The EG-family of algorithms and procedures for solving linear differential and difference higher-order systems

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Part 1: Differential and difference systems with polynomial coefficients

$$A_r(x)\xi^r y + A_{r-1}(x)\xi^{r-1} y + \cdots + A_0(x) y = 0. \qquad (1)$$

$$\xi \in \left\{ \frac{d}{dx}, E \right\}, \quad Ey(x) = y(x+1)$$

$A_i(x) \in \mathrm{Mat}_m(K[x])$, $y = (y_1, y_2, \ldots, y_m)^T$. We suppose that system (1) is of full rank, i.e. its equations are independent over $K[x][\xi]$.

In the general case, the leading matrix $A_r(x)$ is not invertible (is singular) in $\mathrm{Mat}_m(K(x))$, which generates certain difficulties in calculations.

Algorithms EG_δ and EG_σ make it possible to avoid difficulties of this kind: For a system of the form (1), algorithms EG_δ (in the differential case) and EG_σ (in the difference case) construct an l-embracing system of the same form with a nonsingular leading matrix.

For a difference system, algorithm EG_σ allows also to construct a t-embracing system with a nonsingular trailing matrix.

The set of solutions of an embracing system contains all solutions of the original system.

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It is possible that an embracing system has some extra solutions.

However, if we consider *sequential* solutions of a difference system then EG_σ allows to filter out all the "parasitic" solutions of the corresponding embracing system.

For the solutions in the form of Laurent or Newton series, their sequence of coefficients satisfy *induced* recurrent system, which has polynomial coefficients. Thus, by applying EG_σ to the induced system, we will finally be able to construct a basis for the series solutions space.

Note additionally that l- and t-embracing systems which are constructed for the induced system, allow to construct some indicial equations for the original systems. Such equations are very useful for computing valuations of solutions of the original system.

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All this can be used as the base of algorithms for finding solutions of various types.

References:

Abramov: EG–eliminations. *J. of Difference Equations and Applications* **5** (1999) 393–433.

Abramov, Bronstein: On solutions of linear functional systems. *Proc. of ISSAC'2001* (2001) 1–6.

Abramov, Bronstein: Linear algebra for skew-polynomial matrices, *Rapport de Recherche INRIA* RR-4420, March 2002.

Abramov, Bronstein, Khmelnov: Regularization of linear recurrence systems. In: *Transactions of the A.M. Liapunov Institute* **4** (2003) 158–171.

Abramov, Khmelnov: Desingularization of leading matrices of systems of linear ordinary differential equations with polynomial coefficients. In: *International conference "Differential Equations and Related Topics", dedicated to I.G.Petrovskii. Book of Abstracts.* (2011) 5–5.

Implementation

The algorithms are implemented in Maple. Most of the functionality is already available as procedures of `LinearFunctionalSystems` package which is included in the current version of the system (Maple 18 is the latest one). Some of the recent adjustments are not yet publicly available. They are already implemented and submitted to be included in the next versions (subject to the internal Maple release policy and procedures). In the further slides we mark such cases with * and give related comments.

We use the following 2 systems to illustrate the procedures of the package. The systems are in the form of the lists of equations.

```
> sys1:=[x^4*diff(y1(x), x$3)+4*x^3*diff(y1(x), x$2)+2*x^2*diff(y1(x), x)-x^2*y2(x)+diff(y2(x), x)
  = 0, diff(y1(x), x$3)*x^3+4*diff(y1(x), x$2)*x^2-2*x^3*diff(y2(x), x$2)+2*diff(y1(x), x)*x+2*diff
  (y2(x), x) = 0];
```

$$sys1 := \left[x^4\left(\frac{d^3}{dx^3}y1(x)\right)+4x^3\left(\frac{d^2}{dx^2}y1(x)\right)+2x^2\left(\frac{d}{dx}y1(x)\right)-x^2y2(x)+\frac{d}{dx}y2(x)=0, \left(\frac{d^3}{dx^3}y1(x)\right)x^3+4\left(\frac{d^2}{dx^2}y1(x)\right)x^2 - 2x^3\left(\frac{d^2}{dx^2}y2(x)\right)+2\left(\frac{d}{dx}y1(x)\right)x+2\left(\frac{d}{dx}y2(x)\right)=0\right]$$

```
> sys2 := [x*(x+3)*(2*x+3)*y1(x+1)-(x+2)*(x-1)*(-1+2*x)*y2(x), y2(x+1)-y1(x)];
```

$$sys2 := [x\,(x+3)\,(2x+3)\,y1(x+1)-(x+2)\,(x-1)\,(-1+2x)\,y2(x), y2(x+1)-y1(x)]$$

```
> vars := [y1(x), y2(x)];
```

$$vars := [y1(x), y2(x)]$$

Series Solutions

Solutions with the components in the form

$$\sum_{i=v}^{\infty} c_i b_i(x), \tag{2}$$

where $v \in \mathbb{Z}$, $c_i \in K$, $b_i(x)$ are the elements of the compatible power basis B, e.g. $\{x^i\}_{i\in\mathbb{Z}}$ for the differential case and $\left\{ \begin{array}{ll} x^{\underline{i}}, & i \geq 0, \\ 1/x^{\overline{|i|}}, & i < 0 \end{array} \right\}_{i\in\mathbb{Z}}$ for the difference case ($x^{\overline{n}} = \prod_{k=1}^{n}(x+k)$,$x^{\underline{n}} = \prod_{k=1}^{n}(x-k+1)$).

Reference:

Abramov: EG–eliminations. *J. of Difference Equations and Applications* **5** (1999) 393–433.

Procedures: LinearFunctionalSystems[SeriesSolution], LinearFunctionalSystems[ExtendSeries]

EG use: EG_σ for the leading matrix of the induced recurrence to bound the solution valuation and to compute the solution coefficients

```
> SeriesSolution(sys1,vars);
```

$$\left[\frac{\_c_1}{x} + \_c_2 + O(x^2),\ \frac{5}{4}\_c_3 + O(x^3)\right]$$

```
> ExtendSeries(%, 4);
```

$$\left[\frac{\_c_1}{x} + \_c_2 - \frac{5}{24}x^2\_c_3 - \frac{5}{36}x^3\_c_3 - \frac{1}{16}x^4\_c_3 + O(x^5),\ \frac{5}{4}\_c_3 + \frac{5}{12}x^3\_c_3 + \frac{5}{8}x^4\_c_3 + x^5\_c_3 + O(x^6)\right]$$

```
> SeriesSolution(sys2,vars);
```

$$\left[-\frac{648}{11}\_c_2 - \frac{228}{11}\_c_1 + x\left(\frac{228}{11}\_c_2 + \frac{123}{11}\_c_1\right) + O(x^2),\ x\left(-\frac{648}{11}\_c_2 - \frac{228}{11}\_c_1\right) + O(x^2)\right]$$

```
> ExtendSeries(%, 3);
```

$$\left[-\frac{648}{11}\_c_2 - \frac{228}{11}\_c_1 + x\left(\frac{228}{11}\_c_2 + \frac{123}{11}\_c_1\right) + x(x-1)\left(-\frac{9}{11}\_c_1 + \frac{96}{11}\_c_2\right) + x(x-1)(x-2)\left(-\frac{35}{22}\_c_1 - \frac{114}{11}\_c_2\right)\right.$$
$$\left. + O(x^4),\ x\left(-\frac{648}{11}\_c_2 - \frac{228}{11}\_c_1\right) + x(x-1)\left(\frac{351}{22}\_c_1 + \frac{438}{11}\_c_2\right) + x(x-1)(x-2)\left(-\frac{114}{11}\_c_2 - \frac{123}{22}\_c_1\right) + O(x^4)\right]$$

Polynomial Solutions

Solutions with the components in $K[x]$, i.e. in the form

$$\sum_{i=0}^{n} c_i x^i, \tag{3}$$

where $n \in \mathbb{N} \cup \{0\}$, polynomial coefficients c_i are in K.

Reference:

Khmelnov: Search for polynomial solutions of linear functional systems by means of induced recurrences, *Programming and Computer Software* No 2 (2004), 61–67.

Procedure: LinearFunctionalSystems[PolynomialSolution]
EG use: EG_σ for the trailing matrix of the induced recurrence to find an upper bound of the degree of the polynomial solution and to compute the solution coefficients

```
> PolynomialSolution(sys1,vars);
                                   [_c1, 0]
> PolynomialSolution(sys2,vars);
                                   [0, 0]
```

Rational Solutions

Solutions with the components in $K(x)$, i.e. in the form

$$\frac{f(x)}{g(x)}, \tag{4}$$

where $f(x), g(x) \in K[x]$.

References:

Abramov, Barkatou: Rational solutions of first order linear difference systems, *Proc. of ISSAC'98* (1998) 124–131.

Abramov, Gheffar, Khmelnov: Factorization of polynomials and gcd computations for finding universal denominators, *Proc. of CASC'2010*, LNCS 6244 (2010) 4–18.

Abramov, Khmelnov: Denominators of rational solutions of linear difference systems of arbitrary order. *Programming and Computer Software* No 2 (2012) 84–91.

Abramov, Khmelnov: On singular points of solutions of linear differential systems with polynomial coefficients. *Journal of Mathematical Sciences* v.185, No 3 (2012) 347–359.

Procedure*: LinearFunctionalSystems[RationalSolution]
EG use: EG$_\delta$ for the leading matrix of the given system for bounding solution singularities and EG$_\sigma$ for the leading matrix of the induced recurrence for bounding their valuations in the differential case to find a universal denominator; EG$_\sigma$ for the leading and trailing matrices of the given system for finding universal denominator in the difference case; the same use as for polynomial solution when finding the solution numerator

```
> RationalSolution(sys1,vars);
```

$$\left[\frac{\_c_1 + x\,\_c_2}{x}, 0\right]$$

```
> RationalSolution(sys2,vars);
```

$$\left[\frac{\_c_1\,(x+1)}{x\,(x+2)\,(2\,x+1)}, \frac{x\,\_c_1}{(x-1)\,(x+1)\,(-1+2\,x)}\right]$$

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*) The work with the systems of the order higher than 1 is not yet in Maple 18.

Regular Solutions

Solutions with the components in the form

$$\sum_{j=1}^{l} x^{\lambda_j} \sum_{i=0}^{k} g_{ij}(x) \log^i(x), \tag{5}$$

where $\lambda_j \in K$, $g_{ij}(x) \in K[[x]]$, $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$.

Reference:

Abramov, Bronstein, Khmelnov: On regular and logarithmic solutions of ordinary linear differential systems. In *Proc. of CASC'05*, LNCS 3718 (2005) 1–12.

Procedures: LinearFunctionalSystems[RegularSolution], LinearFunctionalSystems[ExtendRegularSolution]
EG use: EG_σ for the leading matrix of the induced recurrence for finding the exponents λ_j and to compute the series coefficients.

```
> RegularSolution(sys1, vars);
```

$$\left[\ln(x)\left(\_c_1 + O(x^2)\right) + \frac{\_c_2}{x} + \_c_3 + O(x), \ln(x)\, O(x^3) + \_c_4 + O(x^2)\right]$$

```
> ExtendRegularSolution(%, 4);
```

$$\left[\ln(x)\left(\_c_1 + O(x^6)\right) + \frac{\_c_2}{x} + \_c_3 - \frac{1}{6}x^2\_c_4 - \frac{1}{9}x^3\_c_4 - \frac{1}{20}x^4\_c_4 + O(x^5), \ln(x)\, O(x^7) + \_c_4 + \frac{1}{3}x^3\_c_4 + \frac{1}{2}x^4\_c_4 + \frac{4}{5}x^5\_c_4 + O(x^6)\right]$$

Rational-logarithmic Solutions

Solutions with the components in $K(x)[\log x]$, i.e. in the form

$$\sum_{i=0}^{k} g_i(x) \log^i(x), \tag{6}$$

where $k \in \mathbb{N} \cup \{0\}$, $g_i(x) \in K(x)$.

Reference:

Abramov, Bronstein, Khmelnov: On regular and logarithmic solutions of ordinary linear differential systems. In *Proc. of CASC'05*, LNCS 3718 (2005) 1–12.

Procedure*: LinearFunctionalSystems[LogarithmicSolution]
EG use: EG_δ for the leading matrix of the given system for finding singularities and EG_σ for the leading matrix of the induced recurrence for bounding valuations in the differential case to find a universal denominator, EG_σ for the leading matrix of the induced recurrence to compute the solution coefficients

```
> LogarithmicSolution(sys1,vars);
```

$$\left[\frac{\ln(x)\, x\, \_c_1 + \_c_2 + x\, \_c_3}{x}, 0\right]$$

---

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Formal Solutions

A space of formal solutions has a basis such that any solution $y(x)$ of this basis can be represented in the parametric form

$$y(t) = e^{Q\left(\frac{1}{t}\right)} t^{\lambda} \Phi(t), \quad x = t^q \tag{7}$$

where $\lambda \in K$; $Q(\frac{1}{t}) \in K[\frac{1}{t}]$; $q \in \mathbb{N}$; $\Phi(t)$ is a column-vector with the component in the form $\sum_{i=0}^{k} g_i(t) \log^i(t)$ and $g_i(t) \in K[[t]]$.

References:

Ryabenko: On exponential-logarithmic solutions of linear differential systems with power series coefficients. *Programming and Computer Software* No 2 (2015) 112–118.

Abramov, Petkovšek, Ryabenko: Resolving sequences of operators for linear differential and difference systems. In preparation.

Procedure*: LinearFunctionalSystems[FormalSolution]
EG use: EG_δ for the leading matrix of the given system to get an embracing system with a nonsingular leading matrix; EG_σ for the leading matrix of the induced recurrences for finding the exponents λ and to compute the series coefficients.

```
> FormalSolution(sys1, vars, t);
```

$$\left[\left[\left[\ln(t)\left(\_c_1 + O(t^2)\right) + \frac{\_c_2}{t} + \_c_3 + O(t), \ln(t)\, O(t^3) + \_c_4 + O(t^2)\right], t = x\right], \left[e^{\frac{1}{6t^3} - \frac{1}{2t^2}}\left[-4\, t^8\, \_c_5 + O(t^9), t^4\, \_c_5 + O(t^5)\right], t = x\right]\right]$$

---

*) The procedure is not yet in Maple 18.

Part 2: Differential systems with computable power series coefficients

In the current context, it is convenient to write differential systems in terms of the operation $\theta = x\frac{d}{dx}$ rather than $\frac{d}{dx}$ (the transition from one notation to the other presents no difficulties). We consider systems of the form

$$A_r(x)\theta^r y + A_{r-1}(x)\theta^{r-1} y + \cdots + A_0(x)y = 0, \tag{8}$$

$A_i(x) \in \mathrm{Mat}_m(K[[x]])$, $y = (y_1, y_2, \ldots, y_m)^T$. We suppose that system (8) is of full rank, i.e. its equations are independent over $K[[x]][\theta]$. We suppose that the entries of the matrices $A_i(x)$ are represented *algorithmically*: for any series $a(x)$ an algorithm Λ_a such that $a(x) = \sum_{i=0}^{\infty} \Lambda_a(i)x^i$ is given.

We are not able, in general, to recognize whether a given series is equal to zero or not. The problem of recognizing whether a given system of the form (8) is of full rank or not is also algorithmically undecidable in the general case.

However, we can construct solutions of various kinds for the case when we know *in advance* that a given system is of full rank.

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However, we can construct solutions of various kinds for the case when we know *in advance* that a given system is of full rank.

The induced recurrent system has the form

$$B_0(n)z(n) + B_{-1}(n)z(n-1) + \cdots = 0, \tag{9}$$

where $B_0(n), B_{-1}(n), \dots \in \mathrm{Mat}_m(K[n])$, each of polynomial entries of these matrices is of degree $\leq r$.

A differential system of the considered form has a Laurent series solution $y(x) = u(v)x^v + u(v+1)x^{v+1} + \dots$ iff the double-sided sequences

$$\dots, 0, 0, u(v), u(v+1), \dots \tag{10}$$

of vector coefficients of $y(x)$ satisfies the induced recurrent system of the form (9):

$$\begin{array}{l} B_0(v)u(v) = 0, \\ B_0(v+1)u(v+1) + B_{-1}(v+1)u(v) = 0, \\ B_0(v+2)u(v+2) + B_{-1}(v+2)u(v+1) + B_{-2}(v+2)u(v) = 0, \\ \dots \end{array}$$

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$$\begin{aligned}
&B_0(v)u(v) = 0,\\
&B_0(v+1)u(v+1) + B_{-1}(v+1)u(v) = 0,\\
&B_0(v+2)u(v+2) + B_{-1}(v+2)u(v+1) + B_{-2}(v+2)u(v) = 0,\\
&\dots
\end{aligned}$$

If the leading matrix $B_0(n)$ is invertible in $\mathrm{Mat}_m(K(n))$ then its determinant can be considered as a kind of the indicial polynomial of the original differential system S (the set of integer roots of $\det B_0(n)$ is a superset of the set of all possible valuations of Laurent series solutions of the original differential system).
However, in many cases the matrix $B_0(n)$ is not invertible even when the leading matrix $A_r(x)$ of the original differential system is invertible in $\mathrm{Mat}_m(K((x)))$.

A special version of EG-eliminations allows to transform the induced recurrent system into a convenient form. Besides, this version provides with a tool which filters out “parasitic” sequential solutions.

Of course, we are not able to work directly with infinite series and recurrent systems of infinite order. Some kind of lazy computation is used in our algorithms and their implementation.

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Of course, we are not able to work directly with infinite series and recurrent systems of infinite order. Some kind of lazy computation is used in our algorithms and their implementation.

Implementation

The algorithms are implemented in Maple as the procedures of external `EG` package (available on `http://www.ccas.ru/ca/doku.php/eg`). The procedures are assumed to be included in the next versions of Maple (subject to the internal Maple release policy and procedures) as the subpackage `LinearFunctionalSystems[SeriesCoefficients]`.

We use the following system to illustrate the procedures of the package. The matrix form for representing operators and systems is used.

```
> L := Matrix([
  [k->piecewise(k=0,-1,k=2,theta+1,theta^2-1), k->piecewise(k=2,-1,0), k->piecewise(k=0,-1,k=1,-1,k=3,0)],
  [k->piecewise(k=2,-1,0), k->piecewise(k=0,theta,theta^2-1), k->piecewise(k=0,-1,k=3,-1,0)],
  [k->piecewise(k=3,-1,0), k->piecewise(k=1,1,0), k->piecewise(k=0,theta-1,theta^2-1)]]);
```

$$L := \begin{bmatrix} k \to piecewise(k=0, -1, k=2, \theta+1, \theta^2-1) & k \to piecewise(k=2, -1, 0) & k \to piecewise(k=0, -1, k=1, -1, k=3, 0) \\ k \to piecewise(k=2, -1, 0) & k \to piecewise(k=0, \theta, \theta^2-1) & k \to piecewise(k=0, -1, k=3, -1, 0) \\ k \to piecewise(k=3, -1, 0) & k \to piecewise(k=1, 1, 0) & k \to piecewise(k=0, \theta-1, \theta^2-1) \end{bmatrix}$$

```
> [map(apply, L, 0), map(apply, L, 1), map(apply, L, 2), map(apply, L, 3), map(apply, L, k)]
  assuming k >= 4;
```

$$\left[\begin{bmatrix} -1 & 0 & -1 \\ 0 & \theta & -1 \\ 0 & 0 & \theta-1 \end{bmatrix}, \begin{bmatrix} \theta^2-1 & 0 & -1 \\ 0 & \theta^2-1 & 0 \\ 0 & 1 & \theta^2-1 \end{bmatrix}, \begin{bmatrix} \theta+1 & -1 & 0 \\ -1 & \theta^2-1 & 0 \\ 0 & 0 & \theta^2-1 \end{bmatrix}, \begin{bmatrix} \theta^2-1 & 0 & 0 \\ 0 & \theta^2-1 & -1 \\ -1 & 0 & \theta^2-1 \end{bmatrix}, \begin{bmatrix} \theta^2-1 & 0 & 0 \\ 0 & \theta^2-1 & 0 \\ 0 & 0 & \theta^2-1 \end{bmatrix}\right]$$

Note: The matrix entry with indices i, j is a function of an integer argument (e.g., k) which computes the coefficient of x^k in the i-th equation of the system for the j-th unknown.

In this case the functions are defined via `piecewise`, but might be also defined as special procedures with any complicated algorithm to compute the coefficients.

The second block of the session above shows the matrix coefficients of x^0, x^1, x^2, x^3 and x^k with $k \geqslant 4$ in the operator L.

Laurent Solutions

Solutions with the components in the form

$$\sum_{i=v}^{\infty} c_i x^i, \tag{11}$$

where $v \in \mathbb{Z}$, $c_i \in K$.

References:

Abramov, Barkatou, Pfluegel: Higher-order linear differential systems with truncated coefficients. *Proc. of CASC' 2011*, LNCS 6885 (2011) 10–24.

Abramov, Barkatou, Khmelnov: On full-rank differential systems with power series coefficients. *J. of Symbolic Computation*, **68** (2015) 120–137.

Procedure: EG[LaurentSolution]

EG use: A special version of EG_σ for the leading matrix of the induced recurrence to bound the solution valuations and to compute the solution coefficients

```
> EG:-LaurentSolution(L, theta, x, 0);
```

$$\left[x\_c_1 + \mathrm{O}(x^2),\ -x\_c_1 + \mathrm{O}(x^2),\ -x\_c_1 + \mathrm{O}(x^2)\right]$$

Regular Solutions

$$\sum_{j=1}^{l} x^{\lambda_j} \sum_{i=0}^{k} g_{ij}(x) \log^i(x), \qquad (12)$$

where $\lambda_j \in K$, $g_{ij}(x) \in K[[x]]$, $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$.

Reference:

Abramov, Khmelnov: Regular solutions of linear differential systems with power series coefficients. *Programming and Computer Software*, No 2 (2014) 98–106.

Procedure: EG[RegularSolution]

EG use: A special version of EG_σ for the leading matrix of the induced recurrence for finding the exponents λ_j and to compute the series coefficients.

```
> EG:-RegularSolution(L, theta, x, 0);
```

$$\left[\ln(x)\left(x\_c_1 + O(x^2)\right) + x\_c_2 + O(x^2), \ln(x)\left(-x\_c_1 + O(x^2)\right) + \_c_1 + x\left(-\_c_2 + 2\_c_1\right) + O(x^2), \ln(x)\left(-x\_c_1 + O(x^2)\right) - x\_c_2 + O(x^2)\right]$$

Formal Solutions

A space of formal solutions has a basis such that any solution $y(x)$ of this basis can be represented in the parametric form

$$y(t) = e^{Q(\frac{1}{t})} t^{\lambda} \Phi(t), \quad x = t^q \tag{13}$$

where $\lambda \in K$; $Q(\frac{1}{t}) \in K[\frac{1}{t}]$; $q \in \mathbb{N}$; $\Phi(t)$ is a column-vector with the component in the form $\sum_{i=0}^{k} g_i(t) \log^i(t)$ and $g_i(t) \in K[[t]]$.

References:

Abramov, Barkatou: Computable infinite power series in the role of coefficients of linear differential systems, *Proc. of CASC'2014*, LNCS 8660 (2014) 1–12.

Ryabenko: On exponential-logarithmic solutions of linear differential systems with power series coefficients. *Programming and Computer Software*, No 2 (2015) 112–118.

Abramov, Petkovšek, Ryabenko: Resolving sequences of operators for linear differential and difference systems. In preparation.

Procedure: EG[FormalSolution]

EG use: EG_δ for the leading matrix of the given system to get an *l*-embracing system; the same use of a special version of EG_σ as for regular solution when finding the regular parts of the solution

```
> Res:=EG:-FormalSolution(L, theta, x, t,
                          'system_order' = 2, 'solution_dimension'=6):
  Res[1];Res[2];Res[3];
```

$$\left[x=t, \left[\ln(t)\left(t\_c_1+O(t^2)\right)+t\_c_2+O(t^2), \ln(t)\left(-t\_c_1+O(t^2)\right)+\_c_1+t\left(-\_c_2+2\_c_1\right)+O(t^2), \ln(t)\left(-t\_c_1+O(t^2)\right)-t\_c_2+O(t^2)\right]\right]$$

$$\left[x=t, e^{\frac{1}{t}}\left[\ln(t)\,O(t^3)-t^2\_c_3+O(t^3), \ln(t)\left(t^2\_c_3+O(t^3)\right)+t^2\_c_4-t\_c_3+O(t^3), \ln(t)\,O(t^3)-t^2\_c_3-t\_c_3+O(t^3)\right]\right]$$

$$\left[x=t^2, e^{-\frac{2}{t}}\left[\sqrt{t}\left(\_c_5+O(t)\right), \sqrt{t}\,O(t), \sqrt{t}\,O(t)\right]\right]$$

Other approaches

It is worthy to note that there exist other approaches to the problem of transformation of a given system into a form which is convenient for finding solutions of one type or another (not necessary of the same types as in our talk).
Such approaches were proposed by

M.Barkatou, T.Cluzeau, C.El Bacha, E.Pfluegel

and

G.Labahn, B.Beckermann, H.Cheng.

An invaluable contribution to the improvement of the first version of the EG-eliminations method was made by Manuel Bronstein (1963–2005).