EXACT SIMPLE SOLUTIONS TO PVI

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Outline

1. **Description of the method**
   - Power Geometry essentials
   - Matching "heads" and "tails" of power expansions
2. **Painlevé equations and their properties**
   - Painlevé equations
   - PVI and its properties
3. **Computed Solutions**
   - Known elementary solution
   - Computed elementary solution
4. **References**

Abstract

We use the method for computing an exact solutions of the ordinary differential equation and apply it to the sixth Painlevé equation (PVI) and other Painlevé equations.

This method is essentially uses algorithms of Power Geometry for computing power expansions of solutions to an ordinary differential equation and computer algebra algorithms (Gröbner basis).

Differential monomial

Let x be independent and y be dependent variables, $x, y \in \mathbb{C}$. A differential monomial $a(x, y)$ is a product of an ordinary monomial $cx^{r_1}y^{r_2}$, where $c = \mathrm{const} \in \mathbb{C}$, $(r_1, r_2) \in \mathbb{R}^2$, and a finite number of derivatives of the form $d^l y/dx^l$, $l \in \mathbb{N}$.

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Differential sum

A sum of differential monomials

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Let a differential equation be given

$$f(x, y) = 0,$$

where $f(x, y)$ is a differential sum.

Power Geometry objects

Power exponent

To each differential monomial $a(x, y)$, we assign its (vector) power exponent $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules:

$$Q(cx^{r_1}y^{r_2}) = (r_1, r_2); \quad Q(d^l y/dx^l) = (-l, 1);$$

when differential monomials are multiplied, their power exponents must be added as vectors $Q(a_1 a_2) = Q(a_1) + Q(a_2)$.

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The Newton–Bruno polygon

The set $\mathbf{S}(f)$ of power exponents $Q(a_i)$ of all differential monomials $a_i(x, y)$ present in the differential sum is called the support of the sum $f(x, y)$. Obviously, $\mathbf{S}(f) \in \mathbb{R}^2$.

The convex hull $\Gamma(f)$ of the support $\mathbf{S}(f)$ is called the Newton–Bruno polygon of the sum $f(x, y)$. The boundary $\partial\Gamma(f)$ of the polygon $\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$.

Power Geometry objects

Truncated sum

Each face $\Gamma_j^{(d)}$ corresponds to the truncated sum

$$\hat{f}_j^{(d)}(x,y) = \sum a_i(x,y) \text{ over } Q(a_i) \in \Gamma_j^{(d)} \cap \mathbf{S}(f).$$

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Normal cone

Let the plane $\mathbb{R}_*^2$ be dual to the plane $\mathbb{R}^2$ such that for $P = (p_1, p_2) \in \mathbb{R}_*^2$ and $Q = (q_1, q_2) \in \mathbb{R}^2$, the scalar product

$$\langle P, Q \rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$$

is defined. Each face $\Gamma_j^{(d)}$ in $\mathbb{R}_*^2$ corresponds to its own normal cone $\mathbf{U}_j^{(d)}$ formed by the outward normal vectors P to the face $\Gamma_j^{(d)}$.

Computation of truncated equations

Each face $\Gamma_j^{(d)}$ corresponds to the normal cone $\mathbf{U}_j^{(d)}$ in the plane $\mathbb{R}_*^2$ and to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.

Theorem (Bruno)

If the power expansion $y = c_r x^r + \sum c_s x^s$, $c_r = \mathrm{const} \in \mathbb{C}$, $c_r \neq 0$, satisfies the equation $f(x, y) = 0$, and $\omega(1, r) \in \mathbf{U}_j^{(d)}$, where $\omega = -1$, if $x \to 0$ or $\omega = 1$, if $x \to \infty$ then the truncation $y = c_r x^r$ of the power expansion is the solution to the truncated equation $\hat{f}_j^{(d)}(x, y) = 0$.

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As truncated equations are quasi–homogeneous it is not difficult to find its power solutions. For each truncated equation $\hat{f}_j^{(d)}(x, y) = 0$, we need to find all its solutions $y = c_r x^r$ which have one of the vectors $\pm(1, r)$ lying in the normal cone $\mathbf{U}_j^{(d)}$.

Matching “heads” and “tails” of power expansions

The modification of method for computing exact solutions, proposed in [Bruno, Gashenenko, 2006] for finding of exact solutions to N. Kovalewski equations, is used.
This method is based on fitting of two power series expansions near the origin and at infinity and getting conditions on the coefficients of expansions in the form of system of algebraic equations. It is possible to get the exact solution in the form of finite sum of power functions with rational degrees by solving this system of equations.

The modification of the method consists in the fact that, using the form of asymptotic expansions of solutions to the equation PVI at the origin and at infinity, the general form of exact solution is composed. After substituting such solution into the equation PVI one can obtain the system of algebraic equations for unknown coefficients of exact solution and parameters of the equation. The obtained system is solved with the help of computer algebra system using Gröbner basis.

Painlevé property

Definition

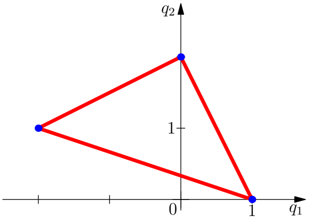
An ordinary non-linear differential equation possesses the Painlevé property if all its solutions have not the movable singularities except poles.

Definition

Painlevé equations are second order non-linear ordinary differential equations which are not generally solvable in term of elementary functions.

List of Painlevé equtions and their polygons

1. $y'' = 6y^2 + x$



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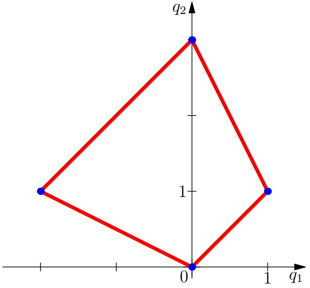

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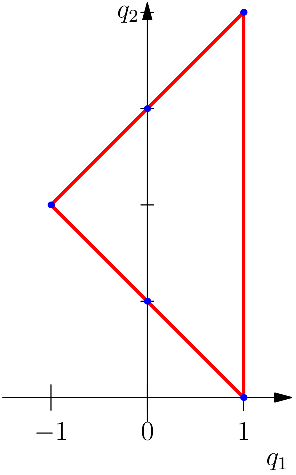
3. $xyy'' = x(y')^2 - yy' + \delta x + \beta y + \alpha y^3 + \gamma xy^4$



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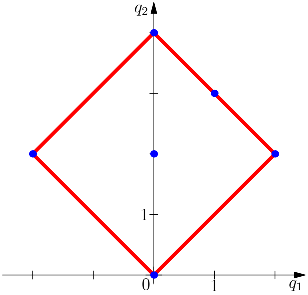
4. $yy'' =$
   $\frac{1}{2}(y')^2 + \beta + 2(x^2 - \alpha)y^2 + 4xy^3 + \frac{3}{2}y^4$



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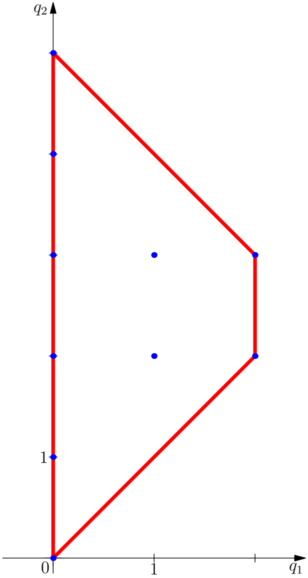
5. $y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{x}y' +$
   $\frac{(y-1)^2}{x}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1}$



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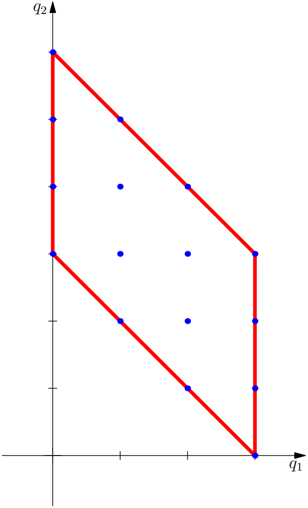
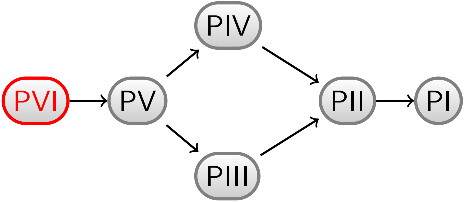
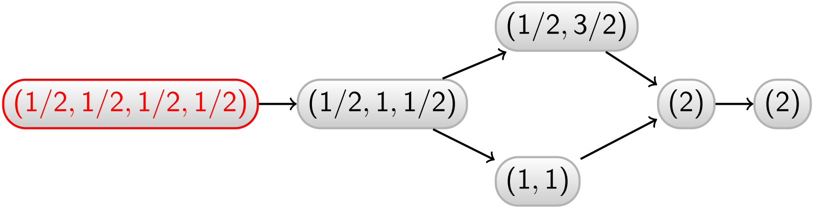
6. $y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\frac{(y')^2}{2} - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right]$



Connections between Painlevé equations





The six Painlevé equation

Original form

$$y'' = \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\frac{(y')^2}{2} - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' +$$
$$+ \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right],$$

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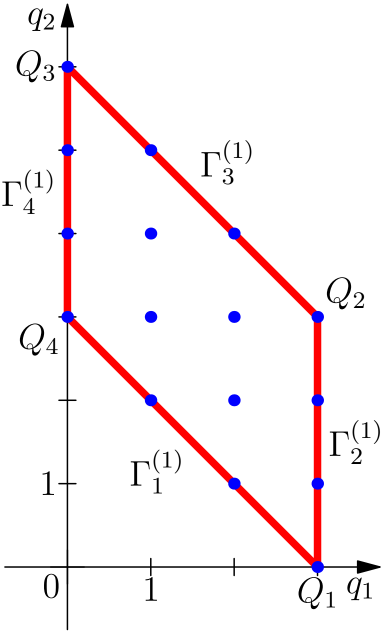
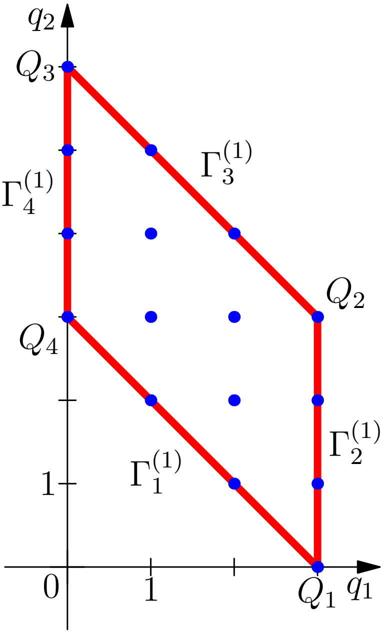
Differential monomials form

$$f(x,y) \stackrel{\mathrm{def}}{=} 2y''x^2(x-1)^2 y(y-1)(y-x) - (y')^2[\, x^2(x-1)^2(y-1)(y-x) + \\ + x^2(x-1)^2 y(y-x) + x^2(x-1)^2 y(y-1)\,] + \\ + 2y'[\, x(x-1)^2 y(y-1)(y-x) + x^2(x-1)y(y-1)(y-x) + \\ + x^2(x-1)^2 y(y-1)\,] - [\, 2\alpha y^2(y-1)^2(y-x)^2 + 2\beta x(y-1)^2(y-x)^2 + \\ + 2\gamma(x-1)y^2(y-x)^2 + 2\delta x(x-1)y^2(y-1)^2\,] = 0.$$


q_2
Q_3
$\Gamma_3^{(1)}$
$\Gamma_4^{(1)}$
Q_2
Q_4
$\Gamma_2^{(1)}$
$\Gamma_1^{(1)}$
1
0
1
Q_1
q_1


q_2
Q_3
$\Gamma_3^{(1)}$
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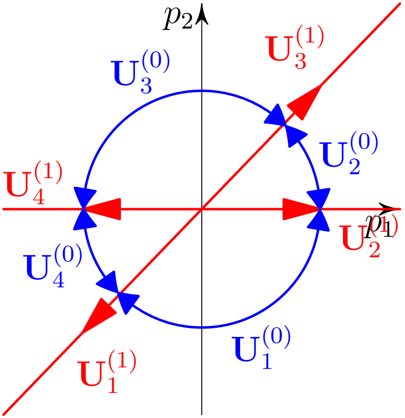
1
$\Gamma_1^{(1)}$
0
1
Q_1
q_1


p_2
$\mathbf{U}_3^{(1)}$
$\mathbf{U}_3^{(0)}$
$\mathbf{U}_2^{(0)}$
$\mathbf{U}_4^{(1)}$
p_1
$\mathbf{U}_2^{(1)}$
$\mathbf{U}_4^{(0)}$
$\mathbf{U}_1^{(0)}$
$\mathbf{U}_1^{(1)}$

Bäcklund transformations

PVI is invariant under the finite group of order 24 with 3 generators of Bäcklund transformations. Let $y(x) = y(x; \alpha, \beta, \gamma, \delta)$ be a solution of PVI for definite values of equation's parameters. The transformation $T_j : y \to y_j$ determines the new solution

1. $y_1(x; -\beta, -\alpha, \gamma, \delta) = 1/y(1/x)$;
2. $y_2(x; -\beta, -\gamma, \alpha, \delta) = 1 - 1/y(1/(1-x))$;
3. $y_3(x; -\beta, -\alpha, 1/2-\delta, 1/2-\gamma) = x/y(x)$;
4. $y_4(x; \alpha, \beta, 1/2-\delta, 1/2-\gamma) = x/y(1/x)$, $T_4 = T_1 \circ T_3$;

References

For more details about Painlevé equations in general and PVI in particular see *Its, Novokshenov, 1986*, *Iwasaki et al., 1991*, *Gromak et al., 2002*, *Conte, Musette, 2008*, *NIST Handbook of Mathematical Functions, §32, 2010*.

Known elementary solution

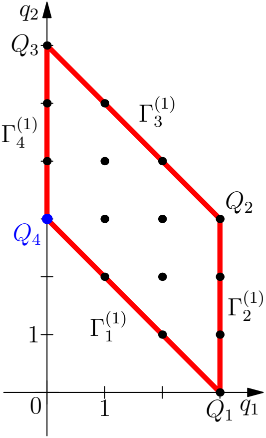
Rational solutions

Some rational solutions of the form $y(x) = \varkappa x^j$, $j = -2, -1, 1, 2$ are presented by Clarkson in *NIST Handbook of Math Functions (2010)*.

Algebraic solutions

Algebraic solutions that determine algebraic curves with different values of genus are presented by *Dubrovin, Mazzocco (2000)*, *Gromak et al. (2002)*, *Hitchin (2003)*, *Boalch (2005, 2006)*. The full list of algebraic solutions and their classification see in *Lisovyy, Tikhyy (2008)*.

Suitable power expansions: Expansion corresponding vertex Q_4



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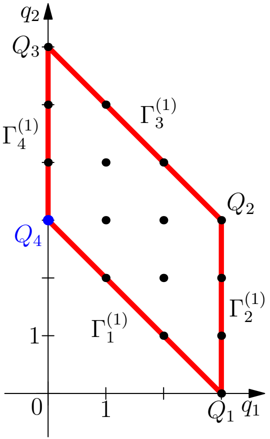

Truncated sum $\hat{f}_4^{(0)}$

$$\hat{f}_4^{(0)} \stackrel{\text{def}}{=} 2x^2yy'^2 - 2xy^2y' - 2x^2y^2y'' = 0$$

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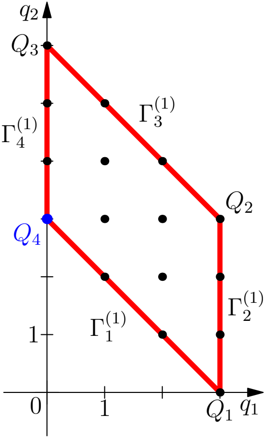

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Normalized cone $\mathbf{U}_4^{(0)}$

$$\mathbf{U}_4^{(0)} = -(1, r),\ r \in \mathbb{C},\ 0 < \operatorname{Re} r < 1$$

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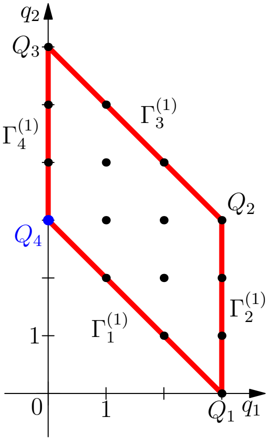

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Truncated solution

$$y = c_r x^r,\ c_r \in \mathbb{C},\ \operatorname{Re} r \in (0, 1)$$

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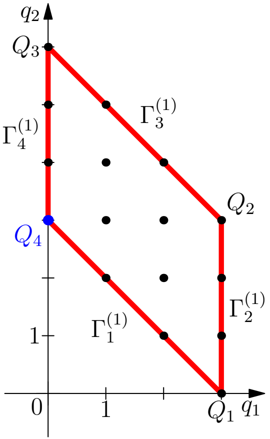

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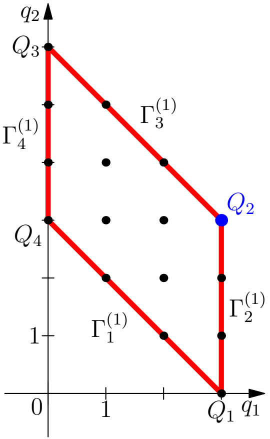
Family $\mathcal{A}_0$ [BrunoGoryuchkina2010]

$\mathcal{A}_0 : y = c_r x^r + \sum c_s x^s$, $r \in \mathbb{C}$, $\operatorname{Re} r \in (0, 1)$, $s \in \{r + kr + m(1 - r); k, m \geqslant 0; k + m > 0; k, m \in \mathbb{Z}\}$

Suitable power expansions: Expansion corresponding vertex Q_2



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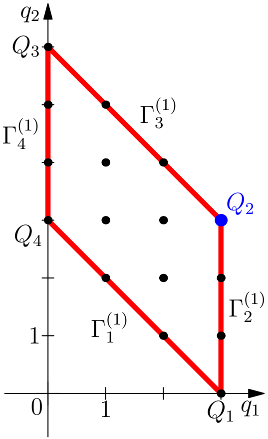

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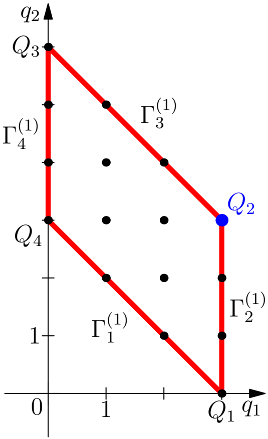
Suitable power expansions: Expansion corresponding vertex Q_2



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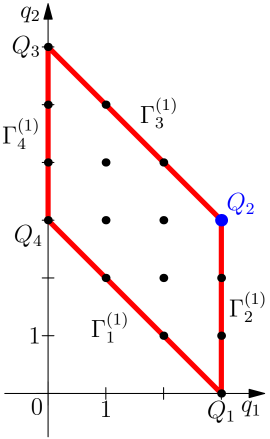

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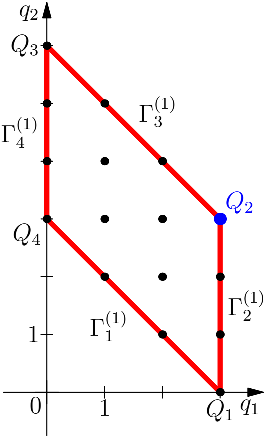

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Family $\mathcal{A}_\infty$

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Search for suitable power expansions

There are only four pairs of power expansions near the origin and near the infinity that suitable for matching procedure:

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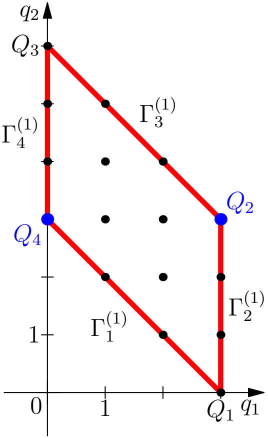

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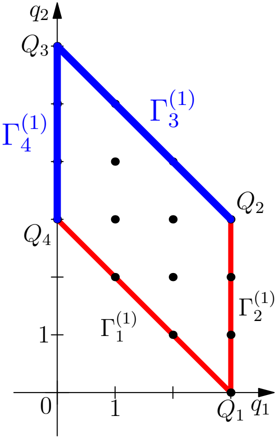
1. the vertex Q_4 and the vertex Q_2;

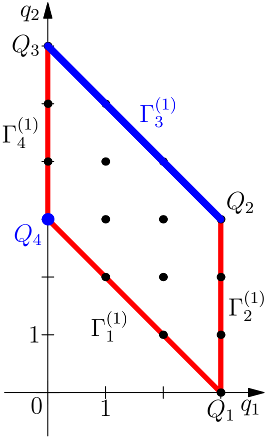
Search for suitable power expansions



There are only four pairs of power expansions near the origin and near the infinity that suitable for matching procedure:

1. the vertex Q_4 and the vertex Q_2;
2. the edge $\Gamma_4^{(1)}$ and the edge $\Gamma_3^{(1)}$;

Search for suitable power expansions



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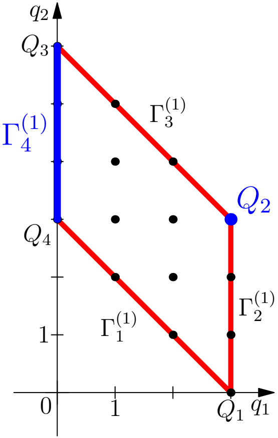

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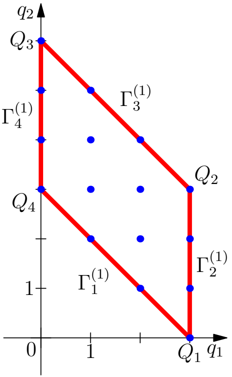

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The exact solutions should be sought in the general form

$$y(x) = \sum_{k=0}^{l} a_k x^{k/l}, \quad l \in \mathbb{N}. \tag{1}$$

Linear solution

If all $a_k = 0$ for $k = 1, \ldots, l-1$ then (1) gives the linear solution

$$y = a_0 + a_1 x$$

of the equation PVI. The following table summaries the families of linear solutions for different values of parameters of the equation PVI.

Sol.	Parameters				Free param.
	α	β	γ	δ	
$ax+1$	$1/2$	$-\dfrac{(a+1)^2}{2}$	0	$a - a^2/2$	a
$ax+1-a$	γ/a^2	$-1/2$	γ	$1/2 - \gamma(1-1/a)^2$	a, γ
$x+b$	$1/2$	$-\dfrac{(b+1)^2}{2}$	$\dfrac{(b-1)^2}{2}$	$1/2$	b

Computed elementary solution

Here we present the list of solutions of the form

$$y(x) = \sum_{k=0}^{l} a_k x^{k/l}$$

for $l = 2, \dots, 6$.
There is an elementary solution $y = ax^n$, $n \in \mathbb{C}$ to the equation PVI for $\alpha = \beta = 0$, $\gamma = n^2/2$, $\delta = n(2-n)/2$, therefore we list only those solutions of the general form, for which at least more then one coefficient $a_k \neq 0$.

Elementary solution for $l = 2$

$l = 2$	Parameters				Coefficients		
Sol. No	α	β	γ	δ	a_0	a_1	a_2
2-1	$1/2$	0	$-5/8$	$1/8$	$1/2$	$\pm 1/2$	0
2-2	$1/2$	0	$9/8$	$3/8$	0	$\pm 1/2$	$1/2$
2-3	$1/32$	$-25/32$	$9/32$	$15/32$	-2	± 3	0
2-4	$1/32$	$-25/32$	$1/32$	$7/32$	0	± 3	-2

One parameter family: $y(x) = a_2(1 \mp \sqrt{x})^2 \pm \sqrt{x}$ for PVI parameters values

$$\alpha = \frac{1}{8a_2^2}, \beta = -\frac{(4a_2 - 1)^2}{8a2^2}, \gamma = \frac{(a_2 - 1)^2}{8a_2^2}, \delta = \frac{3a_2^2 + 2a_2 - 1}{8a_2^2}$$

Pairs of solutions (2-1, 2-2) and (2-3, 2-4) are connected by the Bäcklund transformation

$$T_4 : y_4(x; \alpha, \beta, -\delta + 1/2, -\gamma + 1/2) = xy(1/x; \alpha, \beta, \gamma, \delta)$$

Elementary solution for $l = 3$

$l = 3$	Parameters				Coefficients			
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3
3-1	$1/18$	$-2/9$	0	$1/2$	0	-1	-1	0
3-2	$2/9$	$-1/18$	$1/2$	0	0	$1/2$	$1/2$	0
3-3	$2/9$	$-1/18$	$1/2$	0	0	$-(1+z_0)/2$	$z_0/2$	0
3-4	$1/2$	0	$2/9$	$-7/18$	$1/3$	$1/3$	$1/3$	0
3-5	$1/2$	0	$2/9$	$-7/18$	$1/3$	$-(1+z_0)/3$	$z_0/3$	0
3-6	$1/2$	0	$8/9$	$5/18$	0	$1/3$	$1/3$	$1/3$
3-7	$1/2$	0	$8/9$	$5/18$	0	$-(1+z_0)/3$	$z_0/3$	$1/3$
3-8	$1/18$	$-8/9$	0	$1/2$	1	1	1	1
3-9	$1/18$	$-8/9$	0	$1/2$	1	$-1-z_0$	z_0	1

where z_0, is the root of the equation $z^2 + z + 1 = 0$. Solutions 3-4 and 3-6 are connected by the Bäcklund transformation

$$T_4 : y_4(x; \alpha, \beta, -\delta + 1/2, -\gamma + 1/2) = xy(1/x; \alpha, \beta, \gamma, \delta)$$

Elementary solution for $l = 4$

$l = 4$	Parameters				Coefficients				
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4
4-1	$9/32$	$-1/32$	$1/2$	0	0	$1/3$	$1/3$	$1/3$	0
4-2	$9/32$	$-1/32$	$1/2$	0	0	$\pm 1/3$	0	$\pm 1/3$	0
4-3	$1/32$	$-9/32$	0	$1/2$	0	$\mp i$	1	$\pm i$	0
4-4	$1/2$	0	$9/32$	$-9/32$	$1/4$	$\pm 1/4$	$1/4$	$\pm 1/4$	0
4-5	$1/2$	0	$9/32$	$-9/32$	$1/4$	$\mp i/4$	$-1/4$	$\pm i/4$	0
4-6	$1/2$	0	$25/32$	$7/32$	0	$\pm 1/4$	$1/4$	$\pm 1/4$	$1/4$

Solutions 4-4 and 4-6 are connected by the Bäcklund transformation T_4 as mentioned above.

Elementary solution for $l = 5$

$l = 5$	Parameters				Coefficients					
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4	a_5
5-1	$8/25$	$-1/50$	$1/2$	0	0	$1/4$	$1/4$	$1/4$	$1/4$	0
5-2	$8/25$	$-1/50$	$1/2$	0	0	A_1	$z_0^3/4$	$z_0^2/4$	$z_0/4$	0
5-3	$1/50$	$-8/25$	0	$1/2$	0	-1	-1	-1	-1	0
5-4	$1/50$	$-8/25$	0	$1/2$	0	B_1	z_1^3	z_1^2	z_1	0
5-5	$1/2$	0	$8/25$	$-11/50$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$	0
5-6	$1/2$	0	$8/25$	$-11/50$	$1/5$	C_1	$z_0^3/5$	$z_0^2/5$	$z_0/5$	0

where z_0 is the root of the equation $z^4 + z^3 + z^2 + z + 1 = 0$, $A_1 = -\frac{1}{4} - a_2 - a_3 - a_4$, $C_1 = -1/5 - a_2 - a_3 - a_4$ and z_1 is the root of the equation $z^4 - z^3 + z^2 - z + 1 = 0$, $B_1 = 1 - a_2 - a_3 - a_4$.

Elementary solution for $l = 6$

$l = 6$	Parameters				Coefficients						
Sol. No	α	β	γ	δ	a_0	a_1	a_2	a_3	a_4	a_5	a_6
6-1	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	± 1	1	± 1	1	± 1	0
6-2	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	$-1 - z_0$	$-z_0$	1	$1 + z_0$	z_0	0
6-3	$\frac{1}{72}$	$-\frac{25}{72}$	0	$\frac{1}{2}$	0	$1 - z_1$	z_1	-1	$1 - z_1$	z_1	0
6-4	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$\pm\frac{1}{6}$	$\frac{1}{6}$	$\pm\frac{1}{6}$	$\frac{1}{6}$	$\pm\frac{1}{6}$	0
6-5	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$-\frac{1+z_0}{6}$	$\frac{z_0}{6}$	$\frac{1}{6}$	$-\frac{1+z_0}{6}$	$z_0/6$	0
6-6	$\frac{1}{2}$	0	$\frac{25}{72}$	$-\frac{13}{72}$	$\frac{1}{6}$	$\frac{1-z_1}{6}$	$-\frac{z_1}{6}$	$-\frac{1}{6}$	$\frac{z_1-1}{6}$	$z_1/6$	0

where z_0 is the root of the equation $z^2 + z + 1$, z_1 is the root of the equation $z^2 - z + 1$.

Generalization of computed solutions

Solution 3-1 can be obtained from elementary solutions $y = x^{4/3}$ or $y = x^{1/3}$ by the Bäcklund transformations

$$T_8 : y_8(x; -\delta + 1/2, -\gamma, -\beta, -\alpha + 1/2) = \frac{x(y(x;\alpha,\beta,\gamma,\delta) - 1)}{y(x;\alpha,\beta,\gamma,\delta) - x}$$

and

$$T_9 : y_9(x; \gamma, \delta - 1/2, \alpha, \beta + 1/2) = \frac{y(x;\alpha,\beta,\gamma,\delta) - x}{y(x;\alpha,\beta,\gamma,\delta) - 1},$$

correspondingly. The same transformations connect the solution 5-3 with elementary solutions $y = x^{6/5}$ and $y = x^{1/5}$, correspondingly. It is possible that some other solutions among mentioned above may be obtained from the known elementary solutions of the equation PVI by the Bäcklund transformations but the authors does not known anything about it.
All the found solutions are algebraic solutions and can be written in the polynomial parametrization.

Generalization of computed solutions

Solutions 3-4, 3-8, 4-4, 5-5, 6-1 can be written as the sum of finite geometrical progression and then can be generalized for the case of any power exponents. The direct substitution shows that the function

$$y = b\frac{x-1}{x^b-1}$$

is the exact solution of the equation PVI for $\alpha = 1/2$, $\beta = 0$, $\gamma = (1-b)^2/2$, $\delta = -(2+b)^2/2$, $b \in \mathbb{C}$.

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