# Mechanical interpretation and efficient computation of elliptic integrals of the third kind

Semjon Adlaj
http://SemjonAdlaj.com/

The joint MSU-CCRAS Computer Algebra Seminar

Moscow, Russia
November 27, 2019

## The calculation of the exact period of motion of a simple pendulum

Gauss discovered that complete elliptic integrals of the first kind are readily calculable via the arithmetic-geometric mean. Gauss recorded in his diary, on May 30, 1799, that his discovery "opens an entirely new field of analysis."[1] In particular, the lemniscate integral (that is the quarter length of the lemniscate of Bernoulli whose focal distance is $\sqrt{2}$) is expressible as

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2\,M(\sqrt{2})} \approx 1.31102877714605990523,$$

being merely a special instance of the formula

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-(1-\beta^2)x^2)}} = \frac{\pi}{2\,M(\beta)},$$

where $M(x)$ is the arithmetic-geometric mean of 1 and $x$.

[1] Klein F. Gauß' wissenschaftliches Tagebuch 1796-1814// Mathematische Annalen, Springer Berlin Heidelberg, **57** (1903), 1-34.

# Modified Arithmetic-Geometric Mean



Asked 6 years, 6 months ago Active 3 years, 1 month ago Viewed 428 times

1

Let $\{x_n\}$ and $\{y_n\}$ be defined iteratively, $x_0 := \beta > 1,\ y_0 := 1$and $x_{n+1} = \frac{x_n+y_n}{2}$, $y_{n+1} = (x_n.y_n)^{\frac{1}{2}}$; i.e. they are respectively the arithmetic and geometric mean of the previous terms. We know that their limit is called the arithmetic-geometric mean of $\beta$ and 1 (denoted by $AGM(\beta, 1)$).

Now, let's define $\xi_0 := \beta^2$, $\eta_0 := 1$ and $\zeta_0 := 0$. Then let's define iteratively, $\xi_{n+1} := \frac{\xi_n+\eta_n}{2}$, $\eta_{n+1} := \zeta_n + ((\xi_n - \zeta_n)(\eta_n - \zeta_n))^{\frac{1}{2}}$ and finally $\zeta_{n+1} := \zeta_n - ((\xi_n - \zeta_n)(\eta_n - \zeta_n))^{\frac{1}{2}}$. The common limit of $\xi_n$ and $\eta_n$ is called the modified arithmetic-geometric mean of $\beta^2$ and 1 (denoted by $MAGM(\beta^2, 1)$).

My question is wheter there is an easy way to prove the equality $\xi_n = \beta^2 - \sum_{m=0}^{n-1} 2^m \frac{x_m^2 - y_m^2}{2}$ and if there is, how? Thank you very for any help.

Note: This is from an article in Notices of the AMS, Volume 59, Number 8.

analysis number-theory

share cite edit

asked May 14 '13 at 13:22
btm
41 ▲3

# A demonstration of the MAGM-based formula for calculating the elliptic integrals of the 2nd kind.

For the calculation of the elliptic integral of the 2nd kind, there exists an old algorithm (very well exposed and demonstrated by Jameson in "Elliptic integrals, the arithmetic-geometric mean and the Brent-Salamin algorithm for π") and an easier algorithm, called MAGM, discovered by Semjon Adlaj. Here it is demonstrated they are equivalent.

Let us call 1-S the limit, for $N \to \infty$, of the summation defined in theorem 7.3 by Jameson,

where $\mathrm{E}(k) = \mathrm{J}(1, b) = (1 - S)I(1, b) = (1 - S)I(1, b)$. Let MAGM(1,b) be the result of Semjon Adlaj's MAGM, for $b = \sqrt{1 - k^2}$

Note that after we demonstrate that the two methods are equivalent, E(k) is $\frac{1}{2\pi}$times the perimeter

P(b) of an ellipse with semi-axes 1 and b. So P(b) is $4\ (1 - S)\mathrm{K}(k) = 2\pi\ (1 - S)/\mathrm{AGM}(1, b)$ and

(pending demonstration below) it is also $2\pi\ MAGM(1, b)/\mathrm{AGM}(1, b)$.

# Equivalence of two algorithms, for calculating MAGM, as demonstrated by F. Lamarche

The series 1-S (where S is the convergent running sum of the scaled differences of the squares of the arithmetic and geometric means) and MAGM(b), appear not only have the same limit value, but also appear to be the same at each iteration of the summing (for 1-S) and of modified averaging (for MAGM). Actually, it can relatively easily be checked that the expressions are the same in the first few rows.

| AGM(1,b) and running sum of square differences | | | | MAGM(1,b) | | |
|---|---|---|---|---|---|---|
| Row # | Arithmetic mean | Geometric mean | 1-S running sum up to row# | Arithm. mean | Sum | Diff (*) |
| 0 | 1 | $b$ | $R_0 = 1$ | 1 | $b^2$ | 0 |
| 1 | ½ $+b/2$ | $\sqrt{b}$ | $R_1 = 1 - \frac{1}{2}(1 - b^2)$ | $\frac{1+b^2}{2}$ | $b$ | $-b$ |
| 2 | ¼ $+\frac{b}{4} + \sqrt{b}/2$ | $g_2$ | $R_2 = R_1 -$ 1$((1+b)^2$/4-b) | $A_2 = (\frac{1}{4} + \frac{b^2}{4} + \frac{b}{2})$ | $S_2$ | $D_2$ |
| 3 | $a_3$ | $g_3$ | $R_3 = R_2 - 2(a_2^2 - g_2^2)$ | $A_3$ | $S_3$ | $D_3$ |
| 4 | $a_4$ | $g_4$ | $R_4 = R_3 - 4(a_3^2 - g_3^2)$ | $A_4$ | $S_4$ | $D_4$ |
| … | … | … | … | … | … | … |
| N | $a_N = \frac{a_{N-1}}{2} + \frac{g_{N-1}}{2}$ | $g_N = \sqrt{g_{N-1}a_{N-1}}$ | $R_3 = R_2 - 2^{N-2}(a_{N-1}^2 - g_{N-1}^2)$ | $A_N =$ ½$(A_{N-1} + S_{N-1})$ | $S_N = D_{N-1} + \sqrt{(S_{N-1} - D_{N-1})(A_{N-1} - D_{N-1})}$ | $D_N = 2D_{N-1} - S_N$ |

(*) Note … $D_N = D_{N-1} - \sqrt{(S_{N-1} - D_{N-1})(A_{N-1} - D_{N-1})}$

by simple expansion of the products, it is easy to see that $R_1 = A_1$ and $R_2 = A_2$. Starting with the third row, it is possible to develop the products to demonstrate equality of $R_N$ and $A_N$, but it becomes increasingly difficult.

To prove that the equality stands for any number N, we can build a proof by contradiction. Suppose there is a breaking point of the series of equalities, i.e. a number B where $R_B \neq A_B$, while $R_{B-1} = A_{B-1}$.

One remarkable property of the MAGM series is that one can add a constant K to all the A, S, and D numbers, and it will still be a MAGM, as "K" comes out added on each row via D (for S and D) and via the average (for A), the differences not changing. This means that we can build a MAGM where we add b to each and all the numbers.

Apart from a needed scale factor, the second row (the N=1 row) now looks exactly like a first row (with a different value of b), because the D value is zero, which is the characteristic of the first row of a MAGM.

The different value of b is $b^* = \sqrt{2b/(\frac{1}{2} + \frac{b^2}{2} + b)} = 2\sqrt{b}/(1+b)$.

Another remarkable property of the MAGM is that we can apply any scale factor to a MAGM sequence, and still get a MAGM, because when calculating a new row, if all the values are a factor $S_{MAGM}$ times as large as before, all the values of the new row will simply be a factor $S_{MAGM}$ times as large as before.

The scale factor that we need to make the A of the second row a "1", like the A of the first row was, is easy to figure out: $S_{MAGM} = 1/(\frac{1}{2} + \frac{b^2}{2} + b) = 2/(1+b)^2$

After these two operations, the second row has become a first row (for $b^*$ instead of $b$, but it holds for any value of either in the range ]0,1]).

Looking now at the a, g, and R columns, we notice that the AGM columns a and g can also be scaled by any factor and still will be a AGM.

The increments in the R column are scaled by the square of that factor. If we rescale by $S_{AGM}$=1/((1+b)/2), the second row of the AGM looks like a first row, with the "new b value" $b^*$ being $2\sqrt{b}/(1+b)$, as in the MAGM above, and the running sum increments will be scaled by the square of $S_{AGM}$, 1/((1+b)/2)/((1+b)/2)= 4/(1+b)$^2$. This is twice the scaling of MAGM. In order to use the second row of the R column as a first row, we need to make one further change: divide all the numbers by 2, because $2^{P-2}$ is half of $2^{N-2}$ when P=N-1 is the row number counting from the second row, which differs by one from the row number N counting from the first row.

After setting the new first row R value $R_1$ to 1, we can now see that we are getting a set of values R matching (at least initially) the MAGM's set of values A, since we have set the first values the same, and the increments are the same due to identical scaling factors.

The breaking point of the equality being at B means that as far as the newly built table is concerned, the breaking point of the equality will be at B-1. The breaking point of the equality cannot depend on the value of b, and yet the breaking point of the equality is B when considering b, and B-1, when considering b*. We arrive at the impossible conclusion that B=B-1.

The conclusion being impossible, the hypothesis of a finite B where equality of the R and A series breaks is itself impossible, so $A_n = R_n$ for any n, and thus the limit is the same, and MAGM(1,b)=1-S, where S is the scaled sum of the differences between the square of the arithmetic and geometric means of the plain AGM. QED.

### 4.1 Three Formulas for Calculating Three Kinds of CEI

A CEI of the first kind $I_1$ is defined and calculated as

$$I_1 = I_1(\gamma) := \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}} = \frac{\pi}{2\,M(\beta)}. \tag{1799.05.30}$$

A CEI of the second kind $I_2$ is defined and calculated as

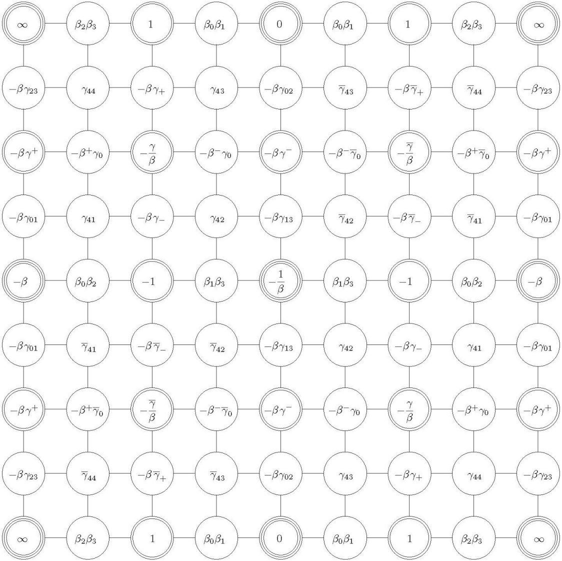
$$I_2 = I_2(\gamma) := \int_0^1 \sqrt{\frac{1-\gamma^2 t^2}{1-t^2}} = \frac{\pi\,N(\beta^2)}{2\,M\,(\beta)}. \tag{2011.12.16}$$

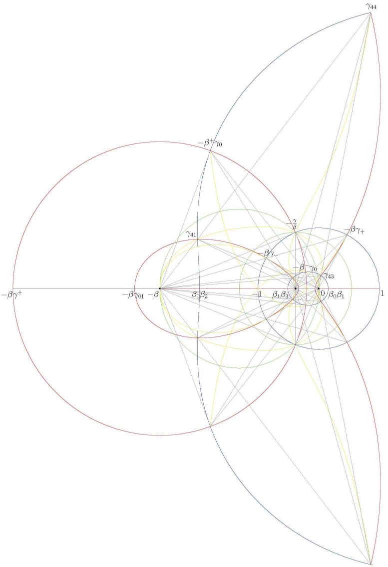
Both formulas (1799.05.30) and (2011.12.16) apply at $\gamma = 0$, with $I_2(0) = I_1(0) = \pi/2$. The second applies, as well, at $\gamma = 1$, with $I_2(1) = 1$, as clarified in [7,8].

A CEI of the third kind $I_3$ is defined and calculated as

$$I_3 = I_3(\gamma, \delta) := \int_0^1 \frac{dt}{(t^2-\delta)\,\sqrt{(1-t^2)\,(1-\gamma^2\,t^2)}} = -\frac{\pi\,\gamma^2\,N(\beta^2,\,1-\delta\gamma^2)}{2\,M(\beta)} =$$

$$= \frac{\pi\,N\,(\beta^2,\,\infty,\,\beta^2-\delta\gamma^2,\,1-\delta\gamma^2)}{2\,M\,(\beta)},\quad \delta \in \mathbb{C}\backslash[0,1]. \tag{2015.09.02}$$


$\infty$ $\beta_2\beta_3$ $1$ $\beta_0\beta_1$ $0$ $\beta_0\beta_1$ $1$ $\beta_2\beta_3$ $\infty$
$-\beta\gamma_{23}$ $\gamma_{44}$ $-\beta\gamma_+$ $\gamma_{43}$ $-\beta\gamma_{02}$ $\overline{\gamma}_{43}$ $-\beta\overline{\gamma}_+$ $\overline{\gamma}_{44}$ $-\beta\gamma_{23}$
$-\beta\gamma^+$ $-\beta^+\gamma_0$ $-\frac{\gamma}{\beta}$ $-\beta^-\gamma_0$ $-\beta\gamma^-$ $-\beta^-\overline{\gamma}_0$ $-\frac{\overline{\gamma}}{\beta}$ $-\beta^+\overline{\gamma}_0$ $-\beta\gamma^+$
$-\beta\gamma_{01}$ $\gamma_{41}$ $-\beta\gamma_-$ $\gamma_{42}$ $-\beta\gamma_{13}$ $\overline{\gamma}_{42}$ $-\beta\overline{\gamma}_-$ $\overline{\gamma}_{41}$ $-\beta\gamma_{01}$
$-\beta$ $\beta_0\beta_2$ $-1$ $\beta_1\beta_3$ $-\frac{1}{\beta}$ $\beta_1\beta_3$ $-1$ $\beta_0\beta_2$ $-\beta$
$-\beta\gamma_{01}$ $\overline{\gamma}_{41}$ $-\beta\overline{\gamma}_-$ $\overline{\gamma}_{42}$ $-\beta\gamma_{13}$ $\gamma_{42}$ $-\beta\gamma_-$ $\gamma_{41}$ $-\beta\gamma_{01}$
$-\beta\gamma^+$ $-\beta^+\overline{\gamma}_0$ $-\frac{\overline{\gamma}}{\beta}$ $-\beta^-\overline{\gamma}_0$ $-\beta\gamma^-$ $-\beta^-\gamma_0$ $-\frac{\gamma}{\beta}$ $-\beta^+\gamma_0$ $-\beta\gamma^+$
$-\beta\gamma_{23}$ $\overline{\gamma}_{44}$ $-\beta\overline{\gamma}_+$ $\overline{\gamma}_{43}$ $-\beta\gamma_{02}$ $\gamma_{43}$ $-\beta\gamma_+$ $\gamma_{44}$ $-\beta\gamma_{23}$
$\infty$ $\beta_2\beta_3$ $1$ $\beta_0\beta_1$ $0$ $\beta_0\beta_1$ $1$ $\beta_2\beta_3$ $\infty$


$\gamma_{44}$
$-\beta^+\gamma_0$
$-\frac{\gamma}{\beta}$
$-\beta\gamma_+$
$\gamma_{41}$
$-\beta\gamma_-$
$-\beta^-\gamma_0$
$\gamma_{43}$
$-\beta\gamma^+$
$-\beta\gamma_{01}$
$-\beta$
$\beta_0\beta_2$
$-1$
$\beta_1\beta_3$
$0$
$\beta_0\beta_1$
$1$

# The "square-root problem"

- Does a choice of a branch (not necessarily continuous) exist for which the equality

$$\frac{1}{\sqrt{x}} = \sqrt{\frac{1}{x}}$$

  holds for all complex values in the punctured plane $\mathbb{C}\backslash\{0\}$?
- If not then does a choice for a branch exist for which the afore-indicated equality might be enforced upon any single (particularly desired) value in $\mathbb{C}\backslash\{0\}$?
- If not again then does there exist any value in $\mathbb{C}\backslash\{0\}$ for which the equality is guaranteed to hold regardless of the choice of a square-root branch?
- The only value which satisfies the condition of the latter question is $1$! Note that the equality NEVER holds for $x = -1$!

Moral: The square root and the reciprocal function do NOT commute.

# Several indefinite integrals of elliptic functions (source I)

Digital Library of Mathematical Functions

Index
Notations
Search
Help?
Citing
Customize
Annotate

## §22.14(iii) Other Indefinite Integrals

In (22.14.13)–(22.14.15), $0 < x < 2K$.

22.14.13
$$\int \frac{\mathrm{d}x}{\operatorname{sn}(x,k)} = \ln\left(\frac{\operatorname{sn}(x,k)}{\operatorname{cn}(x,k)+\operatorname{dn}(x,k)}\right),$$

22.14.14
$$\int \frac{\operatorname{cn}(x,k)\,\mathrm{d}x}{\operatorname{sn}(x,k)} = \frac{1}{2}\ln\left(\frac{1-\operatorname{dn}(x,k)}{1+\operatorname{dn}(x,k)}\right),$$

22.14.15
$$\int \frac{\operatorname{cn}(x,k)\,\mathrm{d}x}{\operatorname{sn}^2(x,k)} = -\frac{\operatorname{dn}(x,k)}{\operatorname{sn}(x,k)}.$$

Alternatively, the preceding integrals might be expressed as

$$\int_{\frac{\pi}{2\sqrt{-1}M(\beta)}}^{x} \frac{dx}{\operatorname{sn}(x,\beta)} = \ln \operatorname{sn}\left(\frac{\sqrt{-1}\,(1+\beta)\,x}{2}, \frac{1-\beta}{1+\beta}\right),$$

$$\int_{\frac{\pi}{2\sqrt{-1}M(-\beta)}}^{x} \frac{dx}{\operatorname{sn}(x,\beta)} = \ln \operatorname{sn}\left(\frac{\sqrt{-1}\,(1-\beta)\,x}{2}, \frac{1+\beta}{1-\beta}\right),$$

$$\int_0^x \beta \operatorname{sn}(x,\beta)\; dx = \ln \operatorname{sn}\left(\frac{1+\beta}{2}\left(\sqrt{-1}\,x + \frac{\pi}{2M(\beta)}\right), \frac{1-\beta}{1+\beta}\right) =$$

$$= -\ln \operatorname{sn}\left(\frac{1-\beta}{2}\left(\sqrt{-1}\,x + \frac{\pi}{2M(-\beta)}\right), \frac{1+\beta}{1-\beta}\right).$$

---

⚠ Commercial software companies such as "Mathematica" are not permitted to use any of the newly presented formulas without an explicit and publically available written permission, signed by its author.

# Several integrals of "degenerate" elliptic functions

An improper special case, corresponding to $\beta = 0$, would be

$$\int_{\infty}^{x} \frac{dx}{\sin(x)} = \ln \tanh\left(\frac{\sqrt{-1}\,x}{2}\right) = \ln \tan\left(\frac{x}{2}\right) + \ln \sqrt{-1}.$$

Definite special cases, corresponding to $\beta = \pm 1$, are

$$\int_{\frac{\pi}{2\sqrt{-1}}}^{x} \frac{dx}{\tanh(x)} = \ln \sin\left(\sqrt{-1}\,x\right) = \ln \sinh(x) + \ln \sqrt{-1},$$

$$\int_{\frac{\pi}{2}}^{x} \frac{dx}{\tan(x)} = \ln \sin(x)\,,$$

$$\int_{0}^{x} \tanh(x)\,dx = \ln \sin\left(\sqrt{-1}\,x + \frac{\pi}{2}\right) = \ln \cosh(x),$$

$$-\int_{0}^{x} \tan(x)\,dx = \ln \sin\left(x + \frac{\pi}{2}\right) = \ln \cos(x).$$

# One "last" integral of a "degenerate" elliptic function

We must reveal that the (square) values

$$\mathrm{sn}\left(\frac{1+\beta}{2}\left(\sqrt{-1}\,x+\frac{\pi}{2M(\beta)}\right),\frac{1-\beta}{1+\beta}\right)^2,\ \mathrm{sn}\left(\frac{1+\beta}{2}\sqrt{-1}\,x,\frac{1-\beta}{1+\beta}\right)^2$$

are interrelated via the inversion

$$x \mapsto t_\beta(x) := \frac{x-1}{x(1-\beta)^2/(1+\beta)^2-1}$$

before we consider "a last" case of "degeneration" for which $\beta$ is (again) tending to zero, so

$$\int_0^x \beta\sin(x)\,dx \approx \frac{1}{2}\ln t_\beta\left(\tanh\left(\frac{\sqrt{-1}\,x}{2}\right)^2\right) = \frac{1}{2}\ln\left(\frac{(1+\beta)^2}{1+\beta^2+2\beta\cos(x)}\right) \approx$$

$\approx 2\beta\sin(x/2)^2$, as long as the upper limit of integration is fixed in $\mathbb{C}$.

# A general analytic formula, describing the motion of the pendulum alleged impossible!


УДК 531
ББК 22.21
Л55

Лидов М.Л. **Курс лекций по теоретической механике.** — 2-е изд., испр. и доп. — М.: ФИЗМАТЛИТ, 2010. — 496 с. — ISBN 978-5-9221-0897-3.


Настоящий курс лекций по теоретической механике был разработан в 1974 г. сотрудником Института прикладной математики им. М.В. Келдыша РАН доктором физико-математических наук, профессором, лауреатом Ленинской премии М.Л. Лидовым (1926–1993 гг.) и в течение ряда лет читался им студентам механико-математического факультета МГУ.

Оригинальный курс лекций представляет собой фундаментальный труд и при жизни автора не издавался. Издание «Лекций по теоретической механике» М.Л. Лидова принесет несомненную пользу студентам, аспирантам, преподавателям, специалистам-механикам.


Второе издание осуществлено при финансовой поддержке Института прикладной математики им. М.В. Келдыша РАН


Когда либрационная кривая приближается к сепаратрисе, $\alpha \to 2\omega^2$ и $k^2 \to 1-0$. Для полного эллиптического интеграла при модуле $k$, близком к 1, можно воспользоваться известным представлением

$$K = \ln\frac{4}{k'} + \left(\frac{1}{2}\right)^2\left(\ln\frac{4}{k'} - \frac{2}{1\cdot 2}\right)k'^2 + \ldots, \qquad (4.8.20)$$

где $k' = \sqrt{1-k^2}$. При $k^2 \to 1-0$ имеем $K \approx \ln\frac{4}{k'}$, $\tau \approx \frac{4}{\omega}\ln\frac{4}{k'} \to \infty$, т.е. при приближении либрационной кривой к сепаратрисе период колебания маятника неограниченно возрастает. При этом основное время маятник будет находиться в окрестностях неустойчивых особых точек $(\pm\pi, 0)$.

б) К сожалению, невозможно единым аналитическим выражением описать траектории движения маятника в либрационной и ротационной зонах. Этому препятствует топологическое различие поведения фазовых траекторий в этих зонах. Для получения аналитического выражения в ротационной зоне $\alpha > 2\omega^2$



преобразуем уравнение (11):

$$\alpha + 2\omega^2\cos\varphi = (\alpha + 2\omega^2)\left(1 - \frac{4\omega^2}{\alpha + 2\omega^2}\sin^2\frac{\varphi}{2}\right).$$

Обозначая

$$k^2 = \frac{4\omega^2}{\alpha + 2\omega^2}, \quad 0 < k^2 < 1 \qquad (****)$$

и $\varphi/2 = \Delta\cdot\psi$, из (11) получаем

$$dt = \frac{2}{\sqrt{\alpha + 2\omega^2}}\,\frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}}. \qquad (4.8.21)$$

Интегрируя (21) с начальными условиям $t = t_0$, $\psi = 0$, находим

$$t - t_0 = \frac{2}{\sqrt{\alpha + 2\omega^2}}\int_0^{\psi}\frac{d\xi}{\sqrt{1 - k^2\sin^2\xi}},$$

откуда для $\psi = n\frac{\pi}{2} + \overline{\psi}$, $0 \leqslant \overline{\psi} < \frac{\pi}{2}$

$$t - t_0 = \frac{2}{\sqrt{\alpha + 2\omega^2}}[nK(k^2) + F(k^2, \overline{\psi})], \qquad (4.8.22)$$

где $n$ – целое.

*Периодом вращения маятника* $\tau$ называется время, за которое угол $\varphi$ изменяется на $2\pi$. За это время угол $\psi$ изменится на $\pi$. Из (22) при $n = 2$ и $\overline{\psi} = 0$ получим

$$\tau = \frac{4K(k^2)}{\sqrt{\alpha + 2\omega^2}}.$$

В случае быстрого вращения

$$|\dot\varphi| \to \infty, \alpha \to \infty, k^2 \to 0, K(k^2) \to \pi/2 \text{ и } \tau \to \frac{2\pi}{\sqrt{\alpha + 2\omega^2}} \approx \frac{2\pi}{\sqrt{\alpha}} \to 0.$$

Если $\alpha \to 2\omega^2 + 0$, то ротационная кривая приближается к сепаратрисе. В этом случае, как следует из (****), $k^2 \to 1-0$, а

$$\tau \to \frac{2}{\omega}\ln\frac{4}{\sqrt{1-k^2}} \to \infty.$$

David Simpson is the first to introduce an exact formula for the period of a simple pendulum in a physics textbook!



We can explicitly write out the first few terms of this series; the result is

$$
\begin{aligned}
T = 2\pi\sqrt{\frac{L}{g}}\Bigg[1 &+ \frac{1}{4}\sin^2\left(\frac{\theta_0}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_0}{2}\right) + \frac{25}{256}\sin^6\left(\frac{\theta_0}{2}\right) \\
&+ \frac{1225}{16384}\sin^8\left(\frac{\theta_0}{2}\right) + \frac{3969}{65536}\sin^{10}\left(\frac{\theta_0}{2}\right) + \frac{53361}{1048576}\sin^{12}\left(\frac{\theta_0}{2}\right) + \frac{184041}{4194304}\sin^{14}\left(\frac{\theta_0}{2}\right) \\
&+ \frac{41409225}{1073741824}\sin^{16}\left(\frac{\theta_0}{2}\right) + \frac{147744025}{4294967296}\sin^{18}\left(\frac{\theta_0}{2}\right) + \frac{2133423721}{68719476736}\sin^{20}\left(\frac{\theta_0}{2}\right) + \cdots\Bigg].
\end{aligned}
\tag{O.13}
$$

If we wish, we can write out a series expansion for the period in another form—one which does not involve the sine function, but only involves powers of the amplitude $\theta_0$. To do this, we expand $\sin(\theta_0/2)$ into a Taylor series:

$$
\sin\frac{\theta_0}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\theta_0^{2n-1}}{2^{2n-1}(2n-1)!} \tag{O.14}
$$

$$
= \frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \frac{\theta_0^5}{3840} - \frac{\theta_0^7}{645120} + \frac{\theta_0^9}{185794560} - \frac{\theta_0^{11}}{81749606400} + \cdots \tag{O.15}
$$

Now substitute this series into the series of Eq. (O.11) and collect terms. The result is

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22931}{1321205760}\theta_0^8 + \frac{1319183}{951268147200}\theta_0^{10}\right.$$
$$+ \frac{233526463}{2009078326886400}\theta_0^{12} + \frac{2673857519}{265928913086054400}\theta_0^{14}$$
$$+ \frac{39959591850371}{4493134915501975142400}\theta_0^{16} + \frac{8797116290975003}{1099919427314883514859520}\theta_0^{18}$$
$$\left. + \frac{4872532317019728133}{668751011807449177034588160000}\theta_0^{20} + \cdots\right). \tag{O.16}$$

An entirely different formula for the exact period of a simple plane pendulum has appeared in a recent paper (Adlaj, 2012). According to Adlaj, the exact period of a pendulum may be calculated more efficiently using the *arithmetic-geometric mean*, by means of the formula

$$T = 2\pi\sqrt{\frac{L}{g}} \times \frac{1}{\operatorname{agm}(1, \cos(\theta_0/2))} \tag{O.17}$$

where $\operatorname{agm}(x, y)$ denotes the arithmetic-geometric mean of $x$ and $y$, which is found by computing the arithmetic and geometric means of $x$ and $y$, then the arithmetic and geometric mean of those two means, then iterating this process over and over again until the two means converge:

$$a_{n+1} = \frac{a_n + g_n}{2} \tag{O.18}$$

$$g_{n+1} = \sqrt{a_n g_n} \tag{O.19}$$

# The stable equilibrium of a simple pendulum

A known solution to the differential equation

$$\frac{\dot{\theta}^2}{2} - \cos\theta = -1 \tag{1}$$

is the trivial solution

$$e^{i\theta} \equiv 1,$$

which might be interpreted as the stable equilibrium position of a simple pendulum.

Is the trivial solution the only solution to equation (1)?!

Two inculcated, innocent-looking and deceiving arguments:

"Matematical": $\dot{\theta}$ must vanish since $\dot{\theta}^2$ is nonnegative.
"Physical": the solution ought to be unique since the equilibrium is stable.

## A non-trivial periodic solution beyond "the stable equilibrium solution"!

Yet, neither argument is worthy of elaboration since the afore-indicated (non-linear) differential equation (1) does possess another non-trivial periodic solution, which period $2\pi$ does coincide with the period of the (so-called) "small angle" pendulum:

$$t \mapsto e^{i\theta} = f(t) := \frac{\sin t + 1}{\sin t - 1} = -(\sec t + \tan t)^2 .$$

Note that the function $f$ might be extended to a map from the Riemann Sphere $\mathbb{C} \cup \infty$ onto itself, via setting $f(\infty) = 1$ (yet, on $\mathbb{C}$, the function $f$ does miss a single value, that is, 1).

Infinitely many other solutions to equation (1) are obtained by shifting the (complex) argument $t$ of the function $f$.

What is the significance of the non-trivial solution $f$? What is its (mechanical) interpretation?

# The unstable equilibrium solution as dual to the stable equilibrium solution

Aside from the unstable equilibrium solution

$$e^{i\theta} \equiv -1,$$

to the differential equation

$$\frac{\dot{\theta}^2}{2} - \cos\theta = 1,$$

the critical solution

$$t \mapsto e^{i\theta} = -f(-it) = \frac{i - \sinh t}{i + \sinh t} = (\operatorname{sech} t + i \tanh t)^2$$

separates oscillatory mode from rotary mode of motion of a simple pendulum. Observe that its range, for real values of the argument $t$, lies on the unit circle (centred at the origin in $\mathbb{C}$). Thus, this critical solution provides the formula of motion of the pendulum, from the stable equilibrium (lowest) position of the pendulum at $t = 0$ to its unstable equilibrium (upmost) position at $t = \infty$, in the critical case when the kinetic energy vanishes at that unstable equilibrium position.

## An analytic unifying formula of oscillatory and rotary motion of a simple pendulum

One might readily verify that the energy conservation law (4) is obeyed by the (dual) functions:

$$\mathcal{E}_{\pm} = \mathcal{E}_{\pm}(t) = \mathcal{E}_{\pm}(t,\, \phi) = -\mathcal{R}\left(\frac{t + u_{\mp}(\phi)}{2},\, e^{i\phi}\right) = \mathcal{R}\left(\frac{t + u_{\mp}(\phi)}{2\,i},\, -e^{i\phi}\right), \tag{5}$$

where the function $\mathcal{E}_+$ would turn out to agree with the function $\mathcal{E}$, which we had defined as the function mapping the current time $t$ to the current pendulum position $e^{i\theta}$. The half-period $u_-(\phi)$ of $\mathcal{R}\left(\cdot, e^{i\phi}\right)$, appearing in two successive expressions on the right hand side, ensures the (natural) choice of the initial condition $\mathcal{E}(0,\, \phi) = 1$. That half-period is added (rather than subtracted) so as to ensure that the angle $\theta$ is moving towards $\phi$ (rather than towards $-\phi$) at the initial instant $t = 0$, and (since $\mathcal{R}$ is an even function) reversing the time corresponds to reversing the direction of motion, that is $\mathcal{E}(t,\, \phi) = \mathcal{E}(-t,\, -\phi)$. The (imaginary) half-period $u_-(\phi)$ can be calculated, along with the (real) half-period $u_+(\phi)$, via applying the second formula of the pair (2):

$$u_+(\phi) = u\left(e^{i\phi}\right) = \frac{\pi}{2M\left(\cos\phi_2\right)},\quad u_-(\phi) = i\,u\left(-e^{i\phi}\right) = \frac{i\,\pi}{2M\left(\sin\phi_2\right)}.$$

As we arrive at discussing concrete examples, the significance of the (dual) function $\mathcal{E}_-$ will become apparent, but we might already indicate its relevance in extending the configuration space of the pendulum, as demonstrated in fig. 1.

The half-periods $u_\pm$ (of $\mathcal{R}$) constitute, of course, quarter-periods of $\mathcal{E}$. The time required for a (single) swing of the pendulum, that is the period, in the oscillatory case, is $4u_+$, whereas the time required for a (single) revolution of the pendulum, that is the period, in the rotary case, is $2w_+$.

## Feynman's wobbling plate

“Within a week I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.

I had nothing to do, so I start to figure out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate two to one. It came out of a complicated equation! Then I thought, "Is there some way I can see in a more fundamental way, by looking at the forces or the dynamics, why it's two to one?" I don't remember how I did it, but I ultimately worked out what the motion of the mass particles is, and how all the accelerations balance to make it come out two to one.”

## Spin to wobble ratio recalculation by Chao

Feynman goes on:

"It was effortless. It was easy to play with these things. It was like uncorking a bottle: Everything flowed out effortlessly. I almost tried to resist it! There was no importance to what I was doing, but ultimately there was. The diagrams and the whole business that I got the Nobel Prize for came from that piddling around with the wobbling plate."

The declared (by Feynman) spin to wobble ratio (2:1) was corrected by Benjamin Chao in 1989 (after Feynman's death):[2]

"A torque free plate wobbles twice as fast as it spins when the wobble angle is slight. The ratio of spin to wobble rates is 1:2 not 2:1!"

[2] Chao B. Feynman's Dining Hall Dynamics// Physics Today, **42**(2), 1989: 15. Having investigated the so-called Chandler wobble phenomenon, Chao knew the correct ratio before he came across Feynman's error.

## Maxwell had vainly asked Feynman to be cautious with the motion of a freely rotating body

Perhaps, Feynman's "wobbling plate" story is even more amusing (and more telling) than Feynman thought, as he (inadvertently) reveals that he did not learn (or know) a (correct) solution to his problem by James Clerk Maxwell who had (incidentally!) cautioned that:[3]

"The theory of the rotation of a rigid system is strictly deduced from the elementary laws of motion, but the complexity of the motion of the particles of a body freely rotating renders the subject so intricate, that it has never been thoroughly understood by any but the most expert mathematicians. Many who have mastered the lunar theory have come to erroneous conclusions on this subject; and even Newton has chosen to deduce the disturbance of the earth's axis from his theory of the motion of the nodes of a free orbit, rather than attack the problem of the rotation of a solid body."

[3] Maxwell C.J. On a Dynamical Top, for exhibiting the phenomena of the motion of a system of invariable form about a fixed point, with some suggestions as to the Earth's motion// Transactions of the Royal Society of Edinburgh, **XXI** (IV), Read on April 20$^{\text{th}}$, 1857: 559-570.

$$w^2 - \frac{h^2}{m^2} = \alpha^2 p^2 - \frac{bc}{m^2} = -\beta^2 q^2 - \frac{ca}{m^2} = \gamma^2 r^2 - \frac{ab}{m^2}, \tag{3}$$

$$\alpha := \sqrt{\frac{(A-B)(A-C)}{BC}},\ \beta := \sqrt{\frac{(A-B)(B-C)}{CA}},\ \gamma := \sqrt{\frac{(A-C)(B-C)}{AB}},$$

yielding the inequalities

$$0 \leq \sqrt{h^2 - ab} \leq h \leq \sqrt{h^2 - bc} \leq mw \leq \sqrt{h^2 - ca}. \tag{4}$$

Therefore, the angular speed $w$ is constant (coinciding with $h/m$) if $c$ vanishes. This is the case of "stable" permanent rotation about either the minor axis (if $b > 0$) or the major axis (if $b < 0$). Note that inequalities (4) might be transformed to inequalities, involving the third component $r$ of the angular velocity, which we might assume to be positive, as it never vanishes, unless $b$ does:

$$0 \leq \frac{\sqrt{ab}}{m\gamma} \leq \sqrt{\frac{Bb}{(B-C)C}} \leq r \leq \sqrt{\frac{Aa}{(A-C)C}}.[6]$$

We must emphasize that the angular speed is not obliged to remain constant for vanishing $b$. In this critical case, which we shall fully explore, the bounds on the angular speed $w$ might be rewritten as:

$$\frac{h}{m} \leq w \leq \frac{\delta h}{m},\ \delta := \sqrt{1+\beta^2} = \sqrt{\frac{B(C+A-B)}{CA}}.$$

The preservation of the angular momentum $m$ implies the (so-called) Euler equations of free motion of a rigid body

$$\frac{\partial m}{\partial t} = m \times w,[7] \tag{5}$$

which, combined with differentiating identity (3), imply the identity

$$\frac{dw^2}{dt} = -\frac{2Vpqr}{ABC}.$$

Thereby, the (elliptic) function $y := w^2 - h^2/m^2$ satisfies the differential equation

$$\dot{y}^2 = -4\left(y + \frac{bc}{m^2}\right)\left(y + \frac{ca}{m^2}\right)\left(y + \frac{ab}{m^2}\right). \tag{6}$$

Recall that the angular velocity $\boldsymbol{w}$ in body rotating frame is (doubly) periodic,[27] which (real) quarter period $T_3$, was calculated in (7).

A formula for calculating the rate of precession $\dot{\psi}$, symmetric in the moments of inertia ($A$, $B$ and $C$),

$$\dot{\psi} = \frac{1}{m}\left(h + \frac{abc}{m^2 y}\right),^{28} \tag{13}$$

was first presented at the PCA annual conference (chaired by Nikolay Vassiliev) on April 20, 2016 [6]. Thus, the generalized spin to wobble ratio might formally be defined and explicitly calculated as

$$\frac{\sigma\pi}{2\psi(T_3)}, \; \psi(t) = \int_0^t \dot{\psi}\, dt. \tag{14}$$

---

[22] This ratio is $-1 : 2$ for "Feynman's wobbling plate". It differs in sign from the ratio $1 : 2$, which Chao had (correctly) calculated. These two seemingly contradictory ratios correspond to two distinct interpretations of precession. According to the first, the axis of symmetry is intrinsic to the body, that is, the axis itself moves with the body. Whereas, according to the second (which Feynman adopts), the axis of symmetry is detached from the body, so the rest of the body "spins" around it "independently" of its own movement. The first ratio, according to the first interpretation means that the spinning and the wobbling possess opposite directions. Adding up two opposed magnitudes $-1$ and $2$ yields $1$, which according to the second interpretation is the relative magnitude of the spinning which is codirected with the wobbling (which relative magnitude is still $2$).

[23] The precession angle $\psi$ ought not be confused with the second Euler angle, that is, the nutation angle $\theta$, which Chao referred to as "the wobble angle". The nutation angle $\theta$ is also, quite frequently, referred to as "the pitch angle".

[24] We were made aware of two interpretations of precession since (unconsciously) adopting the second is inevitably followed by the (seemingly natural) additional assumption that the axis of symmetry "does not spin" (since its own spinning would not then influence the spinning of the rest of the body around it), thereby missing the alternative and important interpretation of the case $m = Cr$ as "pure" wobbling. No problems emerge, if the first interpretation is adopted, since the spinning of an axis (with vanishing thickness) might still be defined as long as the axis is "firmly" attached to the rest of the body. So, in fact, the first interpretation is preferable, although we must keep the second in mind (as Chao did), since it is rather commonly (and unconsciously!) assumed.

[25] This is the limiting case, which in terms adopted by Chao, is described as the case when "the rod does not wobble" and "the plate wobbles twice as fast as it spins".

[26] Chao had (wisely) avoided discussing this (spherically symmetric) case, for which $\dot{\phi} = 0$. Perhaps, he avoided (generally) stating that "the ratio of spin to wobble rates" is $B : C$ in order not to overburden the readers with the inevitable conclusion that the "sphere" spins as fast as it wobbles! Chao's "graciousness" (towards Feynman) somehow precluded a (total) clarification of Feynman's (not so insignificant) error, which was not in the least "a mere slip of memory"! Feynman did not finish deriving the "complicated equation", that is, he did not arrive at the said (simple) ratio $B : C$ which would have protected him from the "pretty obvious" recollection that the spinning of the plate went "faster than the wobbling".

[27] Thereby, the (scalar) function $\boldsymbol{w}$ is, as well, (doubly) periodic (in any reference frame).

[28] The search for this symmetric expression was inspired and guided by Galois. In fact, the search for the invariant axes and their construction is altogether due to Galois, who was undoubtedly able to carry out and surpass all that we have done!

The expression for calculating the precession angle $\psi$ is, of course, the expression obtained on the right hand side of (9). Thus, $\psi(T_3)$ is a complete elliptic integral of the third type, which definition along with its (most efficient) calculation was exhaustively discussed in [2].[29] We might explicitly calculate it as

$$\psi(T_3) = \psi\left(A, B, C, h, m^2\right) := \frac{T_3}{m}\,(h+H)\,,^{30} \tag{15}$$

$$\frac{T_3}{m} = \frac{\pi}{2\,M\left(\sqrt{a(b-c)}, \sqrt{b(a-c)}\right)} = \frac{\pi}{2\,\sqrt{a(b-c)}\,M(k_3)} = \frac{\pi}{2\,\sqrt{b(a-c)}\,M(1/k_3)},\ k_3 = \sqrt{\frac{b\,(a-c)}{a\,(b-c)}},$$

$$H := \frac{abc}{T_3 m^2}\int_0^{T_3}\frac{dt}{y} = \int_{-bc/m^2}^{-ca/m^2}\frac{abc\,dy}{T_3 m^2 y\sqrt{-4(y+bc/m^2)(y+ca/m^2)(y+ab/m^2)}} = aH_1 = bH_2 = cH_3,$$

$$H_1 := \frac{2}{\pi}M\left(\frac{1}{k_3}\right)\int_0^1\frac{dx}{\left((b-a)\,x^2/b-1\right)\sqrt{(1-x^2)(1-k_1^2 x^2)}} = N\left(k_3^2, 0, \frac{a-c}{a+c}, \frac{a-c}{a}\right),$$

$$H_2 := \frac{2}{\pi}M\left(k_3\right)\int_0^1\frac{dx}{\left((a-b)\,x^2/a-1\right)\sqrt{(1-x^2)(1-x^2/k_3^2)}} = N\left(\frac{1}{k_3^2}, 0, \frac{b-c}{b+c}, \frac{b-c}{b}\right),$$

$$H_3 := \frac{2}{\pi}M\left(k_3\right)\int_1^{1/k_3}\frac{dx}{\left((a-c)\,x^2/a-1\right)\sqrt{(x^2-1)(1-k_3^2 x^2)}} = N\left(k_3^2, 0, \frac{a-c}{2a}, \frac{a-c}{a}\right) =$$

$$= N\left(\frac{1}{k_3^2}, 0, \frac{b-c}{2b}, \frac{b-c}{b}\right),^{31}$$

where $M(x)$ is the arithmetic-geometric mean of $1$ and $x$, whereas the function $N(x, \zeta, \eta, \xi)$ is defined recursively via the relation

$$N\left(x_n, \zeta_n, \eta_n, \xi_n\right) = N\Big(x_{n+1} := \sigma(x_n, 1),\, \zeta_{n+1} := \sigma(x_n, \zeta_n, \xi_n),\, \eta_{n+1} := \sigma(x_n, \eta_n, \xi_n),\, \xi_{n+1} := \sigma(x_n, \xi_n)\Big),$$

$$\sigma(x, \xi) := \sigma(x, \xi, \xi),\ \sigma(x, \eta, \xi) := \frac{(\sqrt{x}+\eta)(\sqrt{x}+\xi)}{2\,(\eta+\xi)\,\sqrt{x}},$$

and the value of this recursive function is the limit obtained from successively applying linear fractional transformations

$$L(x, \zeta_n, \eta_n, \xi_n) := \frac{(\eta_n-\xi_n)(x-\zeta_n)}{(\eta_n-\zeta_n)(x-\xi_n)},$$

either to (successive) corresponding first arguments $x_n$, thereby generating the sequence $\{L(x_n, \zeta_n, \eta_n, \xi_n)\}$, or to the (constant) value 1,[32] generating the sequence $\{L(1, \zeta_n, \eta_n, \xi_n)\}$. Both sequences converge quadratically to their common point, that is, the generalized arithmetic-geometric mean, as further clarified in [2].

So is the case here, where (unstable) permanent rotation must be supplemented by two critical separating solutions, given by the (orthogonal) matrix $Q$ (for two signs of $\sigma$). Most importantly, neither one of these two dual solutions "continues by inertia" (in a sense told in [33]) to a solution with reverse orientation (on the same circle) but "continues" to one of the two dual solutions on the "other circle". The latter statement is made precise by noting that the vanishing of the "second integrand" in identity (13) does not imply the vanishing of its corresponding improper "integral" $T_3H/m$ of identity (15) for $\psi$. Such improper integral might be directly calculated as

$$I(A,B,C) := I(A,B,C,x \mapsto +\infty),$$

$$I(A,B,C,x) := \int_1^x \frac{\sqrt{(C-B)(B-A)CA}\,dx}{(B(C-A)x^2-(C-B)A)\sqrt{x^2-1}} = \sigma\,\mathrm{Arctan}\left(\sqrt{\frac{A(B-C)}{C(A-B)}}\frac{\sqrt{x^2-1}}{x}\right),$$

$$\sigma = \begin{cases} 1, & \text{if } A < B < C, \\ -1, & \text{if } A > B > C,^{65} \end{cases}$$

which coincides, modulo $\pi/2$, with the integral $I(C,B,A)$.[66]

Without including the said critical solutions (along with the latter calculation), the (fundamental) problem of rigid body free motion is not entirely solved, so in accordance with the principle "Nil actum reputans si quid superesset agendum", emphasized by Gauss in [17, p. 629], it was not at all ever solved! May all and every credit for (finally) solving it be rightfully and entirely attributed to Évariste Galois![67]

## References


[1] Адлай С.Ф. *Винт Джанибекова.* Available at `http://semjonadlaj.com/SScrew.pdf`.


[60] As we now know, this "fundamental" omission is intrinsic(!) to the construction of Poinsot (as the sliding of the ellipsoid was excluded).

[61] The authors seem oblivious to the significance of picking the "right" signs. Certainly, they must have never heard of Gauss four year struggle with the sign of his quadratic sum, described in a letter to Olbers (dated September 3, 1805), nor they ever seen Anna Johnson sign formula which we already mentioned in footnote 11. Of course (with such sloppiness), the authors never departed from body frame throughout their article. And, like many others, once they determined an infinite period of the separating solution they lost every interest in it!

[62] An enlightening letter, given in [18], from Sir William Rowan Hamilton to the Reverend Charles Graves is recommended here.

[63] A first presentation of these dual solutions was delivered by the author of this paper on October **26th**, 2016 at "the Egorov seminar on the mechanics of space flight" (conducted at the Moscow State University by Victor Sazonov).

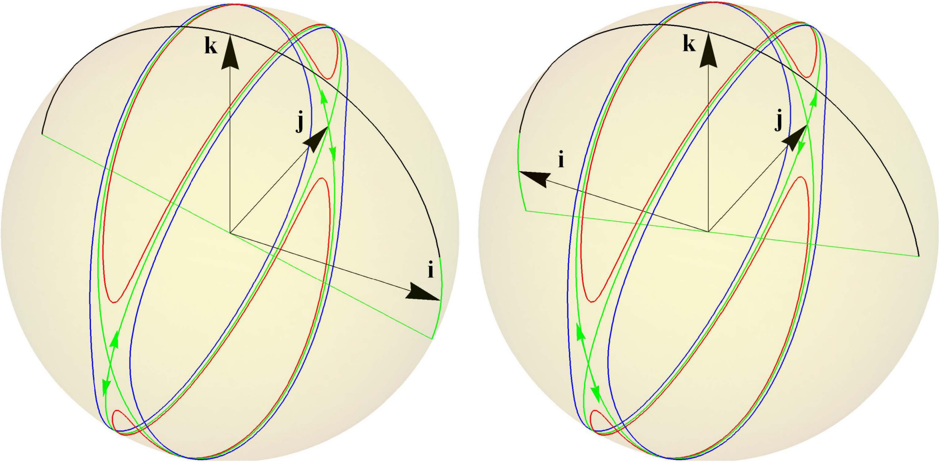
[64] Contrary to common belief (refuted by Dmitry Abramov), the pendulum at the unstable equilibrium does not require any push (however small) in order to yield a separating solution. In other words, no unique single valued function represents a solution to the unstable equilibrium of a simple pendulum. In particular, the (full) solution cannot be limited to a constant function, representing a "standing" pendulum. Two additional solutions correspond to (full) rotations (in infinite time) in either (clockwise or counterclockwise) direction [7].

[65] Aside from the third section (dedicated to the axially symmetric case), we have not violated the lexicographical ordering of the moments of inertia but relabeled them. This is our last violation of our rule.

[66] In particular,

$$I(2,7,8) = \mathrm{Arcsin}\left(\frac{1}{\sqrt{21}}\right) \approx 0.21998797739545 9446,\quad I(8,7,2) = -\mathrm{Arccos}\left(\frac{1}{\sqrt{21}}\right) \approx -1.35080834939943717,$$

$$I(2,7,8) - I(8,7,2) = \mathrm{Arcsin}\left(\frac{1}{\sqrt{21}}\right) + \mathrm{Arccos}\left(\frac{1}{\sqrt{21}}\right) = \frac{\pi}{2}.$$


k
j
i
k
j
i

# Роскосмос назвал причину аварии после пуска с космодрома "Восточный"

Запуск ракеты-носителя "Союз-2.1б" на космодроме Восточный

Москва. 12 декабря. INTERFAX.RU - Авария, произошедшая после запуска с космодрома "Восточный" 28 ноября, стала следствием ошибки в алгоритме разгонного блока "Фрегат", которую невозможно выявить существующими методиками, заявил журналистам глава Роскосмоса Игорь Комаров.

По его словам, "специалисты пришли к выводу, что к нештатной ситуации привело непрогнозирующееся поведение разгонного блока после его отделени от ракеты-носителя".

В свою очередь, первый заместитель гендиректора Роскосмоса Александр Иванов сообщил журналистам, что непосредственной причиной аварии стало несовершенство программного обеспечения разгонного блока, которое не удалось выявить существующими методиками.

"Все члены аварийной комиссии согласились с выводами. Причина установлена однозначно. Причиной аварийной ситуации явилось несовершенство алгоритмов программного обеспечения системы управления разгонного блока "Фрегат", которое проявилось при запуске с космодрома "Восточный". Алгоритм работы системы управления привел к некорректному определению ориентации разгонного блока после отделения от ракеты-носителя при выставленных начальных азимутах ракеты-носителя и разгонного блока на стартовом комплексе космодрома", - сказал Иванов.

При этом "существующие математические методы моделирования выведения космических аппаратов не могли выявить подобную ошибку".

Между тем, претензий к техническому состоянию ракеты-носителя, разгонного блока, а также космодрома не выявлено. В результате работы комиссии было установлено, что все работы, проведенные на стартовом комплексе, связанные с подготовкой к пуску ракеты-носителя не могли привести к подобному результату.

По словам Иванова, эта проблема характерна лишь для запуска ракет-носителей "Союз" с разгонными блоками "Фрегат" с космодрома "Восточный" и устраняется программными методами.

"Это не распространяется ни на "Куру", ни на "Плесецк", ни на "Байконур", поскольку азимутов, которые были бы повернуты на 180 градусов, не существует и не может существовать. Это распространяется на разгонный блок "Фрегат" на "Восточном" и только на солнечно-синхронную орбиту. Это стечение орбиты, азимута старта, районов падения и погодных условий. "Лечится" это программными методами. Это не металл, не двигатели, не перепайка, это разработка программ, отладка и новыми методами проверка", - сказал он.

Он сообщил, что проверки носят трудоемкий характер. В свою очередь, нештатные ситуации такого рода неоднократно раннее случались при запусках в разных странах.

Ракета-носитель "Союз-2.1б" стартовала 28 ноября с космодрома "Восточный". Она должна была вывести на низкую околоземную орбиту спутник дистанционного зондирования Земли (ДЗЗ) "Метеор-М" №2-1 и еще 18 малых космических аппаратов попутной нагрузки. По данным источников в ракетно-космической отрасли, спутник "Метеор-М" должен был отделиться и начать передавать телеметрию, однако телеметрия так и не была получена, поскольку "возникла нештатная ситуация на этапе полета разгонного блока "Фрегат". В итоге, по информации источников, все запущенные спутники упали в Атлантический океан.

Gmail

Semjon Adlaj <semjonadlaj@gmail.com>

# мнение: ошибки классики фатальны - доколе?

Писем: 9

**Abrarov Dmitry** <abrarov@yandex.ru> 12 декабря 2017 г., 08:23

Кому: Semjon Adlaj <semjonadlaj@gmail.com>

Копия: stepsj <stepsj@yandex.ru>, "Vsevolod S. Sergeev" <vssergeev@yandex.ru>, Иван Фёдорович Кожевников <kozhevnikovif@gmail.com>, "Ivan I. Kosenko" <cosenco@mail.ru>, Александру Бурову <jtm@yandex.ru>, АС Сумбатов <necessita@yandex.ru>

Семен, (и коллеги) приведу свое мнение по связи нашей науки с реальной жизнью.

Неверность формул классической механики, в частности, из-за неверной, неполной, и вообще отсутствия склейки знаков "классических эллиптических квадратур" - не просто фундаментальная теоретическая ошибка.

В частности, классические формулы четко пропускают эффекты неориентируемости динамики моделируемых объектов.

Будучи реализованными в ПО "ориентируемые классические формулы" становятся "технологической катастрофической ошибкой"- см. приложение о свежей катастрофе в ближнем космосе при выводе на орбиту спутникового мультиплекса.

Я уверен, что - фундаментальная причина - именно - в "классических расчетных формулах", заложенных в ПО системы управления многоступенчатой ракетой.

Приведу контрольный всем известный пример того как "классика соответствует реальности".

Так вот, ровно такой же эффект - четко и системно пропускается при описании динамики гайки (волчка) Джанибекова:

его динамика описывается "классиками" как динамика на ориентируемой поверхности (торе Арнольда), хотя он четко движется по заведомо неориентируемой поверхности -

его угловая скорость на "полупериоде" его периодической фазовой траектории четко меняет знак. Это просто визуализировано экспериментально.

И это соответствует сонаправленности вектора угловой скорости гайки и вектора его кинетического момента.

Следовательно, фазовая траектория гайки Джанибекова - неориентируема. (В итоге - это диагональный цикл на 3d-аналоге бутылки Клейна).

В частности, она никак не может лежать на торе - ориентируемой поверхности.

Не говоря о том, что нет прямого отношения динамики гайки Джанибекова к волчку Эйлера - как пишут в "признанных работах группы Журавлева" -(гайка движется в поле гравитации системы "Земля-Луна", а не в классическом плоско-параллельном поле гравитации и нет у нее "классической неподвижной точки" - центр масс гайки подвижен в системе "Земля-Луна", производящей силовое поле и его моменты, учитываемые при определении динамики гайки.)

Написал это - просто потому, что кажется, что человечество неумолимо смотрит на мир сквозь кривое зеркало формального времени t (не обратимого), даже не смотря на то, что видит явно происходящее. А ученые-механики все учат и учат - студентов, инженеров, ...

С уважением,
ДА,

# A proposal for collaboration

Уважаемые коллеги!

Мы, преподаватели кафедры теоретической механики Уральского Федерального университета имени первого Президента России Б.Н. Ельцина, приглашаем вас вступить в коллаборацию по подготовке научной публикации «Периодические движения свободного твёрдого тела в отсутствии внешних сил. Численный анализ и точное решение».

Цель работы состоит в популяризации современных математических методов в приложении к задачам динамики твёрдого тела, предложенных российским учёным Семёном Франковичем Адлай. В частности, получению точных формул для определения периодов поворота вектора углового момента по отношению к главным осям (http://semjonadlaj.com/TFRBM.pdf) с использованием процедуры вычисления арифметико-геометрического среднего значения. В работе на примере сравнения приближённых и точных решений планируется продемонстрировать принципиальную ограниченность первых и высокую эффективность вторых при чрезвычайно высоких порядках малости начальных возмущений.

К письму прилагается набросок статьи, которую все участники коллаборации могут исправлять и дополнять по их усмотрению. После окончательного редактирования и согласования с соавторами статья будет послана для публикации в журнал «Компьютерные инструменты в образовании».

Приглашение к коллаборации можете распространить среди своих заинтересованных коллег.

Полупериоды в зависимости от начальных условий

$A=8,\quad B=7,\quad C=2$, q0=1, $r_0=0$

| p0 | $T/2$ | Вычисление T/2 точной формулой |
|---|---|---|
| $10^{-2}$ | 20,9 | 20.870107428337133790003 |
| $10^{-3}$ | 29,1 | 29.108670400337497927777 |
| $10^{-4}$ | 37,3 | 37.346657905086988922244 |
| $10^{-5}$ | 45,6 | 45.584636886485782560092 |
| $10^{-6}$ | 53,8 | 53.822615754968570727770 |
| $10^{-7}$ | 62,1 | 62.060594622045374496667 |
| $10^{-8}$ | 68,1 | 70.298573489105350178332 |
| $10^{-9}$ | 67,8 | 78.536552356165129896666 |
| $10^{-10}$ | 72,3 | 86.774531223224907378354 |
| $10^{-11}$ | 66,4 | 95.012510090284684835299 |
| $10^{-12}$ | 68,6 | 103.25048895734446229118 |
| $10^{-13}$ | 70,9 | 111.48846782440423974822 |
| $10^{-14}$ | 67,5 | 119.72644669146401720477 |
| $10^{-15}$ | 67,4 | 127.96442555852379466122 |
| $10^{-16}$ | 67,8 | 136.20240442558357211766 |
| $10^{-17}$ | 69,5 | 144.44038329264334957411 |
| $10^{-18}$ | 66,8 | 152.67836215970312703066 |
| $10^{-19}$ | 67,5 | 160.91634102676290448700 |
| $10^{-20}$ | 70,19 в начале | 169.15431989382268194355 |
| $10^{-20}$ | 67,6 через 600с | За ничтожную долю секунды! |