i(g_i)$ for some $g_i \in K$.

Structure theorem

Lemma. Let (K, D) be a differential field with $D(x) = 1$ and $f \in K$. Then

(*i*) $f = D^n(g)$ for some $g \in K$ iff for any i with $0 \leq i \leq n-1$, $\exists h_i \in K$ s.t. $x^i f = D(h_i)$.

(*ii*) f is stable iff for all $i \in \mathbb{N}$, $x^i f = D(g_i)$ for some $g_i \in K$.

Theorem. Let (K, D) be a differential field with $D(x) = 1$. Then Stab(D, K) forms a differential $C_K[x]$-module.

Problem. Is Stab(D, K) always a free $C_K[x]$-module?

Example. $\exp(c \cdot x)$ is stable, so are

$$x^n \exp(c \cdot x), \quad x^n \sin(c \cdot x), \quad x^n \cos(c \cdot x), \quad \ldots$$

Stable elementary functions

Let $\mathscr{E}_{\mathbb{C}(x)}$ be the field of all elementary functions over $\mathbb{C}(x)$.

Theorem. Let $D = d/dx$ and $f, g \in \mathbb{C}(x)$ with $g \notin \mathbb{C}$. Then

(i) f is always stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$.

(ii) f is stable in $(\mathbb{C}(x), D)$ iff $f \in \mathbb{C}[x]$.

(iii) $f \cdot \log(x)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x, x^{-1}]$.

(iv) $f \cdot \exp(g)$ is stable in $(\mathscr{E}_{\mathbb{C}(x)}, D)$ iff $f \in \mathbb{C}[x]$ and $g = ax + b$ with $a, b \in \mathbb{C}$ with $a \neq 0$.

Examples. Stable elementary functions: $f(x) \in \mathbb{C}(x)$, $\exp(ax+b)$,

$$\log(f(x)), \quad \sin(x), \quad \cos(x), \quad \arcsin(x) \quad \arccos(x), \quad \arctan(x), \ldots$$

Non-stable elementary functions: $\tan(x)$, $\cot(x)$, $\sec(x)$, $\csc(x), \ldots$

D-finite power series and exact integration

Definition. $f(x) \in \mathbb{C}[[x]]$ is D-finite over $\mathbb{C}(x)$ if $\exists\ L = \sum_{i=0}^{r} \ell_i \cdot D_x^i$ in $\mathbb{C}(x)\langle D_x \rangle$ with $\ell_r \neq 0$ s.t. $L(f) = 0$, equivalently

$$\dim_{\mathbb{C}(x)} \left(\text{span}_{\mathbb{C}(x)} \{ D_x^i(f) \mid i \in \mathbb{N} \} \right) < +\infty.$$

If L is monic and of minimal order r, then call L the minimal annihilator for f and call r the order of f, denoted by $\text{ord}(f)$.

Remark. In general, the formal integral int(f):= $\int f(x)dx$ has the minimal annihilator of order $\text{ord}(f) + 1$.

Exact Integration. In 1997, Abramov and van Hoeij gave an algorithm to decide whether $\int f(x)dx$ has an annihilator of the same order as that of f.

Stable D-finite power series

Let $f(x) \in \mathbb{C}[[x]]$ be a D-finite power series.

Definition. $f(x)$ is stable if $\exists \, \{g_i\}_{i\in\mathbb{N}} \in \mathbb{C}[[x]]$ s.t. $g_0 = f$ and

$$g_i = D_x(g_{i+1}) \text{ and } \operatorname{ord}(g_i) = \operatorname{ord}(f) \quad \text{for all } i \in \mathbb{N}.$$

$f(x)$ is eventually stable if $\exists \, m \in \mathbb{N}$ s.t. $\operatorname{int}^m(f)$ is stable.

Theorem. Any D-finite power series is eventually stable.

Example (Z.-W. Guo). The Airy function $\operatorname{Ai}(x)$ satisfies

$$y''(x) = xy(x).$$

By Abramov-van Hoeij's algorithm, we have $\operatorname{Ai}(x)$ is not stable, but is eventually stable with $\operatorname{ord}(\operatorname{int}^m(\operatorname{Ai}(x))) = 3$ for all $m \geq 2$.

Open problems

Problem. Characterizing stable algebraic functions in $(\overline{\mathbb{C}(x)}, d/dx)$.

Conjecture. An algebraic function $f(x)$ is stable in $(\overline{\mathbb{C}(x)}, d/dx)$ iff

$$f(x) = \sum_{i=1}^{n} p_i \cdot (x - c_i)^{r_i},$$

where $p_i \in \mathbb{C}[x]$, $c_i \in \mathbb{C}$ and $r_i \in \mathbb{Q} \setminus \{-1, -2, \ldots\}$.

Problem. Characterizing stable elementary functions over $\mathbb{C}(x)$.

Conjecture. Let $f(x)$ be an elementary function over $\mathbb{C}(x)$. Then

$$\{i \in \mathbb{N} \mid x^i f(x) \text{ is elementary integrable over } \mathbb{C}(x)\}$$

is a union of finitely many arithmetic progressions.

Thank You!