Applying Sophus Lie algebraic approach to investigating elliptic functions

Semjon Adlaj

CC RAS, Moscow, Russia

September 19, 2012

Major Swindon: What will history say, sir?

General Burgoyne: History, sir, will tell lies, as usual!

From "The Devil's Disciple" by George Bernard Shaw.

An essential elliptic function and its associated curve

Introduce, for a parameter $\alpha > 1$,

- a polynomail $p_\alpha(x) := x^2 + 2\alpha x + 1$
- an elliptic function $\mathcal{R}_\alpha$, with a (double) pole at zero, satisfying the differential equation

$$x'^2 = 4\,x\,p_\alpha(x)$$

- Λ_α: the lattice of $\mathcal{R}_\alpha$
- a complex projective elliptic curve (associated with $\mathcal{R}_\alpha$)

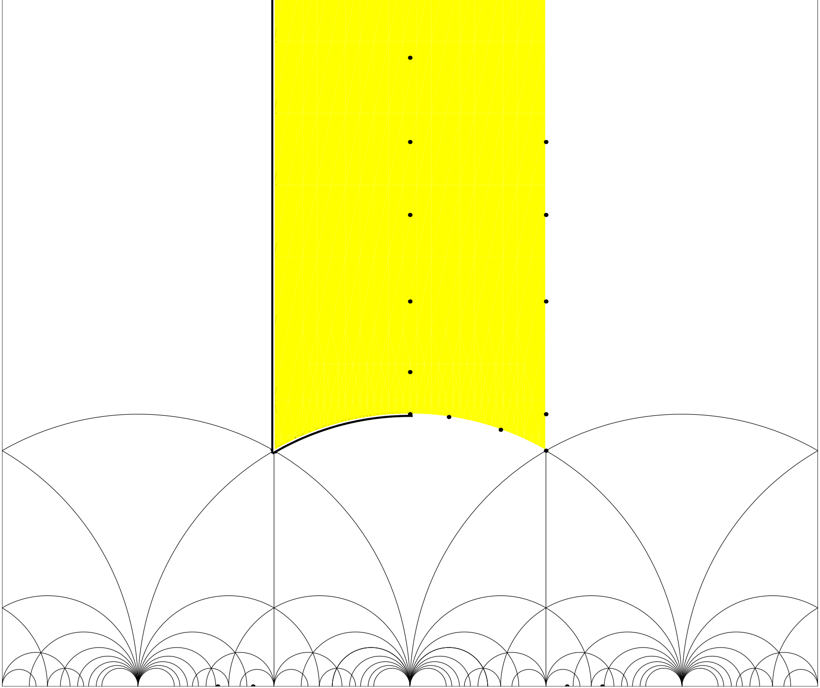
$$E_\alpha : y^2 = 4\,x\,p_\alpha(x)$$

Two correspondences

The map

$$\mathbb{C}/\Lambda_\alpha \to E_\alpha$$
$$\mathrm{z} \mapsto (\mathcal{R}_\alpha(\mathrm{z}), \mathcal{R}'_\alpha(\mathrm{z})),$$

which turns out being an isomorphism of Riemann surfaces, as well as, an isomorphism of groups, enables an identification (exploiting the j-invariant) of isomorphism classes of projective complex elliptic curves with the homothety classes of lattices $\mathcal{L}/\mathbb{C}^\times$, which might, in turn, be identified with the fundamental domain for the action of the modular group upon the (extended) upper half plane $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$.



In a previous paper [1], a justification for defining an *essential elliptic function* was made. Yet, enabling an inversion of the modular invariant is, perhaps, even more convincing. We shall not elaborate upon describing previous attempts for inverting the modular invariant aside from mentioning two typical references [2, 3]. The first reference provides a glimpse upon Ramanujan latest efforts, whereas the appendix of the second concludes with a well-known expression for a point τ in the fundamental domain as a ratio of hypergeometric functions, thereby linking τ with an intermediate variable λ. Formula (3.3), in the same paper, yields the modular invariant j as a (well-known) fractional transformation of λ, of degree 6. We point out this transformation so as to suggest that verifying a formula for an inverse of the modular invariant is as straightforward as verifying a root of a given hexic.

An inversion of the modular invariant is afforded via successively composing the functions

$$k_0(x) = \frac{iG\left(\sqrt{1-x^2}\,\right)}{G(x)},\ k_1(x) = \frac{\sqrt{x+4}-\sqrt{x}}{2},\ k_2(x) = \frac{3}{2}\left(\frac{x}{k_3(x)} + k_3(x)\right) - 1,$$

where

$$k_3(x) = \sqrt[3]{\sqrt{x^2-x^3}-x}$$

and $G(x)$ is the arithmetic-geometric mean of 1 and x. In other words, the function

$$k = k_0 \circ k_1 \circ k_2$$

is an inverse of the modular invariant, which (we need not point out) is not single-valued.

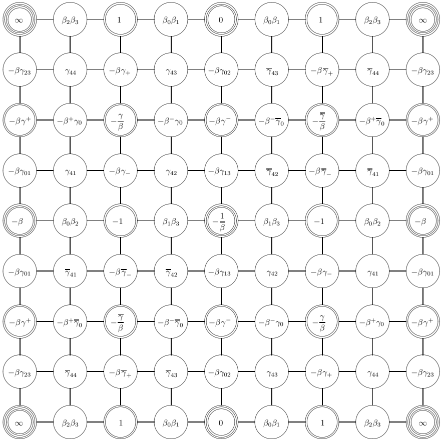
23.7 Quarter Periods

23.7.1
$$\wp\left(\tfrac{1}{2}\omega_1\right) = e_1 + \sqrt{(e_1 - e_3)(e_1 - e_2)} = e_1 + \omega_1^{-2}(K(k))^2 k',$$

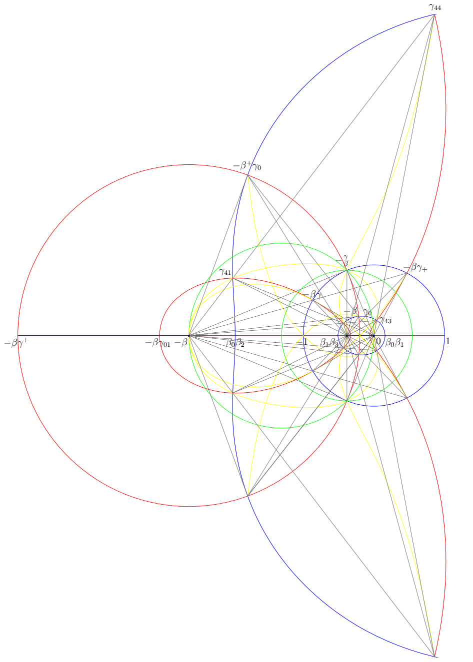
23.7.2
$$\wp\left(\tfrac{1}{2}\omega_2\right) = e_2 - i\sqrt{(e_1 - e_2)(e_2 - e_3)} = e_2 - i\omega_1^{-2}(K(k))^2 k k',$$

23.7.3
$$\wp\left(\tfrac{1}{2}\omega_3\right) = e_3 - \sqrt{(e_1 - e_3)(e_2 - e_3)} = e_3 - \omega_1^{-2}(K(k))^2 k,$$

where k, k' and the square roots are real and positive when the lattice is rectangular; otherwise they are determined by continuity from the rectangular case.



The values of the essential elliptic function $\mathcal{R}$, satisfying the differential equation
$\mathcal{R}'^2 = 4\mathcal{R}\,(\mathcal{R}+\beta)\,(\mathcal{R}+1/\beta)$, for $\beta > 1$, whose lattice is Λ,
at the nodes of the lattice $\Lambda/8$.



Final Answers

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Perimeter of an Ellipse

(See abridged version at original location.)

Related topics on this website include:

- Hypergeometric functions.
- Arithmetic-geometric mean.
- Surface of an ellipse.
- Volume of an ellipsoid.
- Ellipses and Hyperbolae.
- Elliptic arc: Length of the arc of an ellipse between two points.
- Perimeter of an ellipse. Exact formulas and simple ones.
  - Circumference of an ellipse: Unabridged discussion.
- Surface area of an ellipsoid of revolution (oblate or prolate).
  - Surface area of a general ellipsoid.
- Volume of a hypersphere in any number of dimensions. Hyper-surface area too!

Related Links (Outside this Site)

Ellipse by Dr. James B. Calvert, University of Denver (Colorado).
Circumference of an Ellipse by Robert L. Ward in "MathForum@Drexel".

Perimeter of an Ellipse by *Stanislav Sýkora* (2005-05-30).
On the Perimeter of an Ellipse (pdf) by *Paul Abbott* (Avignon, June 2006).
How Euler Did It by *Ed Sandifer* (Western Connecticut State)

Related posts :

2009-02-08 : Arithmetic-Geometric Mean & Elliptic Integrals by Michael Press.

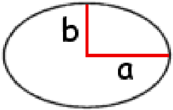
A few articles posted by

David W. Cantrell :

2001-05-08 : New Approximation for [the] Perimeter of an Ellipse
2004-05-23 : Two New Approximations, in a Certain Form, for the Perimeter of an Ellipse
2004-05-24 : Modifying Ramanujan's Second Approximation for the Perimeter of an Ellipse
2006-01-12 : Arithmetic Approximations of the Perimeter of an Ellipse

Circumference of an Ellipse

(Jaleigh. B. of Minonk, IL. 2000-11-26 twice)
What is the formula for the perimeter of an ellipse? [oval]
(S. H. of United Kingdom. 2001-01-25)
What is the formula for the circumference of an ellipse?



There is no simple exact formula: There are simple formulas but they are not exact, and there are exact formulas but they are not simple. Here, we'll discuss many approximations, and 3 or 4 exact expressions (infinite sums).

The *complementary* convergence properties of two such sums allow an *efficient* computation, at any prescribed precision, of the perimeter of *any* ellipse, by using one series for eccentricities below 0.96 [say] and the other one for higher eccentricities.

For an ellipse of cartesian equation $x^2/a^2 + y^2/b^2 = 1$ with $a > b$:

The arithmetic-geometric mean and a modification thereof

Introduce a sequence of pairs $\{x_n,\ y_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := \sqrt{x_n y_n}.$$

Define *the arithmetic-geometric mean* (AGM) of two positive numbers x and y as the (common) limit of the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$.

Introduce, next, a sequence of triples $\{x_n,\ y_n,\ z_n\}_{n=0}^{\infty}$:

$$x_{n+1} := \frac{x_n + y_n}{2},\ y_{n+1} := z_n + \sqrt{(x_n - z_n)(y_n - z_n)},$$

$$z_{n+1} := z_n - \sqrt{(x_n - z_n)(y_n - z_n)}.$$

Define *the modified arithmetic-geometric mean* (MAGM) of two positive numbers x and y as the (common) limit of the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ with $x_0 = x$, $y_0 = y$ and $z_0 = 0$.

Calculating complete elliptic integrals of the first kind

Assume, unless indicated otherwise, that β and γ are two positive numbers whose squares sum to one: $\beta^2 + \gamma^2 = 1$.

Gauss had discovered a highly efficient (unsurpassable) method for calculating complete elliptic integrals of the first kind:

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\gamma^2 x^2)}} = \frac{\pi}{2\,M(\beta)}, \tag{1}$$

where $M(x)$ is the arithmetic-geometric mean of 1 and x. In particular, equality (1) holds if (in violation of the assumption, otherwise imposed) $\gamma^2 = -1$:

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\pi}{2\,M(\sqrt{2})} \approx 1.31102877714605990523.$$

The integral on the left hand side of the latter equation is referred to as the lemniscate integral and is interpreted as the quarter length of the lemniscate of Bernoulli whose focal distance is $\sqrt{2}$.

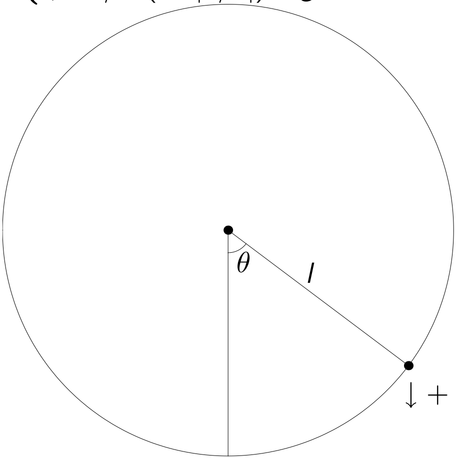
Real and imaginary periods of a simple pendulum

$$T = 2\pi k\sqrt{\frac{l}{g}}, \text{ where } k = k(\theta) = \begin{cases} 1/M(\cos(\theta/2)) : & g > 0 \\ \\ \sqrt{-1}/M(\sin|\theta/2|) : & g < 0 \end{cases}$$

l is the length of the pendulum,

g is the acceleration due to gravity (positive or negative),

θ is the angle of the maximal inclination from the vertical (whose positive direction might be chosen to be pointing downwards as shown).



Calculating the perimeter of an ellipse

A recent survey[2] of formulae (approximate and exact) for calculating the perimeter of an ellipse is erroneously resuméd:

"There is no simple exact formula: There are simple formulas but they are not exact, and there are exact formulas but they are not simple."

The formula for calculating complete elliptic integrals of the second kind (which refutes the preceding statement) be now known:

$$\int_0^1 \sqrt{\frac{1-\gamma^2 x^2}{1-x^2}}\, dx = \frac{\pi N(\beta^2)}{2\, M(\beta)}, \tag{2}$$

where $N(x)$ is the modified arithmetic-geometric mean of 1 and x. The integral on the left hand side of equation (2) is interpreted as the quarter length of an ellipse with a semi-major axis of unit length and a semi-minor axis of length β (and eccentricity γ).

[2]Michon G. P. Final Answers: Perimeter of an Ellipse //www.numericana.com/answer/ellipse.htm (updated May 17, 2011)

Computing π via power series

Ramanujan's formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}},$$

producing 8 digits of π per term, was used by Bill Gosper in 1985 to set a record of 17 million digits.

The Chudnovsky formula (developed in 1987)

$$\frac{426880\sqrt{10005}}{\pi} = \sum_{k=0}^{\infty} \frac{(6k)!(13591409 + 545140134k)}{(3k)!(k!)^3(-640320)^{3k}}$$

producing 14 digits of π per term, was used for calculating over one billion digits in 1989 by the Chudnovsky brothers, 2.7 trillion digits by Fabrice Bellard in 2009, and 10 trillion digits in 2011 by Alexander Yee and Shigeru Kondo.

Computing π via iterative methods

Legendre relation (relating complementary complete elliptic integrals of the first and the second kind to each other) might now be rewritten yielding a parametric (uncountably infinite) family of identities for π:

$$\pi = \frac{2M(\beta)M(\gamma)}{N(\beta^2) + N(\gamma^2) - 1}, \tag{3}$$

and, in particular, yielding a countably infinite family of identities (where the ratio of $M(\gamma)$ to $M(\beta)$ is an integer power of $\sqrt{2}$) from which, setting $c := \sqrt{2} - 1$, we list a few:

$$\pi = \frac{M(\sqrt{2})^2}{N(2) - 1} =$$

$$= \frac{M\left(2\sqrt{\sqrt{2}}\,c\right)^2/2}{N\left(4\sqrt{2}\,c^2\right) - 2c} = \frac{M\left(\sqrt{2c}\right)^2}{\sqrt{2}N(2c) - 1} = \frac{2M(c)^2}{\sqrt{2}N\left(c^2\right) - c} = \frac{2M\left(c^2\right)^2}{N\left(c^4\right) - c^2},$$

where the first of the latter chain of identities for π might be inferred from a special case (where $\beta = \gamma$), of Legendre relation, discovered by Euler. Iteratively calculating for the sequences $\{x_n\}$ and $\{y_n\}$ (converging to the AGM of 1 and $\sqrt{2}$), one arrives at the (so-called) Brent-Salamin algorithm for computing π[3]. Setting

$$\pi_n := \frac{\left(\sqrt{2} + 1 - \sum_{m=1}^{n-1} x_m - y_m\right)^2}{2\sqrt{2} - 1 - \sum_{m=1}^{n-1} 2^m (x_m - y_m)^2}, \ n \in \mathbb{N},$$

we enlist, for $n \leq 4$, approximations for the ratios π_n (descendingly and quadratically converging to π):

$$\pi_1 \approx 3.18, \ \pi_2 \approx 3.1416, \ \pi_3 \approx 3.1415926538,$$

$$\pi_4 \approx 3.14159265358979323846 6.$$

[3]Evidently, "Gauss-Euler algorithm" would be a naming less exotic, yet restoring the credit to whom it rightfully belongs.

интерполяцию полиномов и так называемое «быстрое преобразование Фурье». Объём вычислений по этому алгоритму двух целых n-разрядных чисел по сравнению с методом умножения «в столбик» уменьшается в $\frac{n}{\log_2 n \cdot \log_2 \log_2 n}$ раз. Например, поиск произведения двух 2^{16}-разрядных сомножителей ускоряется более чем в тысячу (2^{10}) раз по сравнению с обычным способом умножения. Довольно существенная экономия для электронных вычислителей точных знаков числа π!

«Сверхэффективный» алгоритм Джонатана и Питера Борвейнов

Канадские математики Джонатан и Питер Борвейны в 1987 году нашли удивительный ряд:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \left\{ \frac{(-1)^n (6n)!}{(n!)^3 (3n)! \, (5280(236674 + 30303\sqrt{61}))^{3n+\frac{3}{2}}} \times \right.$$
$$\times (212\,175\,710\,912\sqrt{61} + 1\,657\,145\,277\,365 +$$
$$\left. + n(13\,773\,980\,892\,672\sqrt{61} + 107\,578\,229\,802\,750)) \right\},$$

где $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, а $0! = 1$.

Последовательность стоящих под знаком суммы слагаемых при $n = 0, 1, 2, \ldots$ добавляет около 25 точных цифр числа π с каждым новым членом. Первый член (соответствующий $n = 0$) даёт число, совпадающее с π в 24 десятичных знаках [6].

Джонатан и Питер Борвейны предложили также алгоритм расчёта десятичных знаков числа π, имеющий фантастическую эффективность: каждый новый шаг выполнения этого алгоритма уточняет количество верных цифр в разложении числа π более чем вчетверо! [6, 7]. Вот этот удивительный алгоритм.

Вначале положим $y_0 = \sqrt{2} - 1$, $a_0 = 6 - 4\sqrt{2}$, а затем каждое новое значение y_{n+1} будем находить, отправляясь от предыдущего значения по формуле

$$y_{n+1} = \frac{1 - \sqrt[4]{1 - y_n^4}}{1 + \sqrt[4]{1 + y_n^4}}, \quad n = 0, 1, 2, \ldots$$

Похожим образом будем находить члены последовательности a_0, a_1,

a_2, ..., вычисляя их по формуле

$$a_{n+1} = (1 + y_{n+1})^4 a_n - 2^{2n+3} y_{n+1} (1 + y_{n+1} + y_{n+1}^2), \quad n = 0, 1, 2, \ldots$$

Оказывается, по мере увеличения номера шага n величина $\frac{1}{a_n}$ очень быстро приближается к π, а именно, имеет место оценка

$$0 < a_n - \frac{1}{\pi} < 2^{2n+5} \cdot e^{-2^{2n+1}\pi}.$$

Так, уже a_4 даёт 694 верных знаков числа $\frac{1}{\pi}$.

У истоков открытия этого алгоритма лежали исследования в области так называемых *эллиптических интегралов* и *тета-функций* — высших разделов современной математики [7]. Авторы этого поразительного алгоритма также утверждают, что им помогли некоторые идеи гениального индийского математика Сринивазы Рамануджана (1887—1920).

Продолжение «марафона»

Удивительный «марафон», начатый с вычисления Архимедом трёх точных знаков числа π, сегодня так же далёк от завершения, как и две тысячи лет назад.

По алгоритму Джонатана и Питера Борвейнов в январе 1986 года Дэвид Х. Бейли получил 29 360 000 десятичных знаков π на суперкомпьютере Cray-2, а в 1987 году Я. Канада и его сотрудники — 134 217 000 знаков на суперкомпьютере NEC SX-2. Результат Дэвида и Грегори Чудновски из Колумбийского университета в Нью-Йорке, вычисливших в 1989 году 1 011 196 691 знак числа π, попал даже в книгу рекордов Гиннесса. Для своих расчётов они использовали суперкомпьютер Cray-2 и сеть компьютеров IBM-3090. К октябрю 1995 года сотрудниками Токийского университета Ясумасой Канадой и Дайсуке Такахаши было вычислено свыше 6 миллиардов цифр. Они же в 1999 году на компьютере HITACHI SR 8000 вычислили 206 158 430 000 цифр числа π [8].

В конце прошлого столетия посетители сайта [9] встречали объявление, приглашающее их принять участие в глобальном проекте «Pi-Hex». Любой житель Земли, подключив свой компьютер к сети Интернет, мог стать участником коллективных вычислений отдельных цифр двоичной записи числа π. Координатором этого глобального проекта выступил студент университета Симона Фрезера (США)

Pi: Difference between revisions

From Wikipedia, the free encyclopedia

Revision as of 13:19, 4 April 2012 (view source)
Noleander (talk | contribs)
m (→Monte Carlo methods: punctuation)
← Previous edit

Line 131:

:$\scriptstyle \pi \approx \frac{(a_n + b_n)^2}{4 t_n}.\!$

}}

− The iterative algorithms **used following 1980** were independently published in 1976 by [[Eugene Salamin]] and [[Richard Brent]].<ref>Arndt, p 87.</ref> These algorithms were unique because they utilized an iterative approach rather than an infinite series. However, Salamin and Brent were not the first to discover the approach: it was actually invented over 160 years earlier by [[Carl Friedrich Gauss]], in what is now termed the [[AGM method|**Arithmetic**-geometric mean method]] (AGM method) or [[Gauss–Legendre algorithm]]. <ref>Arndt, p 87.</ref> The algorithm, as modified by Salamin and Brent, is also referred to as the "Brent-Salamin algorithm".

− Whereas series typically increase the accuracy with a fixed amount for each added term, **there exist** iterative algorithms **that** "multiply" the number of correct digits at each step, with the downside that each step generally requires an expensive calculation. **A breakthrough was made in 1975, when [[Richard** Brent **(scientist)|Richard Brent]] and [[Eugene Salamin (mathematician)|Eugene Salamin]] independently discovered the Brent–Salamin** algorithm**, which uses only arithmetic to double** the number of **correct** digits **at** each step.<ref **name="brent"**>{{Cite **news**|

Revision as of 13:32, 4 April 2012 (view source)
Noleander (talk | contribs)
(→Computer era and the AGM algorithm: clarify wording; fix date)
Next edit →

Line 131:

:$\scriptstyle \pi \approx \frac{(a_n + b_n)^2}{4 t_n}.\!$

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\+ **The iterative algorithms have been widely used by {{pi}} hunters following 1980 because they tend to be faster than infinite series algorithms:** Whereas series typically increase the accuracy with a fixed amount for each added term, **some** iterative algorithms "multiply" the number of correct digits at each step, with the downside that each step generally requires an expensive calculation. **For example, the** Brent-**Salamin algorithm doubles the number of digits in each iteration. An iterative** algorithm **that quadruples** the number of digits **in** each step **was discovered by [[Jonathan Borwein|Jonathan]] and [[Peter Borwein]].**

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of the American Mathematical Society

September 2012

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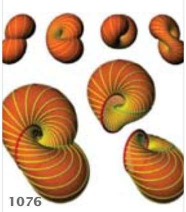

The September issue features an article examining the role of Francesco Severi in the Italian mathematics of the Mussolini era. The photographs that accompany that article are noteworthy. There is also a remembrance of the remarkable mathematician Benoît Mandelbrot. We have an article exhibiting new (and old) ways of viewing the Klein bottle. And an article examining elliptic functions in an unusual light. As usual, the issue is rounded out by the Doceamus and Scripta Manent columns.

—Steven G. Krantz, Editor

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of the American Mathematical Society

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$$\frac{\partial \{e_1, e_2, e_3\}}{\partial g_3} = \frac{1}{2\left(g_2^3 - 27\, g_3^2\right)} \left(12\, g_2\, \{e_1, e_2, e_3\}^2 - 18\, g_3\, \{e_1, e_2, e_3\} - 2\, g_2^2 + (6\, g_2\, \{\eta_1, \eta_2, \eta_3\} - g_3\, \{\omega_1, \omega_2, \omega_3\}) \left\{e_1', e_2', e_3'\right\}\right)$$

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_2} = \frac{1}{8\left(g_2^3 - 27\, g_3^2\right)} \left(18\, g_3 \left\{e_1', e_2', e_3'\right\} - g_2\, (3\, g_3 + 2\, g_2\, \{e_1, e_2, e_3\})\, \{\omega_1, \omega_2, \omega_3\} + 2 \left(g_2^2 + 18\, g_3\, \{e_1, e_2, e_3\}\right) \{\eta_1, \eta_2, \eta_3\}\right)$$

$$\frac{\partial \{\eta_1, \eta_2, \eta_3\}}{\partial g_3} = \frac{1}{4\left(g_2^3 - 27\, g_3^2\right)} \left(\left(g_2^2 + 18\, g_3\, \{e_1, e_2, e_3\}\right) \{\omega_1, \omega_2, \omega_3\} - 6\, g_2 \left\{e_1', e_2', e_3'\right\} - 6\, (3\, g_3 + 2\, g_2\, \{e_1, e_2, e_3\})\, \{\eta_1, \eta_2, \eta_3\}\right),$$

where $\left\{e_1', e_2', e_3'\right\} = \{\wp'(\omega_1;\, g_2, g_3),\, \wp'(\omega_2;\, g_2, g_3),\, \wp'(\omega_3;\, g_2, g_3)\}$ are the values of the derivative of the Weierstrass elliptic $\wp$ function $\frac{\partial \wp(z;\, g_2, g_3)}{\partial z} = \wp'(z;\, g_2, g_3)$ at half-period points $z = \omega_j \,/;\, j = 1, 2, 3$.

Введём обозначения

$$h_{23} = \frac{h_2}{3},\ h_{32} = \frac{h_3}{2},\ h_{23}^* = \frac{h_2^*}{3},\ h_{32}^* = \frac{h_3^*}{2}$$

и установим соотношения:

$$h_{23}^3 - h_{32}^2 = \frac{d^2}{108},\ 3h_{23}'h_{23} - 2h_{32}'h_{32} = \frac{\alpha}{6},\ 3h_{23}'h_{32} - 2h_{32}'h_{23} = \frac{1}{3},$$

$$h_{23}^{*3} - h_{32}^{*2} = \frac{4d^4}{27},\ 3h_{23}^{*\prime}h_{23}^{*2} - 2h_{32}^{*\prime}h_{32}^* = \frac{16\alpha d^2}{3},\ 3h_{23}^{*\prime}h_{32}^* - 2h_{32}^{*\prime}h_{23}^* = \frac{8d^2}{3}.$$

Пользуясь тождеством диффереациальных операторов

$$\partial/\partial\alpha = h_2'\ \partial/\partial h_2 + h_3'\ \partial/\partial h_3 = h_2^{*\prime}\ \partial/\partial h_2^* + h_3^{*\prime}\ \partial/\partial h_3^*$$

и известными формулами для частных производных от ζ, $\wp$ и $u_\pm$ по h_2 и h_3 [65][3]

$$\begin{pmatrix}\partial\zeta(t)/\partial h_2\\ \partial\zeta(t)/\partial h_3\end{pmatrix} = \frac{9}{d^2}\begin{pmatrix}\left(h_{23}^2 + h_{32}\wp(t)\right)\zeta(t) - \left(h_{32} + h_{23}\wp(t)/2\right)h_{23}t + h_{32}\wp'(t)/2\\ \left(h_{23}^2 + h_{32}\right)t - \left(h_{32} + h_{23}\wp(t)\right)\zeta(t) - h_{23}\wp'(t)/2\end{pmatrix},$$

$$\begin{pmatrix}\partial\wp(t)/\partial h_2\\ \partial\wp(t)/\partial h_3\end{pmatrix} = \frac{9}{d^2}\begin{pmatrix}2\left(h_{23}^2 - h_{32}\wp(t)\right)\wp(t) + 4h_{23}h_{32} + \left(h_{23}^2t - h_{32}\zeta(t)\right)\wp'(t)\\ 2\left(h_{23}\wp(t) - h_{32}\right)\wp(t) - 4h_{23}^2 - \left(h_{32}t - h_{23}\zeta(t)\right)\wp'(t)\end{pmatrix},$$

$$\begin{pmatrix}\partial u_\pm/\partial h_2\\ \partial u_\pm/\partial h_3\end{pmatrix} = \frac{9}{d^2}\begin{pmatrix}h_{32}\zeta\left(u_\pm\right) - h_{23}^2u_\pm\\ h_{32}u_\pm - h_{23}\zeta\left(u_\pm\right)\end{pmatrix},$$

выведем формулы для производных от ζ, ζ_*, $\wp$, $\wp_*$ и $u_\pm$ по параметру α:

$$\partial/\partial\alpha\begin{pmatrix}\zeta(t)\\ \zeta_*(t)\\ \wp(t)\\ \wp_*(t)\end{pmatrix} = \frac{3}{2d^2}\begin{pmatrix}(2\wp(t) + \alpha)\zeta(t) - (\alpha\wp(t) + 2h_{23})t + \wp'(t)\\ (\wp_*(t) + 2\alpha)\zeta_*(t) - (2\alpha\wp_*(t) + h_{23}^*)t + \wp_*'(t)/2\\ \left(2\alpha - 4\wp(t)\right)\wp(t) + 8h_{23} + \left(\alpha t - 2\zeta(t)\right)\wp'(t)\\ \left(4\alpha - 2\wp_*(t)\right)\wp_*(t) + 4h_{23}^* + (2\alpha t - \zeta_*(t))\wp_*'(t)\end{pmatrix},$$

$$d/d\alpha\begin{pmatrix}u_+\\ u_-\end{pmatrix} = \frac{3}{d^2}\begin{pmatrix}\zeta\left(u_+\right) - \alpha u_+/2\\ \zeta\left(u_-\right) - \alpha u_-/2\end{pmatrix} = \frac{3}{d^2}\begin{pmatrix}\zeta_*\left(u_+/2\right) - \alpha u_+\\ \zeta_*\left(u_-\right)/2 - \alpha u_-\end{pmatrix},$$

[3]Удачный подбор обозначений позволил легко обнаружить и исправить ошибку в одной из формул, предоставленных указанным источником. Эта ошибка остаётся неустранённой в последней версии "v. 8.0.1" программного пакета "Wolfram Mathematica", выпущенной 7-го марта 2011 г.

где штрих над функциями $\wp(\cdot)$ и $\wp_*(\cdot)$ означает взятие производной по аргументу t при фиксированном значении параметра α.

Вычислим величины $w_\pm$ методом Гаусса [5]:

$$\frac{w_+(\alpha)}{\pi} = G\left(\frac{2-3\alpha}{4}\right) = \sqrt{\beta}\ G(1-\beta^2) = \frac{\sqrt{\beta}}{M(\beta)},$$

$$\frac{w_-(\alpha)}{i\pi} = \sqrt{\sqrt{\frac{\beta}{d}}}\ G\left(\frac{2-\sqrt{\beta d} - 1/\sqrt{\beta d}}{4}\right) = \sqrt{\beta}\ G(\beta^2),\ \beta < 1,\ \beta + \frac{1}{\beta} = 3\alpha.$$

Здесь $M(x)$ – арифметико-геометрическое среднее чисел 1 и x, а $G(\cdot)$ – гипергеометрическая функция Гаусса с регулярной особенностью в нуле и удовлетворяющая дифференциальному тождеству

$$4x\,(x-1)\,G''(x) + 4\,(2x-1)\,G'(x) + G(x) \equiv 0.$$

Этому тождеству соответствует дифференциальное уравнение для $u_\pm$ как функции параметра α:

$$\left(2d/3\right)^2\ u_\pm'' + 8\alpha u_\pm' + u_\pm = 0.$$

Приведём и дифференциальное уравнение, которому удовлетворяет функция $\theta_\pm$ – логарифмическая производная функции $u_\pm$:

$$\theta_\pm' = -3\left(\left(\frac{\theta_\pm}{d}\right)^2 + \frac{1}{4}\right),\ \theta_\pm := \frac{\zeta\left(u_\pm\right)}{u_\pm} - \frac{\alpha}{2}. \tag{3.12}$$

Последнее уравнение эквивалентно дифференциальному тождеству для функции параметра γ

$$y'(\gamma) + y(\gamma)^2 + \operatorname{csch}(2\gamma)^2 \equiv 0.$$

Эта эквивалентость основывается на преобразованиях параметров α и γ друг в друга:

$$\gamma = \frac{1}{4}\ln\left(\frac{3\alpha-2}{3\alpha+2}\right),\ \alpha = -\frac{2}{3}\coth\left(e^{2\gamma}\right).$$

Здесь выразим отношение l длины нити к расстоянию между двумя точками её крепления к оси (в задаче Аппеля) как функцию параметра α (см. приложение А)

$$l = l(\alpha) = l_\alpha(0,1) = \theta_+ + \frac{3\alpha}{2} = N(1/\beta,\ \beta), \tag{3.13}$$

и отметим, что эта функция строго монотонно возрастающая.

В дальнейшем будет удобным пользоваться следующими сокращёнными обозначениями

$$A = A(\alpha,\ s,\ t) := \frac{\zeta_*\left(u_-\right) + 2s\zeta_*\left(u_+/2\right) - \zeta_*\left(u_- + u_+t\right)}{w_+(t-s)} + \alpha, \tag{3.14}$$

$$B = B(\alpha,\ t) := \frac{\wp_*\left(u_- + u_+t\right)}{2} + \alpha,\ C = C(\alpha,\ s,\ t) := \frac{u_+(t-s)\,\wp_*'\left(u_- + u_+t\right)}{4},$$

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