Infinite series as the input of certain algebraic procedures

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Infinite power series play an important role in mathematical studies. Those series may appear as inputs for certain mathematical problems. In order to be able to discuss the corresponding algorithms, we must agree on representation of the infinite series (algorithm inputs are always objects, represented by specific finite words in some alphabet). This talk examines two possible solutions to the problem of representation of power series.

First, we consider the algorithmic representation. For each series, an algorithm is specified that, given an integer i, finds the coefficient of x^i. Any deterministic algorithms are allowed (any such algorithm defines a so called constructive series). For example, suppose that a linear ordinary differential system S of arbitrary order with infinite formal power series coefficients is given, decide whether the system has non-zero Laurent series, regular, or formal exponential-logarithmic solutions, and find all such solutions if they exist. If the coefficients of the original systems are arbitrary formal power series represented algorithmically (thus we are not able, in general, to recognize whether a given series is equal to zero or not) then these three problems are algorithmically undecidable, and this can be deduced from the classical results of A. Turing. But, it turns out that the first two problems are decidable in the case when we know in advance that a given system S is of full rank (Abramov, Barkatou, Khmelnov, 2014). However, the third problem (finding formal exponential-logarithmic solutions) is not decidable even in this case (Abramov, Barkatou, 2014). It is shown that, despite the fact that such a system has a basis of formal exponential-logarithmic solutions involving only computable (i.e., algorithmically represented) series, there is no algorithm to construct such a basis. However, it is possible to specify a limited version of the third problem, for which there is an algorithm of the desired type: namely, if S and a positive integer d are such that for the system S the existence of d linearly independent solutions is guaranteed, we can construct these d solutions (Ryabenko, 2015).

It is shown also that the algorithmic problems connected with the ramification indices of irregular formal solutions of a given system are mostly undecidable even if we fix a conjectural value r of the ramification index. However, there is nearby an algorithmically decidable problem: if a system S of full rank and positive integers r, d are such that for S the existence of d linearly independent formal solutions of ramification index r is guaranteed then one can compute such d solutions of S (Abramov, 2017).

We prove additionally that the width of a given full-rank system S with formal power series coefficients (represented algorithmically) can be found, where the width of S is the smallest non-negative integer w such that any l-truncation of S with $l \geqslant w$ is a full-rank system. It is shown as well that the above-mentioned value w exists for any full-rank system (Abramov, Barkatou, Khmelnov, 2014).

Thus, when we use the algorithmic way of power series representation, a neighborhood of algorithmically solvable and unsolvable problems is observed.

For the solvable problems mentioned above, a Maple implementation was proposed (Khmelnov, Ryabenko, 2014-15). We report some experiments. Note that the ring of constructive formal power series is smaller than the ring of all formal power series because not every sequence of coefficients can be represented algorithmically. Indeed, the set of elements of the constructive formal power series is countable (each of the algorithms is a finite word in some fixed alphabet) while the set of all power series is uncountable.

Second, we consider an "approximate" representation. A well-known example is the result of Lutz and Schäfke (1985) related to the number of terms in M that can influence some components of formal exponential-logarithmic solutions of a differential system $x^s y' = My$, where s is a given non-negative integer, M is a matrix whose entries are power series. As a new example (Abramov, Barkatou, 2017), we consider matrices with infinite power series in the role of their entries and suppose that those series are represented in an approximate form, namely, in a truncated form. Thus, it is assumed that a polynomial matrix P which is the l-truncation (l is a non-negative integer, $\deg P = l$) of a power series matrix M is given, and P is non-singular, i.e., $\det P \neq 0$. We prove that the the question of strong non-singularity, i.e., the question whether P is not the l-truncation of a singular matrix having power series entries, is algorithmically decidable. Assuming that a non-singular power series matrix M (which is not known to us) is represented by a strongly non-singular polynomial matrix P, we give a tight lower bound for the number of initial terms of M^{-1} which can be determined from P^{-1}. In addition, we prove that if $M \in \mathrm{Mat}_n(K[[x]])$, $\det M \neq 0$ then there exists and can be constructed algorithmically a non-negative integer l such that $M^{\langle l \rangle}$ is a strongly non-singular polynomial matrix.

We discuss the possibility of applying the proposed approach to "approximate" linear higher-order differential systems: if a system is given in the approximate truncated form and the leading matrix is strongly non-singular then the result mentioned above of Lutz and Schäfke and its generalization can be used, and the number of reliable terms of Laurent series solution can be estimated by the algorithm proposed by Abramov, Barkatou, Pflügel (2011).

The information that can be extracted from truncated series, matrices, systems, etc. may be sufficient to obtain certain characteristics of the original (untruncated) objects. Naturally, these characteristics are incomplete, but may suffice for some purposes.