# On systems of linear ordinary differential equations with formal power series coefficients.

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# PART 1: GENERALITIES

$K, \partial$

Char $K = 0$

$Const(K) = \{c \in K \mid \partial c = 0\}$

$K_0 = Const(K)$ $(= \overline{K_0})$

$L = A_r \partial^r + \cdots + A_1 \partial + A_0, \quad A_i \in \mathrm{Mat}_m(K)$

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$y' = Ay \quad L = I_m\partial - A$

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rank $L$ over $K[\partial]$

if rank $L = m$:

(a) dim $V_L$

(b) unimodularity of $L$ (dim $V_L = 0$)

(c) $L^{-1}$

(d) the Jacobson form of $L$:

$$SLT = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & p \end{pmatrix}, \quad p \in K[\partial] \setminus \{0\}.$$

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if the zero testing problem in $K$ is undecidable then the problem of recognizing whether a given $L \in \mathrm{Mat}_m(K[\partial])$ is of full rank is undecidable.

Indeed, let $u \in K$, then the operator

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is of full rank iff $u \neq 0$, and any algorithm to recognize whether a given $L \in \mathrm{Mat}_m(K[\partial])$ is of full rank can be used for zero testing in $K$.

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if the zero testing problem in $K$ is undecidable then even we know in advance that operators under consideration are of full rank, many questions related to those operators are undecidable.

### Theorem 1

*Let the zero testing problem in $K$ be undecidable.*
*Then for $m \geq 2$ the following problems on a full rank operator $L \in \mathrm{Mat}_m(K[\partial])$ are undecidable:*
*(a) computing* $\dim V_L$,
*(b) testing unimodularity of $L$,*
*(c) constructing the Jacobson form of $L$.*

(a) Let $u \in K$ and

$$L = \begin{pmatrix} u\partial + 1 & \partial \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} u & 1 \\ 0 & 0 \end{pmatrix} \partial + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (1)$$

If $u = 0$ then $L$ is unimodular:

$$\begin{pmatrix} 1 & \partial \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\partial \\ 0 & 1 \end{pmatrix}$$

and therefore $\dim V_L = 0$.
We can check that

$$\dim V_L = \begin{cases} 0 & \text{if } u = 0, \\ 1 & \text{if } u \neq 0 \end{cases}.$$

This implies that if we have an algorithm for computing the dimension then we have an algorithm for the zero testing problem.
(b) As we have seen the operator $L$ of form (1) is unimodular iff $u = 0$.
(c) We are not able in general to construct algorithmically the Jacobson form of $L$ since $\dim V_L = \operatorname{ord} p$ (recall that the leading coefficient of $p$ must be equal to 1).

Why the fields that are not constructive can be of interest of computer algebra?

Computable Power Series:

Let $K$ be the field $K_0((x))$ where $K_0$ is a constructive field of characteristic 0.
This field contains the set of *computable* series, whose sequences of coefficients are represented algorithmically.
We will denote this set by $K|_{\mathrm{C}}$.

To consider this set as a constructive differential subfield of $K$, it would be necessary to define algorithmically on $K|_{\mathrm{C}}$ the field operation of the field $K$, the unary operation $\frac{d}{dx}$, and a zero testing algorithm as well.

However, if series are represented algorithmically, i.e., when each series $a(x) \in K|_{\mathrm{C}}$ is represented by some algorithm $\Xi_a$ for computing the coefficient $a_i$ for a given $i$, then in accordance with the classical A.Turing results we are not able to solve algorithmically the zero testing problem in $K|_{\mathrm{C}}$.

The field $K|_C$ is smaller than the field $K$ because of not any sequence of coefficients can be represented algorithmically: the set of elements of $K|_C$ is countable (each of algorithms is a finite word in some fixed alphabet) while the cardinality of the set of elements of $K$ is continuum.

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Representation:

If $a(x) \in K|_C$ is represented only by an algorithm $\Xi_a$ then the problem of finding $\mathrm{val}\, a(x)$ for a given $a(x) \in K|_C$ is undecidable even in the case when it is known in advance that $a(x)$ is not the zero series.

This implies that when we work with elements of $K|_C$, i.e., with computable Laurent series, we cannot compute $a^{-1}(x)$ for a given non-zero $a(x) \in K|_C$, since the coefficient of $x^{-1}$ of the series $a'(x)a^{-1}(x) \in K|_C$ is equal to $\mathrm{val}\, a(x)$, i.e., is equal to the value that we are not able to find algorithmically knowing only $\Xi_a$.

This means that a suitable representation has to contain some additional information besides the corresponding algorithm.

The value $\operatorname{val} a(x)$ cannot close the gap, since we have no algorithm to compute the valuation of the sum of two series.
However, we can use a lower bound of the valuation instead: observe that if we know that a series $a(x)$ is non-zero then using a valuation lower bound we can compute the exact value of $\operatorname{val} a(x)$.

Thus, we can use as the representation of $a(x) \in K|_C$ a pair of form

$$(\Xi_a, \mu_a), \tag{2}$$

where $\Xi_a$ is an algorithm for computing the coefficient $a_i$ for a given $i$, and an integer $\mu_a$ is a lower bound for the valuation of $a(x)$.

A computable Laurent series $a(x)$, represented by a pair of form (2) is equal to

$$\sum_{i=\mu_a}^{\infty} \Xi_a(i) x^i.$$

In the situation when we know in advance that a Laurent series is non-zero, representation (2) allows to compute the valuation of $a(x)$ and to perform the division operation.

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We can define the field structure on $K|_C$. Since we do not have an algorithm for solving the zero testing problem in $K|_C$, we use for $K|_C$ the term "semi-constructive field".

**Definition 1**

*A ring (field) is* semi-constructive *if there are algorithms to perform the ring (field) operations and the differentiation, but there is no algorithm to solve the zero testing problem.*

Considering for the ring $R = K_0[[x]]$ its semi-constructive sub-ring $R|_C$ of computable power series, we do not need to include a lower bound of the valuation into a representation of a series $a(x) \in R|_C$, since 0 is such bound for $a(x)$.

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# PART 2: SYSTEMS WITH COMPUTABLE POWER SERIES COEFFICIENTS

Suppose that $K_0$ is a constructive field of characteristic 0,

$$K = K_0((x)) \;\; R = K_0[[x]],$$

and

$$K|_{\mathsf{C}}, \;\; R|_{\mathsf{C}}$$

are semi-constructive field and, resp., ring.

Consider systems of form

$$L(y) = 0, \;\; L \in \mathrm{Mat}_m\left(R|_{\mathsf{C}}\left[\frac{d}{dx}\right]\right). \tag{3}$$

It follows from Theorem 1 that the problems (a), (b), (c) listed in that theorem are undecidable if $L$ is as in (3).

At first glance it seems that such undecidability is mostly due to we cannot distinguish zero and nonzero coefficients of operators and systems. However the situation is even worse.

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At first glance it seems that such undecidability is mostly due to we cannot distinguish zero and nonzero coefficients of operators and systems. However the situation is even worse.

Even if for an operator $L$ we know in advance which of its coefficients equal to zero, we, nevertheless, cannot solve problems (a), (b) and (c) algorithmically.
Let $u \in R|_{\mathsf{C}}$ and

$$L = \begin{pmatrix} (u(x)x+1)\frac{d}{dx}+1 & \frac{d}{dx} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} u(x)x+1 & 1 \\ 0 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For the operator $L$ we know in advance which of its coefficients equal to zero, but we do not know whether the power series $u(x)$ is equal to zero. It is easy to see that

$$\dim V_L = \begin{cases} 0 & \text{if } u(x) = 0, \\ 1 & \text{if } u(x) \neq 0. \end{cases}$$

It is known (Schlesinger, 1895) that if $K_0$ is an algebraically closed subfield of the complex numbers field $\mathbb{C}$ and $K$ is the field $K_0((x))$ of formal Laurent series with coefficients from $K_0$ then the universal differential field extension $\Lambda$ is the quotient field of the ring generated by expressions of form

$$e^{P(x)}x^{\gamma}(\psi_0 + \psi_1 \ln x + \cdots + \psi_s (\ln x)^s), \tag{4}$$

where in any such expression

- $P(x) \in K_0[x^{-1/q}]$, $q$ is a positive integer,
- $\gamma \in K_0$,
- $s$ is a non-negative integer and

$$\psi_i \in K_0[[x^{1/q}]], \tag{5}$$

$i = 0, 1, \ldots, s$.

In fact, system

$$\partial y = Ay, \ \ A \in \mathrm{Mat}_n(K_0((x)))$$

has $n$ linearly independent solutions $b_1(x), \ldots, b_n(x)$ with

$$b_i(x) = e^{P_i(x)} x^{\gamma_i} \Psi_i(x), \qquad (6)$$

where the factor $e^{P_i(x)} x^{\gamma_i}$ is common for all components of $b_i$, and $\gamma_i \in K_0$, $q_i$ is a positive integer, $P_i(x) \in K_0[x^{-1/q_i}]$, $\Psi_i(x) \in K_0^n[[x^{1/q_i}]][\ln x]$, $i = 1, \ldots, n$.

**Definition 2**

*A solution of form (6) is called a* (formal) logarithmic-exponential *solution. If* $q = 1$ *and* $P(x) = 0$ *then solution (6) is* regular.

If $K_0$ is not algebraically closed then for any concrete system, solutions (6) will exist if we consider instead of $K_0$ some simple algebraic extension $K_1$ of $K_0$ (such extensions are different for different systems).

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### Theorem 2

*Let $m$ be an integer number, $m \geq 2$, $K_0$ be a constructive subfield of $\mathbb{C}$. In this case for a given full rank system of form (3),*
*(i) the existence problem of Laurent series solutions and regular solutions are decidable;*
*(ii) the existence problem of formal logarithmic-exponential solutions testing problem is algorithmically undecidable;*
*(iii) the existence problem of formal logarithmic-exponential solutions which are not regular solutions is algorithmically undecidable.*

Concerning (i):
An implementation in Maple is available from
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## Theorem 3

*Let $m$ be an integer number, $m \geq 2$, $K_0$ be a constructive subset of $\mathbb{C}$. Let $L(y) = 0$ be a full rank system of form (3), and $d = \dim V_L$. Then $V_L$ has a basis $b_1(x), \ldots, b_d(x)$ consisting of logarithmic-exponential solutions such that any $\Psi_i(x)$ from (6) is of form $\Psi_i(x) = \Phi_i(x^{1/q_i})$ where $q_i$ is a non-negative integer,*

$$\Phi_i(x) \in (K_1^m[[x]])\,|_C\,[\ln x], \tag{7}$$

*and $K_1$ is a simple algebraic extension of $K_0$, $\gamma_i \in K_1$, $P_i(x) \in K_1[x]$, $i = 1, \ldots, d$.*

Thus, the series that are involved into representation of solutions are constructive (Theorem 3), but we cannot find them algorithmically (Theorem 2).

It is proven that if the dimension $d$ of the space of logarithmic-exponential solutions is known in advance then the basis $b_1, \ldots, b_d$ which is mentioned in Theorem 3 can be constructed algorithmically. (The corresponding algorithm is implemented in Maple.)

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As we see, if the algorithmic way of series representation is used then some of problems related to linear ordinary differential systems are undecidable, while others are decidable.
There is a subtle border between them, and a careful formulation of each of problems under consideration is absolutely necessary.
A small change in a decidable problem formulation can transform it into undecidable, and vice versa.