EXACT SIMPLE SOLUTIONS TO PVI

Alexander BATKHIN

Keldysh Institute of Applied Mathematics of RAS, Moscow, Russia

Seminar on Computer Algebra, CMC faculty of MSU & CCAS
Outline

1. Description of the method
   - Power Geometry essentials
   - Matching “heads” and “tails” of power expansions

2. Painlevé equations and their properties
   - Painlevé equations
   - PVI and its properties

3. Computed Solutions
   - Known elementary solution
   - Computed elementary solution

4. References
We use the method for computing an exact solutions of the ordinary differential equation and apply it to the sixth Painlevé equation (PVI) and other Painlevé equations. This method is essentially uses algorithms of Power Geometry for computing power expansions of solutions to an ordinary differential equation and computer algebra algorithms (Gröbner basis).
Differential monomial

Let \( x \) be independent and \( y \) be dependent variables, \( x, y \in \mathbb{C} \). A **differential monomial** \( a(x, y) \) is a product of an ordinary monomial \( cx^{r_1}y^{r_2} \), where \( c = \text{const} \in \mathbb{C} \), \( (r_1, r_2) \in \mathbb{R}^2 \), and a finite number of derivatives of the form \( d^l y / dx^l \), \( l \in \mathbb{N} \).
Differential monomial

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Differential sum

A sum of differential monomials

\[
f(x, y) = \sum a_i(x, y)
\]

is called the **differential sum**.
Differential monomial

Let $x$ be independent and $y$ be dependent variables, $x, y \in \mathbb{C}$. A differential monomial $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \text{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l, l \in \mathbb{N}$.

Differential sum

A sum of differential monomials

$$f(x, y) = \sum a_i(x, y)$$

is called the differential sum.

Let a differential equation be given

$$f(x, y) = 0,$$

where $f(x, y)$ is a differential sum.
To each differential monomial \(a(x, y)\), we assign its (vector) power exponent \(Q(a) = (q_1, q_2) \in \mathbb{R}^2\) by the following rules:

\[
Q(cx^{r_1} y^{r_2}) = (r_1, r_2); \quad Q(d^l y/dx^l) = (-l, 1);
\]

when differential monomials are multiplied, their power exponents must be added as vectors \(Q(a_1 a_2) = Q(a_1) + Q(a_2)\).
Description of the method

Power Geometry objects

Power exponent

To each differential monomial \( a(x, y) \), we assign its (vector) power exponent \( Q(a) = (q_1, q_2) \in \mathbb{R}^2 \) by the following rules:

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when differential monomials are multiplied, their power exponents must be added as vectors \( Q(a_1a_2) = Q(a_1) + Q(a_2) \).

The Newton–Bruno polygon

The set \( S(f) \) of power exponents \( Q(a_i) \) of all differential monomials \( a_i(x, y) \) present in the differential sum is called the support of the sum \( f(x, y) \). Obviously, \( S(f) \in \mathbb{R}^2 \).

The convex hull \( \Gamma(f) \) of the support \( S(f) \) is called the Newton–Bruno polygon of the sum \( f(x, y) \). The boundary \( \partial \Gamma(f) \) of the polygon \( \Gamma(f) \) consists of generalized faces \( \Gamma^{(d)}_j \).
Power Geometry objects

**Truncated sum**

Each face $\Gamma_j^{(d)}$ corresponds to the truncated sum

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ over } Q(a_i) \in \Gamma_j^{(d)} \cap S(f).$$
**Power Geometry objects**

### Truncated sum

Each face $\Gamma_j^{(d)}$ corresponds to the truncated sum

$$\hat{f}_j^{(d)}(x, y) = \sum a_i(x, y) \text{ over } Q(a_i) \in \Gamma_j^{(d)} \cap S(f).$$

### Normal cone

Let the plane $\mathbb{R}_2^*$ be dual to the plane $\mathbb{R}^2$ such that for $P = (p_1, p_2) \in \mathbb{R}_2^*$ and $Q = (q_1, q_2) \in \mathbb{R}^2$, the scalar product

$$\langle P, Q \rangle \overset{\text{def}}{=} p_1 q_1 + p_2 q_2$$

is defined. Each face $\Gamma_j^{(d)}$ in $\mathbb{R}_2^*$ corresponds to its own normal cone $U_j^{(d)}$ formed by the outward normal vectors $P$ to the face $\Gamma_j^{(d)}$. 
Computation of truncated equations

Each face $\Gamma_j^{(d)}$ corresponds to the normal cone $U_j^{(d)}$ in the plane $\mathbb{R}^2$ and to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.

**Theorem (Bruno)**

*If the power expansion $y = c_r x^r + \sum c_s x^s$, $c_r = \text{const} \in \mathbb{C}$, $c_r \neq 0$, satisfies the equation $f(x, y) = 0$, and $\omega(1, r) \in U_j^{(d)}$, where $\omega = -1$, if $x \to 0$ or $\omega = 1$, if $x \to \infty$ then the truncation $y = c_r x^r$ of the power expansion is the solution to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.***
Computation of truncated equations

Each face $\Gamma_j^{(d)}$ corresponds to the normal cone $U_j^{(d)}$ in the plane $\mathbb{R}^2_*$ and to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.

**Theorem (Bruno)**

If the power expansion $y = c_r x^r + \sum c_s x^s$, $c_r = \text{const} \in \mathbb{C}$, $c_r \neq 0$, satisfies the equation $f(x, y) = 0$, and $\omega(1, r) \in U_j^{(d)}$, where $\omega = -1$, if $x \to 0$ or $\omega = 1$, if $x \to \infty$ then the truncation $y = c_r x^r$ of the power expansion is the solution to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.

As truncated equations are quasi–homogeneous it is not difficult to find its power solutions. For each truncated equation $\hat{f}_j^{(d)}(x, y) = 0$, we need to find all its solutions $y = c_r x^r$ which have one of the vectors $\pm (1, r)$ lying in the normal cone $U_j^{(d)}$. 
Matching “heads” and “tails” of power expansions

The modification of method for computing exact solutions, proposed in [Bruno, Gashenenko, 2006] for finding of exact solutions to N. Kovalewski equations, is used. This method is based on fitting of two power series expansions near the origin and at infinity and getting conditions on the coefficients of expansions in the form of system of algebraic equations. It is possible to get the exact solution in the form of finite sum of power functions with rational degrees by solving this system of equations.

The modification of the method consists in the fact that, using the form of asymptotic expansions of solutions to the equation PVI at the origin and at infinity, the general form of exact solution is composed. After substituting such solution into the equation PVI one can obtain the system of algebraic equations for unknown coefficients of exact solution and parameters of the equation. The obtained system is solved with the help of computer algebra system using Gröbner basis.
**Painlevé property**

**Definition**

An ordinary non-linear differential equation possesses the **Painlevé property** if all its solutions have not the movable singularities except poles.

**Definition**

**Painlevé equations** are second order non-linear ordinary differential equations which are not generally solvable in term of elementary functions.
List of Painlevé equations and their polygons

1. \( y'' = 6y^2 + x \)
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2. $y'' = 2y^3 + xy + \alpha$
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2. \( y'' = 2y^3 + xy + \alpha \)
3. \( xyy''' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma xy^4 \)
List of Painlevé equations and their polygons

1. \( y'' = 6y^2 + x \)
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3. \( xyy'' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma xy^4 \)
4. \( yy'' = \frac{1}{2} (y')^2 + \beta + 2(x^2 - \alpha)y^2 + 4xy^3 + \frac{3}{2}y^4 \)
List of Painlevé equations and their polygons

1. \( y''' = 6y^2 + x \)
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3. \( xyy''' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma xy^4 \)
4. \( yy''' = \frac{1}{2}(y')^2 + \beta + 2(x^2 - \alpha)y^2 + 4xy^3 + \frac{3}{2}y^4 \)
5. \( y''' = \left( \frac{1}{2y} + \frac{1}{y-1} \right)(y')^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{x} + \delta \frac{y(y+1)}{y-1} \)
List of Painlevé equations and their polygons

1. \[ y'' = 6y^2 + x \]
2. \[ y'' = 2y^3 + xy + \alpha \]
3. \[ xy y'' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma xy^4 \]
4. \[ yy'' = \frac{1}{2}(y')^2 + \beta + 2(x^2 - \alpha)y^2 + 4xy^3 + \frac{3}{2}y^4 \]
5. \[ y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{1}{x} y' + \frac{(y-1)^2}{x} \left( \alpha y + \frac{\beta}{y} \right) + \gamma \frac{y}{x} + \delta \frac{y(y+1)}{y-1} \]
6. \[ y'' = \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \frac{(y')^2}{2} - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right] \]
Connections between Painlevé equations

Painlevé equations

PV → PVI

PIV

PII → PI

PIII

(1/2, 1/2, 1/2) → (1/2, 1, 1/2) → (2) → (2)

(1/2, 1/2, 1/2, 1/2)
The six Painlevé equation

Original form

\[ y'' = \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) \frac{(y')^2}{2} - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y' + \\
\quad + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right], \]
The six Painlevé equation

Original form

\[ y'' = \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - x} \right) \left( \frac{y'}{2} \right)^2 - \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{y - x} \right) y' + \]
\[ + \frac{y(y - 1)(y - x)}{x^2(x - 1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x - 1}{(y - 1)^2} + \delta \frac{x(x - 1)}{(y - x)^2} \right], \]

Differential monomials form

\[ f(x, y) \overset{\text{def}}{=} 2y''x^2(x - 1)^2y(y - 1)(y - x) - (y')^2[ x^2(x - 1)^2(y - 1)(y - x) + \]
\[ + x^2(x - 1)^2y(y - x) + x^2(x - 1)^2y(y - 1) ] + \]
\[ + 2y'[ x(x - 1)^2y(y - 1)(y - x) + x^2(x - 1)y(y - 1)(y - x) + \]
\[ + x^2(x - 1)^2y(y - 1) ] - [ 2\alpha y^2(y - 1)^2(y - x)^2 + 2\beta x(y - 1)^2(y - x)^2 + \]
\[ + 2\gamma(x - 1)y^2(y - x)^2 + 2\delta x(x - 1)y^2(y - 1)^2 ] = 0. \]
Bäcklund transformations

PVI is invariant under the finite group of order 24 with 3 generators of Bäcklund transformations. Let \( y(x) = y(x; \alpha, \beta, \gamma, \delta) \) be a solution of PVI for definite values of equation’s parameters. The transformation \( T_j : y \to y_j \) determines the new solution

1. \( y_1(x; -\beta, -\alpha, \gamma, \delta) = 1/y(1/x) \);
2. \( y_2(x; -\beta, -\gamma, \alpha, \delta) = 1 - 1/y(1/(1-x)) \);
3. \( y_3(x; -\beta, -\alpha, 1/2 - \delta, 1/2 - \gamma) = x/y(x) \);
4. \( y_4(x; \alpha, \beta, 1/2 - \delta, 1/2 - \gamma) = x/y(1/x), \ T_4 = T_1 \circ T_3 \);

References

Known elementary solution

Rational solutions

Some rational solutions of the form \( y(x) = \kappa x^j, \ j = -2, -1, 1, 2 \) are presented by Clarkson in *NIST Handbook of Math Functions* (2010).

Algebraic solutions

Suitable power expansions: Expansion corresponding vertex $Q_4$
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Truncated sum $\hat{f}_4^{(0)}$

$\hat{f}_4^{(0)} \overset{\text{def}}{=} 2x^2yy'^2 - 2xy^2y' - 2x^2y^2y'' = 0$
Suitable power expansions: Expansion corresponding vertex $Q_4$

**Truncated sum $\hat{f}_4^{(0)}$**

$$\hat{f}_4^{(0)} \overset{\text{def}}{=} 2x^2yy'^2 - 2xy^2y' - 2x^2y^2y'' = 0$$

**Normalized cone $U_4^{(0)}$**

$$U_4^{(0)} = -(1, r), \ r \in \mathbb{C}, \ 0 < \text{Re} \ r < 1$$
Suitable power expansions: Expansion corresponding vertex $Q_4$

Truncated sum $\hat{f}_4^{(0)}$

$$\hat{f}_4^{(0)} \overset{\text{def}}{=} 2x^2yy'' - 2xy^2y' - 2x^2y^2y'' = 0$$

Normalized cone $U_4^{(0)}$

$$U_4^{(0)} = -(1, r), \ r \in \mathbb{C}, \ 0 < \text{Re} \ r < 1$$

Truncated solution

$$y = c_r x^r, \ c_r \in \mathbb{C}, \ \text{Re} \ r \in (0, 1)$$
Suitable power expansions: Expansion corresponding vertex $Q_4$

- **Truncated sum** $\hat{f}_4^{(0)}$
  \[
  \hat{f}_4^{(0)} \overset{\text{def}}{=} 2x^2yy'^2 - 2xy^2y' - 2x^2y^2y'' = 0
  \]

- **Normalized cone** $U_4^{(0)}$
  $U_4^{(0)} = -(1, r), r \in \mathbb{C}, 0 < \Re r < 1$

- **Truncated solution**
  $y = c_r x^r, c_r \in \mathbb{C}, \Re r \in (0, 1)$

**Family** $A_0$ [BrunoGoryuchkina2010]

- $A_0 : y = c_r x^r + \sum c_s x^s, r \in \mathbb{C}, \Re r \in (0, 1), s \in \{r + kr + m(1-r); k, m \geq 0; k + m > 0; k, m \in \mathbb{Z}\}$

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- **Truncated sum** $\hat{f}_2^{(0)}$
  
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  \hat{f}_2^{(0)} \overset{\text{def}}{=} xyy'' - x(y')^2 + yy' = 0
  \]

- **Normalized cone** $U_2^{(0)}$
  
  $U_2^{(0)} = (1, r), r \in \mathbb{C}, 0 < \text{Re} r < 1$

- **Truncated solution**
  
  $y = c_r x^r, c_r \in \mathbb{C}, \text{Re} r \in (0, 1)$
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Normalized cone $U_2^{(0)}$

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Truncated solution

$$y = c_r x^r, \ c_r \in \mathbb{C}, \ \Re r \in (0, 1)$$

Family $A_\infty$

$$A_\infty : y = c_r x^r + \sum c_s x^s, \ r \in \mathbb{C}, \ \Re r \in (0, 1), \ s \in \{ r - kr + m(r-1); \ k, m \geq 0; \ k + m > 0; \ k, m \in \mathbb{Z} \}$$
Search for suitable power expansions

There are only four pairs of power expansions near the origin and near the infinity that suitable for matching procedure:

1. the vertex $Q_4$ and the vertex $Q_2$;
2. the edge $\Gamma_1(4)$ and the edge $\Gamma_1(3)$;
3. the vertex $Q_4$ and the edge $\Gamma_1(3)$;
4. the edge $\Gamma_1(4)$ and the vertex $Q_2$. 

The exact solutions should be sought in the general form 

$$y(x) = \sum_{k=0}^{l} a_k x^k, \quad l \in \mathbb{N}.$$ 

(1)
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The exact solutions should be sought in the general form

$$y(x) = \sum_{k=0}^{l} a_k x^k, \quad l \in \mathbb{N}. \quad (1)$$
There are only four pairs of power expansions near the origin and near the infinity that suitable for matching procedure:

1. the vertex $Q_4$ and the vertex $Q_2$;
2. the edge $\Gamma_4^{(1)}$ and the edge $\Gamma_3^{(1)}$;
3. the vertex $Q_4$ and the edge $\Gamma_3^{(1)}$;
4. the edge $\Gamma_4^{(1)}$ and the vertex $Q_2$.

The exact solutions should be sought in the general form

$$y(x) = \sum_{k=0}^{l} a_k x^{k/l}, \quad l \in \mathbb{N}.$$  \hfill (1)
If all $a_k = 0$ for $k = 1, \ldots, l - 1$ then (1) gives the linear solution

$$y = a_0 + a_1 x$$

of the equation PVI. The following table summaries the families of linear solutions for different values of parameters of the equation PVI.

<table>
<thead>
<tr>
<th>Sol.</th>
<th>Parameters</th>
<th>Free param.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ax + 1$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-\frac{(a + 1)^2}{2}$</td>
<td>0</td>
</tr>
<tr>
<td>$ax + 1 - a$</td>
<td>$\gamma/a^2$</td>
<td>$-1/2$</td>
</tr>
<tr>
<td>$x + b$</td>
<td>$1/2$</td>
<td>$-\frac{(b + 1)^2}{2}$</td>
</tr>
</tbody>
</table>
Computed elementary solution

Here we present the list of solutions of the form

$$y(x) = \sum_{k=0}^{l} a_k x^{k/l}$$

for $l = 2, \ldots, 6$.

There is an elementary solution $y = ax^n$, $n \in \mathbb{C}$ to the equation PVI for $\alpha = \beta = 0$, $\gamma = n^2/2$, $\delta = n(2 - n)/2$, therefore we list only those solutions of the general form, for which at least more then one coefficient $a_k \neq 0$. 
**Elementary solution for \( l = 2 \)**

<table>
<thead>
<tr>
<th>Sol. No</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>1/2</td>
<td>0</td>
<td>-5/8</td>
<td>1/8</td>
<td>1/2</td>
<td>( \pm 1/2 )</td>
<td>0</td>
</tr>
<tr>
<td>2-2</td>
<td>1/2</td>
<td>0</td>
<td>9/8</td>
<td>3/8</td>
<td>0</td>
<td>( \pm 1/2 )</td>
<td>1/2</td>
</tr>
<tr>
<td>2-3</td>
<td>1/32</td>
<td>-25/32</td>
<td>9/32</td>
<td>15/32</td>
<td>-2</td>
<td>( \pm 3 )</td>
<td>0</td>
</tr>
<tr>
<td>2-4</td>
<td>1/32</td>
<td>-25/32</td>
<td>1/32</td>
<td>7/32</td>
<td>0</td>
<td>( \pm 3 )</td>
<td>-2</td>
</tr>
</tbody>
</table>

One parameter family: \( y(x) = a_2(1 \mp \sqrt{x})^2 \pm \sqrt{x} \) for PVI parameters values

\[
\alpha = \frac{1}{8a_2^2}, \beta = -\left(\frac{4a_2 - 1}{8a_2^2}\right) , \gamma = \left(\frac{a_2 - 1}{8a_2^2}\right), \delta = \left(\frac{3a_2^2 + 2a_2 - 1}{8a_2^2}\right)
\]

Pairs of solutions (2-1, 2-2) and (2-3, 2-4) are connected by the Bäcklund transformation

\[
T_4 : y_4(x; \alpha, \beta, -\delta + 1/2, -\gamma + 1/2) = xy(1/x; \alpha, \beta, \gamma, \delta)
\]
## Elementary solution for $l = 3$

<table>
<thead>
<tr>
<th>$l = 3$</th>
<th>Parameters</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sol. No</td>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>3-1</td>
<td>1/18</td>
<td>-2/9</td>
</tr>
<tr>
<td>3-2</td>
<td>2/9</td>
<td>-1/18</td>
</tr>
<tr>
<td>3-3</td>
<td>2/9</td>
<td>-1/18</td>
</tr>
<tr>
<td>3-4</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>3-5</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>3-6</td>
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</tr>
<tr>
<td>3-7</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>3-8</td>
<td>1/18</td>
<td>-8/9</td>
</tr>
<tr>
<td>3-9</td>
<td>1/18</td>
<td>-8/9</td>
</tr>
</tbody>
</table>

where $z_0$, is the root of the equation $z^2 + z + 1 = 0$. Solutions 3-4 and 3-6 are connected by the Bäcklund transformation

$$T_4 : y_4(x; \alpha, \beta, -\delta + 1/2, -\gamma + 1/2) = xy(1/x; \alpha, \beta, \gamma, \delta)$$
### Elementary solution for \( l = 4 \)

<table>
<thead>
<tr>
<th>( l = 4 )</th>
<th>Parameters</th>
<th>Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sol. No</td>
<td>( \alpha )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>4-1</td>
<td>( \frac{9}{32} )</td>
<td>( -\frac{1}{32} )</td>
</tr>
<tr>
<td>4-2</td>
<td>( \frac{9}{32} )</td>
<td>( -\frac{1}{32} )</td>
</tr>
<tr>
<td>4-3</td>
<td>( \frac{1}{32} )</td>
<td>( -\frac{9}{32} )</td>
</tr>
<tr>
<td>4-4</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>4-5</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
<tr>
<td>4-6</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
</tr>
</tbody>
</table>

Solutions 4-4 and 4-6 are connected by the Bäcklund transformation \( T_4 \) as mentioned above.
## Elementary solution for \( l = 5 \)

<table>
<thead>
<tr>
<th>Sol. No</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( a_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-1</td>
<td>8/25</td>
<td>-1/50</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>0</td>
</tr>
<tr>
<td>5-2</td>
<td>8/25</td>
<td>-1/50</td>
<td>1/2</td>
<td>0</td>
<td>0</td>
<td>( A_1 )</td>
<td>( z_0^3/4 )</td>
<td>( z_0^2/4 )</td>
<td>( z_0/4 )</td>
<td>0</td>
</tr>
<tr>
<td>5-3</td>
<td>1/50</td>
<td>-8/25</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>5-4</td>
<td>1/50</td>
<td>-8/25</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
<td>( B_1 )</td>
<td>( z_1^3 )</td>
<td>( z_1^2 )</td>
<td>( z_1 )</td>
<td>0</td>
</tr>
<tr>
<td>5-5</td>
<td>1/2</td>
<td>0</td>
<td>8/25</td>
<td>-11/50</td>
<td>1/5</td>
<td>1/5</td>
<td>1/5</td>
<td>1/5</td>
<td>1/5</td>
<td>0</td>
</tr>
<tr>
<td>5-6</td>
<td>1/2</td>
<td>0</td>
<td>8/25</td>
<td>-11/50</td>
<td>1/5</td>
<td>( C_1 )</td>
<td>( z_0^3/5 )</td>
<td>( z_0^2/5 )</td>
<td>( z_0/5 )</td>
<td>0</td>
</tr>
</tbody>
</table>

where \( z_0 \) is the root of the equation \( z^4 + z^3 + z^2 + z + 1 = 0 \), \( A_1 = -\frac{1}{4} - a_2 - a_3 - a_4 \), \( C_1 = -\frac{1}{5} - a_2 - a_3 - a_4 \) and \( z_1 \) is the root of the equation \( z^4 - z^3 + z^2 - z + 1 = 0 \), \( B_1 = 1 - a_2 - a_3 - a_4 \).
## Elementary solution for \( l = 6 \)

<table>
<thead>
<tr>
<th>Sol. No</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( a_5 )</th>
<th>( a_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6-1</td>
<td>( \frac{1}{72} )</td>
<td>( -\frac{25}{72} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \pm 1 )</td>
<td>1</td>
<td>( \pm 1 )</td>
<td>1</td>
<td>( \pm 1 )</td>
<td>0</td>
</tr>
<tr>
<td>6-2</td>
<td>( \frac{1}{72} )</td>
<td>( -\frac{25}{72} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( -1 - z_0 )</td>
<td>( -z_0 )</td>
<td>1</td>
<td>( 1 + z_0 )</td>
<td>( z_0 )</td>
<td>0</td>
</tr>
<tr>
<td>6-3</td>
<td>( \frac{1}{72} )</td>
<td>( -\frac{25}{72} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( 1 - z_1 )</td>
<td>( z_1 )</td>
<td>( -1 )</td>
<td>( 1 - z_1 )</td>
<td>( z_1 )</td>
<td>0</td>
</tr>
<tr>
<td>6-4</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{25}{72} )</td>
<td>( -\frac{13}{72} )</td>
<td>( \frac{1}{6} )</td>
<td>( \pm \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \pm \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \pm \frac{1}{6} )</td>
<td>0</td>
</tr>
<tr>
<td>6-5</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{25}{72} )</td>
<td>( -\frac{13}{72} )</td>
<td>( \frac{1}{6} )</td>
<td>( -\frac{1 + z_0}{6} )</td>
<td>( \frac{z_0}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( -\frac{1 + z_0}{6} )</td>
<td>( z_0/6 )</td>
<td>0</td>
</tr>
<tr>
<td>6-6</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{25}{72} )</td>
<td>( -\frac{13}{72} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1 - z_1}{6} )</td>
<td>( -\frac{z_1}{6} )</td>
<td>( -\frac{1}{6} )</td>
<td>( \frac{z_1 - 1}{6} )</td>
<td>( z_1/6 )</td>
<td>0</td>
</tr>
</tbody>
</table>

where \( z_0 \) is the root of the equation \( z^2 + z + 1 \), \( z_1 \) is the root of the equation \( z^2 - z + 1 \).
Generalization of computed solutions

Solution 3-1 can be obtained from elementary solutions $y = x^{4/3}$ or $y = x^{1/3}$ by the Bäcklund transformations

$$T_8 : y_8(x; -\delta + 1/2, -\gamma, -\beta, -\alpha + 1/2) = \frac{x(y(x; \alpha, \beta, \gamma, \delta) - 1) - y}{y(x; \alpha, \beta, \gamma, \delta) - x}$$

and

$$T_9 : y_9(x; \gamma, \delta - 1/2, \alpha, \beta + 1/2) = \frac{y(x; \alpha, \beta, \gamma, \delta) - x}{y(x; \alpha, \beta, \gamma, \delta) - 1}.$$

correspondingly. The same transformations connect the solution 5-3 with elementary solutions $y = x^{6/5}$ and $y = x^{1/5}$, correspondingly. It is possible that some other solutions among mentioned above may be obtained from the known elementary solutions of the equation PVI by the Bäcklund transformations but the authors does not known anything about it.

All the found solutions are algebraic solutions and can be written in the polynomial parametrization.
Generalization of computed solutions

Solutions 3-4, 3-8, 4-4, 5-5, 6-1 can be written as the sum of finite geometrical progression and then can be generalized for the case of any power exponents. The direct substitution shows that the function

$$y = b \frac{x - 1}{x^b - 1}$$

is the exact solution of the equation PVI for $\alpha = 1/2$, $\beta = 0$, $\gamma = (1 - b)^2/2$, $\delta = -(2 + b)^2/2$, $b \in \mathbb{C}$. 


Bruno, A. D., Goruchkina, I. V. *Asymptotic expansions of solutions to the sixth Painlevé equation* //


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- to Prof. Yu. Brezhnev for references
- to all

FOR ATTENTION!