Calculation of Power Series Expansions with a Finite Principal or Regular Part for Solutions of Systems of ODEs

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$$dp/dt = qr, \quad dq/dt = -pr, \quad dr/dt = -k^2 pq.$$

with first integrals

$$q^2{+}p^2{=}1, \quad r^2{+}k^2p^2{=}1.$$

Polynomial form

$$0 = -dp/dt + qr, \quad 0 = -dq/dt - pr, \quad 0 = -dr/dt - k^2 pq.$$

Power series substitution

$$p{=}t^{\alpha_1}\left(p_0{+}\sum_{j=1}^{\infty} p_j t^{j\Delta}\right),\ q{=}t^{\alpha_2}\left(q_0{+}\sum_{j=1}^{\infty} q_j t^{j\Delta}\right),\ r{=}t^{\alpha_3}\left(r_0{+}\sum_{j=1}^{\infty} r_j t^{j\Delta}\right)$$

Transform the equations to expansions

$$0{=}t^{\beta_1}\left(f_0{+}\sum_{j=1}^{\infty} f_j t^{j\Delta}\right),\ 0{=}t^{\beta_2}\left(g_0{+}\sum_{j=1}^{\infty} g_j t^{j\Delta}\right),\ 0{=}t^{\beta_3}\left(h_0{+}\sum_{j=1}^{\infty} h_j t^{j\Delta}\right).$$

$$\begin{aligned}
0&{=}{-}\alpha_1 p_0 t^{\alpha_1-1}{+}q_0 r_0 t^{\alpha_2+\alpha_3}{+}\cdots,\\
0&{=}{-}\alpha_2 q_0 t^{\alpha_2-1}{-}p_0 r_0 t^{\alpha_1+\alpha_3}{+}\cdots,\\
0&{=}{-}\alpha_3 r_0 t^{\alpha_3-1}{-}k^2 p_0 q_0 t^{\alpha_1+\alpha_2}{+}\cdots.
\end{aligned}$$

$\beta_1 = \min(\alpha_1 - 1, \alpha_2 + \alpha_3)$, $\beta_2 = \min(\alpha_2 - 1, \alpha_1 + \alpha_3)$, $\beta_3 = \min(\alpha_3 - 1, \alpha_1 + \alpha_2)$

$$\begin{array}{lll} \beta_1 \leqslant \alpha_1 - 1, & \beta_2 \leqslant \alpha_2 - 1, & \beta_3 \leqslant \alpha_3 - 1 \\ \beta_1 \leqslant \alpha_2 + \alpha_3, & \beta_2 \leqslant \alpha_1 + \alpha_3, & \beta_3 \leqslant \alpha_1 + \alpha_2 \end{array}$$

$$\begin{array}{ccc} \sum_j \nu_j V_j, & \sum_j \eta_j V_j, \eta > 0, & \sum_j \mu_j V_j, 0 < \mu < 1, \sum_j \mu_j = 1 \\ \text{Linear hull} & \text{Cone hull} & \text{Convex hull} \end{array}$$

Solution $\alpha_1 = \alpha_2 = \alpha_3 = -1$, $\beta_1 = \beta_2 = \beta_3 = -2$

$$p = p_0 t^{-1} + p_1 t^{-1+\Delta} + \cdots,\ q = q_0 t^{-1} + q_1 t^{-1+\Delta} + \cdots,\ r = r_0 t^{-1} + r_1 t^{-1+\Delta} + \cdots,$$

$$\begin{array}{ll} 0 = (f_0 = p_0 + q_0 r_0 = 0) t^{-2} + (f_1 = 0) t^{-2+\Delta} + \cdots, & p_0 = \pm 1/k, \\ 0 = (g_0 = q_0 - p_0 r_0 = 0) t^{-2} + (g_1 = 0) t^{-2+\Delta} + \cdots, & q_0 = \pm i/k, \ \ i^2 = -1. \\ 0 = (h_0 = r_0 - k^2 p_0 q_0 = 0) t^{-2} + (r_1 = 0) t^{-2+\Delta} + \cdots. & r_0 = \pm i, \end{array}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \left(\begin{bmatrix} f_1 \\ g_1 \\ h_1 \end{bmatrix} = \begin{bmatrix} 1-\Delta & \pm i & \pm i/k \\ \mp i & 1-\Delta & \mp i/k \\ \mp ik & \mp k & 1-\Delta \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} t^{-2+\Delta} \\ t^{-2+\Delta} \\ t^{-2+\Delta} \end{bmatrix} + \cdots$$

Determinant $-(\Delta - 2)^2 (\Delta + 1) = 0$ if $\Delta = 2$. Then $r_1 = -k(ip_1 + q_1)$. With integrals $p_1 = \pm(k^2+1)/(6k)$, $q_1 = \pm i(1 - 2k^2)/(6k)$, $r_1 = \pm i(k-2)/6$.

Euler-Poisson Equations

V.V. Golubev, Lectures on Integration of Equations Motion of a Heavy Rigid Body near Fixed Point, Moscow: GITTL, 1953.

$$
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr - Mg(z_0\gamma_2 - y_0\gamma_3),\\
B\frac{dq}{dt} &= (C-A)rp - Mg(x_0\gamma_3 - z_0\gamma_1),\\
C\frac{dr}{dt} &= (A-B)pq - Mg(y_0\gamma_1 - x_0\gamma_2),\\
\frac{d\gamma_1}{dt} &= r\gamma_2 - q\gamma_3,\\
\frac{d\gamma_2}{dt} &= p\gamma_3 - r\gamma_1,\\
\frac{d\gamma_3}{dt} &= q\gamma_1 - p\gamma_2,
\end{aligned}
$$

Parameters of Euler-Poisson Equations

where t - time, A, B, C - principal moments of inertia, which satisfy triangle inequalities

$$A > 0,\ B > 0,\ C > 0,$$
$$A + B \geq C,\ A + C \geq B,\ B + C \geq A,$$

Mg - the body weight, x_0, y_0, z_0 - coordinates of the center of gravity of the rigid body in the body frame, p, q, r - projections of the angular velocity vector onto the body frame axes, $\gamma_1, \gamma_2, \gamma_3$ - direction cosines of the vertical in the body frame.

First Integrals of Euler-Poisson Equations

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const},$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

These are energy, momentum and geometry integrals.
We take a system of units where $Mg = 1$.

Powers and Coefficients of minimal power Terms

Ssubstitute terms $p_0 t^{-1}$, $q_0 t^{-1}$, $r_0 t^{-1}$, $\gamma_{1,0} t^{-2}$, $\gamma_{2,0} t^{-2}$, $\gamma_{3,0} t^{-2}$ to Euler-Poisson system

$$
\begin{aligned}
0 &= (f_{1,0} = Ap_0 + (B - C)q_0 r_0 - z_0\gamma_{2,0} + y_0\gamma_{3,0} = 0)t^{-2} + \dots,\\
0 &= (f_{2,0} = Bq_0 + (C - A)p_0 r_0 - x_0\gamma_{3,0} + z_0\gamma_{1,0} = 0)t^{-2} + \dots,\\
0 &= (f_{3,0} = Cr_0 + (A - B)p_0 q_0 - y_0\gamma_{1,0} + x_0\gamma_{2,0} = 0)t^{-2} + \dots,\\
0 &= (f_{4,0} = 2\gamma_{1,0} + r_0\gamma_{2,0} - q_0\gamma_{3,0} = 0)t^{-3} + \dots,\\
0 &= (f_{6,0} = 2\gamma_{2,0} + p_0\gamma_{3,0} - r_0\gamma_{1,0} = 0)t^{-3} + \dots,\\
0 &= (f_{6,0} = 2\gamma_{3,0} + q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0)t^{-3} + \dots.
\end{aligned}
$$

Powers and Coefficients of minimal power Terms

Substitute terms
$p{=}t^{-1}\left(p_0{+}p_1 t^{\Delta}{+}\cdots\right), q{=}t^{-1}\left(q_0{+}q_1 t^{\Delta}{+}\cdots\right), r{=}t^{-1}\left(r_0{+}r_1 t^{\Delta}{+}\cdots\right),$
$\gamma_1{=}t^{-2+\eta}\left(\gamma_{1,0}{+}\gamma_{1,1} t^{\Delta}{+}\cdots\right), \gamma_2{=}t^{-2+\eta}\left(\gamma_{2,0}{+}\gamma_{2,1} t^{\Delta}{+}\cdots\right), \gamma_3{=}t^{-2+\eta}\left(\gamma_{3,0}\right.$
$\eta > 0$

$$
\begin{aligned}
0 &= (f_{1,0} = Ap_0 + (B-C)q_0 r_0 = 0)t^{-2} - (z_0\gamma_{2,0} - y_0\gamma_{3,0})t^{-2+\eta} + \ldots,\\
0 &= (f_{2,0} = Bq_0 + (C-A)p_0 r_0 = 0)t^{-2} - (x_0\gamma_{3,0} - z_0\gamma_{1,0})t^{-2+\eta} + \ldots,\\
0 &= (f_{3,0} = Cr_0 + (A-B)p_0 q_0 = 0)t^{-2} - (y_0\gamma_{1,0} - x_0\gamma_{2,0})t^{-2+\eta} + \ldots,\\
0 &= (f_{4,0} = (2-\eta)\gamma_{1,0} + r_0\gamma_{2,0} - q_0\gamma_{3,0} = 0)t^{-3+\eta} + \ldots,\\
0 &= (f_{5,0} = (2-\eta)\gamma_{2,0} + p_0\gamma_{3,0} - r_0\gamma_{1,0} = 0)t^{-3+\eta} + \ldots,\\
0 &= (f_{6,0} = (2-\eta)\gamma_{3,0} + q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0)t^{-3+\eta} + \ldots.
\end{aligned}
$$

$$p_0=\frac{i\sqrt{BC}}{\sqrt{(A-B)(A-C)}},\ q_0=\frac{-\sqrt{AC}}{\sqrt{(A-B)(B-C)}},\ r_0=\frac{i\sqrt{AB}}{\sqrt{(B-C)(A-C)}},$$

$$\gamma_{2,0}=\frac{i\gamma_{1,0}\sqrt{B(A-C)}}{\sqrt{A(B-C)}},\ \gamma_{3,0}=\frac{\gamma_{1,0}\sqrt{C(A-B)}}{\sqrt{A(B-C)}},\ \eta=1$$

Determinant $0=ABC(\Delta-2)^3(\Delta-1)\Delta(\Delta+1)$

$$z_0=\frac{-i(y_0\sqrt{AB(B-C)(C-A)}-x_0A(C-B)}{\sqrt{AC(B-A)(B-C)}}$$

For $z_0=0$ $y_0=\dfrac{-x_0\sqrt{A(B-C)}}{\sqrt{B(C-A)}}$

Gess-Appelrot condition.

Powers and Coefficients of minimal power Terms

Substitute terms
$p{=}t^{-1}\Big(p_0{+}p_1t^{\Delta}{+}\cdots\Big)\,,q{=}t^{-1}\Big(q_0{+}q_1t^{\Delta}{+}\cdots\Big)\,,r{=}t^{-1}\Big(r_0{+}r_1t^{\Delta}{+}\cdots\Big),$
$\gamma_1{=}t^{-2}\Big(\gamma_{1,0}{+}\gamma_{1,1}t^{\Delta}{+}\cdots\Big)\,,\gamma_2{=}t^{-2}\Big(\gamma_{2,0}{+}\gamma_{2,1}t^{\Delta}{+}\cdots\Big)\,,\gamma_3{=}t^{-2+\eta}\Big(\gamma_{3,0}{+}\gamma_{3,1}$
$\eta > 0$

$$
\begin{aligned}
0 &= (f_{1,0} = Ap_0 + (A - C)q_0r_0 = 0)t^{-2} + y_0\gamma_{3,0}t^{-2+\eta} + \ldots,\\
0 &= (f_{2,0} = Aq_0 + (C - A)p_0r_0 = 0)t^{-2} - x_0\gamma_{3,0}t^{-2+\eta} + \ldots,\\
0 &= (f_{3,0} = Cr_0 - (y_0\gamma_{1,0} - x_0\gamma_{2,0}) = 0)t^{-2} + \ldots,\\
0 &= (f_{4,0} = 2\gamma_{1,0} + r_0\gamma_{2,0} = 0)t^{-3} - q_0\gamma_{3,0}t^{-3+\eta} + \ldots,\\
0 &= (f_{5,0} = 2\gamma_{2,0} - r_0\gamma_{1,0} = 0)t^{-3} + p_0\gamma_{3,0}t^{-3+\eta} + \ldots,\\
0 &= (f_{6,0} = q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0)t^{-3} + (2 - \eta)\gamma_{3,0}t^{-3+\eta} + \ldots.
\end{aligned}
$$

Powers and Coefficients of minimal power Terms

$q_0 = ip_0,\ r_0 = 2i,\ \gamma_{1,0} = \dfrac{-A}{iy_0 + x_0},\ \gamma_{2,0} = \dfrac{-iA}{iy_0 + x_0},\ C = A/2.$

Determinant $(A^3(\Delta - 4)(\Delta - 3)(\Delta - 2)(\Delta - 1)\Delta(\Delta + 1))/2$

Powers and Coefficients of minimal power Terms

Substitute terms

$p{=}t^{-1+3\mu}\left(p_0{+}p_1t^{\Delta}{+}\cdots\right), q{=}t^{-1+3\mu}\left(q_0{+}q_1t^{\Delta}{+}\cdots\right), r{=}t^{-1}\left(r_0{+}r_1t^{\Delta}{+}\cdots\right.$

$\gamma_1{=}t^{-2}\left(\gamma_{1,0}{+}\gamma_{1,1}t^{\Delta}{+}\cdots\right), \gamma_2{=}t^{-2}\left(\gamma_{2,0}{+}\gamma_{2,1}t^{\Delta}{+}\cdots\right), \gamma_3{=}t^{-2+3\mu}\left(\gamma_{3,0}{+}\gamma_{3}\right.$

$0 < \mu < 1$

$$
\begin{aligned}
0 &= (f_{1,0} = (1-3\mu)Ap_0 + (B-C)q_0r_0 = 0)t^{-2+3\mu} + \ldots,\\
0 &= (f_{2,0} = (1-3\mu)Bq_0 + (C-A)p_0r_0 - x_0\gamma_{3,0} = 0)t^{-2+3\mu} + \ldots,\\
0 &= (f_{3,0} = Cr_0 + x_0\gamma_{2,0} = 0)t^{-2} + (A-B)p_0q_0t^{-2+6\mu} + \ldots,\\
0 &= (f_{4,0} = 2\gamma_{1,0} + r_0\gamma_{2,0} = 0)t^{-3} - q_0\gamma_{3,0}t^{-3+6\mu} + \ldots,\\
0 &= (f_{6,0} = 2\gamma_{2,0} - r_0\gamma_{1,0} = 0)t^{-3} + p_0\gamma_{3,0}t^{-3+6\mu} + \ldots,\\
0 &= (f_{6,0} = (2-3\mu)\gamma_{3,0} + q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0)t^{-3+3\mu} + \ldots.
\end{aligned}
$$

Powers and Coefficients of minimal power Terms

$A=\frac{4C(C-B)}{3B\mu(3\mu-1)+2(C-B)}$, $q_0=\frac{2iC(3\mu-1)p_0}{3B\mu(3\mu-1)+2(C-B)}$, $r_0=2i$,
$\gamma_{1,0}=-\frac{2C}{x_0}$, $\gamma_{2,0}=-\frac{2iC}{x_0}$. $\gamma_{3,0}=\frac{2iC(3B\mu-2(C+B))p_0}{(3B\mu(3\mu-1)+2(C-B))x_0}$,
Determinant

$$\frac{4BC^2(C-B)(\Delta-4)(\Delta-2)\Delta(\Delta+1)(3\mu+\Delta-3)(6\mu+\Delta-1)}{3B\mu(3\mu-1)+2(C-B)} \quad \Delta<6\mu, \Delta=6\mu$$

$\Delta=1$, $\mu=1/6$, $A=16C(C-B)/(8C-9B)$ Goraychev case
$\Delta=2/3$, $\mu=1/9$, $A=18C(C-B)/(9C-10B)$ N.Kowalewski case