On full rank differential systems with power series coefficients

S. Abramov, M. Barkatou, D. Khmelnov

$K, \quad K[x], \quad K[[x]], \quad K(x), \quad K((x))$

We write θ for $x\frac{d}{dx}$ and consider differential systems of the form

$$A_r(x)\theta^r y + A_{r-1}(x)\theta^{r-1} y + \cdots + A_0(x) y = 0 \tag{1}$$

where $y = (y_1, y_2, \ldots, y_m)^T$ is a column vector of unknown functions of x.

For the coefficient matrices

$$A_0(x), A_1(x), \ldots, A_r(x) \tag{2}$$

we have $A_i(x) \in \mathrm{Mat}_m(K[[x]])$, and $A_r(x)$ is non-zero.
We suppose that system (1) is of full rank, i.e. its equations are independent over $K[[x]][\theta]$.

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We suppose that system (1) is of full rank, i.e. its equations are independent over $K[[x]][\theta]$.

If $a(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is a formal power series, l is a non-negative integer then the polynomial $a^{\langle l \rangle}(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_l x^l$ is the *l-truncation* of $a(x)$.

The l-truncation is also defined for Laurent series.

If S is an arbitrary-order linear differential system with formal power series coefficients then the l-truncation $S^{\langle l \rangle}$ of S is the system whose coefficients are the l-truncations of corresponding coefficients of S.

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We define V_S as the space of Laurent series solutions of S, and $V_S^{\langle l \rangle}$ as the space whose elements are the l-truncations of the corresponding elements of V_S (thus $V_S^{\langle l \rangle}$ consists of Laurent polynomials). Let $l_0 \in \mathbb{Z} \cup \{-\infty\}$ be such that the l_0-truncation mapping

$$V_S \to V_S^{\langle l_0 \rangle} \tag{3}$$

is bijective (if $V_S = \{0\}$ then, e.g., $-\infty$ can be taken as l_0).

For a given system S of the form (1) of full rank we are concerned with three problems. The first two of them are as follows:

P1. Compute $l_0 \in \mathbb{Z} \cup \{-\infty\}$ such that the l_0-truncation mapping (3) is bijective (i.e., it preservs the space dimension).

P2. Given $l \geq l_0$ (see **P1**), construct a basis for $V_S^{\langle l \rangle}$.

The third problem will be formulated below.

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The third problem will be formulated below.

If S is of full rank then the minimal integer w such that $S^{\langle l \rangle}$ is of full rank for all $l \geq w$ is called the *width* of S.

An example of a system S of full rank and a non-negative integer l such that $S^{\langle l \rangle}$ is of full rank while $S^{\langle l+1 \rangle}$ is not, can be given; however we prove that the width is defined for any system of full rank.

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We suppose that the entries of the matrices (2) are represented *algorithmically*: for any series $u(x)$ an algorithm Λ_u such that $u(x) = \sum_{i=0}^{\infty} \Lambda_u(i) x^i$ is given (thus we are not able, in general, to recognize whether a given series is equal to zero or not).

The problem of recognizing whether a given system of the form (1) is of full rank or not (i.e., whether the equations of S are independent over $K[[x]][\theta]$ or not) is algorithmically undecidable in the general case.

However, we show that the problems **P1**, **P2** and **P3** can be solved algorithmically for the case when we know *in advance* that a system S is of full rank.

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Operators related to systems

System (1) can be written as $L(y) = 0$ where

$$L = A_r(x)\theta^r + A_{r-1}(x)\theta^{r-1} + \cdots + A_0(x), \tag{4}$$

$L \in \mathrm{Mat}_m(K[[x]])[\theta]$,
the leading matrix $A_r(x)$ supposed to be nonzero.

We say that L is of full rank if system (1) is of full rank.

Note that the operator L can be also represented in the matrix form, i.e., in the form

$$\begin{pmatrix} L_{11} & \dots & L_{1m} \\ \dots & \dots & \dots \\ L_{m1} & \dots & L_{mm} \end{pmatrix}, \tag{5}$$

where $L_{ij} \in K[[x]][\theta]$, $i, j = 1, 2, \dots, m$, and $\max_{i,j} \operatorname{ord} L_{ij} = r$. (The transformation of (4) to such representation and the inverse transformation present no difficulties.)
The operator L is of full rank iff the rows of (5) are linearly independent over $K[[x]][\theta]$.

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On induced recurrent systems

Let E denote the shift operator: $Ez(n) = z(n+1)$ for any sequence $z(n)$. The mapping

$$x \rightarrow E^{-1}, \ \ x^{-1} \rightarrow E, \ \ \theta \rightarrow n \tag{6}$$

transforms an original differential system S into the *induced* recurrent system for the sequences of coefficients of Laurent series solutions.

This mapping (6) produces the ring isomorphism

$$\mathcal{M}: \ \mathrm{Mat}_m(K((x)))[\theta] \rightarrow \mathrm{Mat}_m(K[n])((E^{-1})).$$

The induced recurrent system can be written as $R(z) = 0$ with $R = \mathcal{M}(L)$,

$$R = B_0(n) + B_{-1}(n)E^{-1} + B_{-2}(n)E^{-2} + \dots$$

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The induced recurrent system $R(z) = 0$ has the form

$$B_0(n)z(n) + B_{-1}(n)z(n-1) + \cdots = 0, \tag{7}$$

where

- $z(n) = (z_1(n), \ldots, z_m(n))^T$ is a column vector of unknown sequences such that $z_i(n) = 0$ for all negative integer n with $|n|$ large enough, $i = 1, 2, \ldots, m$.
- $B_0(n), B_{-1}(n), \cdots \in \mathrm{Mat}_m(K[n])$, each of polynomial entries of these matrices is of degree $\leq r$.
- $B_0(n)$ is a non-zero matrix, it is called the leading matrix of the operator R and of the system (7).

The original differential system $L(y) = 0$ is of full rank iff the induced system $R(z) = 0$ is of full rank (i.e., the equations of $R(z) = 0$ are independent over $K[n][[E^{-1}]]$).

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A system $L(y) = 0$ has a Laurent series solution $y(x) = u(v)x^v + u(v+1)x^{v+1} + \dots$ iff the double-sided sequences

$$\dots, 0, 0, u(v), u(v+1), \dots$$

of vector coefficients of $y(x)$ satisfies the induced recurrent system of the form (7):

$$\begin{aligned}
&B_0(v)u(v) = 0,\\
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If the leading matrix $B_0(n)$ is invertible in $\mathrm{Mat}_m(K(n))$ then its determinant can be considered as a kind of the indicial polynomial of the original differential system S (the set of integer roots of $\det B_0(n)$ is a superset of the set of all possible valuations of Laurent series solutions of $L(y) = 0$).

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However in many cases the matrix $B_0(n)$ is not invertible even when the leading matrix $A_r(x)$ of $L(y) = 0$ is invertible in $\mathrm{Mat}_m(K(x))$.

Theorem. *Let $R = \mathcal{M}L$ where L is a full rank operator from $\mathrm{Mat}_m(K[[x]])[\theta]$. In this case*

(i) There exists $F \in \mathrm{Mat}_m(K[n])[E]$ such that the leading matrix $\bar{B}_0(n)$ of FR is invertible.

(ii) Such an operator F can be constructed algorithmically.

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It follows from this theorem that

(a) We can construct any finite number of the terms of the recurrent operator

$$FR = \bar{R} = \bar{B}_0(n) + \bar{B}_{-1}(n)E^{-1} + \bar{B}_{-2}(n)E^{-2} + \dots \tag{8}$$

(b) The width is defined correctly for any full rank operator $L = \mathrm{Mat}_m(K[[x]])[\theta]$, and if F is as in (i) then $s = \operatorname{ord} F$ (a finite number!) is an upper bound for the width of L.
(c) The exact value of the width can be computed algorithmically.

Thus (b) and (c) give a solution of **P3**.

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Remark on a naive (wrong!) algorithm for computing the width.

Example. *It is possible that for a system S of the form (1) and some positive l the system $S^{\langle l\rangle}$ is of full rank while $S^{\langle l+1\rangle}$ is not: let d be a positive integer, and*

$$A(x)=\begin{pmatrix}1 & 0 & 0\\ 1 & x^d & x^{2d}+x^{2d+1}\\ 1 & 1 & x^d\end{pmatrix}.$$

Let S be the system $A_1(x)\theta y + A_0(x)y = 0$, where

$$A_0(x)=A(x)+M_0(x),\ \ A_1(x)=A(x)+M_1(x), \tag{9}$$

$M_0(x), M_1(x) \in \mathrm{Mat}_3(K[[x]])$,
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Then the system $S^{\langle l\rangle}$ is of full rank iff

$$l = d, d+1, \ldots,\ 2d-1,\ 2d+1,\ 2d+2,\ \ldots \tag{10}$$

($2d$ is not in this list; thus the width of S is not d but $2d+1$). Observe that θy_1 has the coefficient 1 in each equation of $S^{\langle l\rangle}$, $l=0,1,\ldots$, and $B_0(n)$ in (7) has no zero row.

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As for **P1**, **P2**, we need to discard the extra “parasitic” solutions which can appear due to the left multiplication by the operator F.

Linear constraints

In many cases, it is natural to consider the system (7) together with a finite set of *linear constraints*, i.e. linear relations, each of which contains a set of variables $z_i(j)$ with constant coefficients. Let $R(z) = 0$ and $R'(z) = 0$ be systems of the form (7) and C and C' be finite sets of linear constraints.

We then say that the systems $(R(z) = 0, C)$ and $(R'(z) = 0, C')$ are equivalent if the space of solutions of $R(z) = 0$ that satisfy C is the same than the space of solutions of $R'(z) = 0$ that satisfy C'.

Finitely many linear constraints are easily taken into account when determining various properties of a system and when computing its solutions.

We study the problems of regularization ("invertibilization") of the leading matrix of the system R, by considering a system

$(\bar{R}(z) = 0, \mathcal{C}), \quad \bar{R} = FR,$

which is equivalent to $(R(z) = 0, \emptyset)$ and such that the leading matrix $\bar{B}_0(n)$ of $\bar{R}$ is invertible in $\mathrm{Mat}_m(K(n))$.

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which is equivalent to $(R(z) = 0, \emptyset)$ and such that the leading matrix $\bar{B}_0(n)$ of $\bar{R}$ is invertible in $\mathrm{Mat}_m(K(n))$.

Linear constraints

In many cases, it is natural to consider the system (7) together with a finite set of *linear constraints*, i.e. linear relations, each of which contains a set of variables $z_i(j)$ with constant coefficients. Let $R(z) = 0$ and $R'(z) = 0$ be systems of the form (7) and C and C' be finite sets of linear constraints.
We then say that the systems $(R(z) = 0, C)$ and $(R'(z) = 0, C')$ are equivalent if the space of solutions of $R(z) = 0$ that satisfy C is the same than the space of solutions of $R'(z) = 0$ that satisfy C'.

Finitely many linear constraints are easily taken into account when determining various properties of a system and when computing its solutions.

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Algorithms for P1 and P2

If $\det \bar{B}_0(n) = 0$ has no integer roots then the original differential system has no non-zero Laurent solution. Then $-\infty$ and $\{0\}$ are solutions of **P1** and **P2**.

Otherwise, let e^*, e_* be the maximal and the minimal integer roots of this equation. Then $l_0 = e^*$ is a solution of **P1**.

Let $\mathcal{C}$ be the set of the linear constraints, and $n_{\mathcal{C}}$ be the maximal of values of indices n in $\mathcal{C}$. If $l \geq \max\{l_0, n_{\mathcal{C}}\}$ then a basis for the space $V_S^{\langle l \rangle}$ can be computed with the linear algebraic system of the equations

$$
\begin{array}{l}
\bar{B}_0(e_*)z(e_*) = 0, \\
\bar{B}_0(e_*+1)z(e_*+1) + \bar{B}_{-1}(e_*+1)z(e_*) = 0, \\
\bar{B}_0(e_*+2)z(e_*+2) + \bar{B}_{-1}(e_*+2)z(e_*+1) + \bar{B}_{-2}(e_*+2)z(e_*) = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
\bar{B}_0(l)z(l) + \bar{B}_{-1}(l)z(l-1) + \quad \dots \quad + \bar{B}_{e_*-l}(l)z(e_*) = 0,
\end{array}
$$

and of all constraints from $\mathcal{C}$. This solves **P2**.

If a given l is such that $l_0 \leq l < \max\{l_0, n_{\mathcal{C}}\}$ then **P2** can be solved for $\tilde{l} = \max\{l_0, n_{\mathcal{C}}\}$ followed by the l-truncation of the result.

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\dots\dots\dots\dots\dots\dots\dots\dots \\
\bar{B}_0(l)z(l) + \bar{B}_{-1}(l)z(l-1) + \quad \dots \quad + \bar{B}_{e_*-l}(l)z(e_*) = 0,
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Implementation

Of course, we are not able to work directly with infinite series and recurrent systems of infinite order. Some kind of lazy computation is used in our algorithms and their implementation.

The algorithms are implemented in Maple. The implementation is partially based on the implementation of algorithm EG' for the finite case (2003). In our implementation the matrix form for representing operators and systems is used.

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Example of Laurent series solution computation

Let L be the operator

$$\begin{pmatrix} \theta^3 + \sum\limits_{k=2k_0+2}^{\infty} (k\theta + k^2) x^k & \sum\limits_{k=2k_0+2}^{\infty} 3k\theta x^k & \sum\limits_{k=2k_0+2}^{\infty} (2\theta + k) x^k \\ \theta^2 + \sum\limits_{k=2k_0+2}^{\infty} (k\theta^2 + 3) x^k & x^{k_0}\theta^2 + \sum\limits_{k=2k_0+2}^{\infty} (\theta + k) x^k & (x^{2k_0} + x^{2k_0+1})\theta^2 + \sum\limits_{k=2k_0+2}^{\infty} (k\theta + k) x^k \\ \theta + \sum\limits_{k=2k_0+2}^{\infty} (k+1) x^k & \theta + \sum\limits_{k=2k_0+2}^{\infty} k\theta^2 x^k & x^{k_0}\theta + \sum\limits_{k=2k_0+2}^{\infty} (k\theta^2 + 3k) x^k \end{pmatrix}$$

and let $k_0 = 2$.

The system to be solved represented in Maple as the following:

```
> k0:=2:L:=Matrix([[k->piecewise(k=0,theta^3,k<2*k0+2,0,k*theta+k^2),k->piecewise(k<2*k0+2,0,3*k*
  theta),k->piecewise(k<2*k0+2,0,2*theta+k)],[k->piecewise(k=0,theta^2,k<2*k0+2,0,k*theta^2+3),
  k->piecewise(k=k0,theta^2,k<2*k0+2,0,theta+k),k->piecewise(k=2*k0,theta^2,k=2*k0+1,theta^2,k<2*
  k0+2,0,k*theta+k)],[k->piecewise(k=0,theta,k<2*k0+2,0,k+1),k->piecewise(k=0,theta,k<2*k0+2,0,
  theta^2*k),k->piecewise(k=k0,theta,k<2*k0+2,0,theta^2+3*k)]]);
```

$$L := \Big[\Big[k\rightarrow piecewise\big(k=0,\theta^3,k<2\,k0+2,0,k\,\theta+k^2\big), k\rightarrow piecewise\big(k<2\,k0+2,0,3\,k\,\theta\big), k\rightarrow piecewise\big(k<2\,k0+2,0,2\,\theta+k\big)\Big],$$
$$\Big[k\rightarrow piecewise\big(k=0,\theta^2,k<2\,k0+2,0,k\,\theta^2+3\big), k\rightarrow piecewise\big(k=k0,\theta^2,k<2\,k0+2,0,\theta+k\big), k\rightarrow piecewise\big(k=2\,k0,\theta^2,k=2\,k0+1,\theta^2,k<2\,k0+2,0,k\,\theta+k\big)\Big],$$
$$\Big[k\rightarrow piecewise\big(k=0,\theta,k<2\,k0+2,0,k+1\big), k\rightarrow piecewise\big(k=0,\theta,k<2\,k0+2,0,k\,\theta^2\big), k\rightarrow piecewise\big(k=k0,\theta,k<2\,k0+2,0,\theta^2+3\,k\big)\Big]\Big]$$

```
> [map(apply,L,0),map(apply,L,1),map(apply,L,2),map(apply,L,3),map(apply,L,4),map(apply,L,5),map
  (apply,L,k) assuming (k>=6)];
```

$$\left[\begin{bmatrix}\theta^3&0&0\\\theta^2&0&0\\\theta&\theta&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&\theta^2&0\\0&0&\theta\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&\theta^2\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&\theta^2\\0&0&0\end{bmatrix}, \begin{bmatrix}k\,\theta+k^2&3\,k\,\theta&2\,\theta+k\\k\,\theta^2+3&\theta+k&k\,\theta+k\\k+1&k\,\theta^2&\theta^2+3\,k\end{bmatrix}\right]$$

Note: The matrix entry with indices i, j is a function of an integer argument (e.g. k) which is computing the coefficient of x^k in the operator $L_{i,j}$.

In this case the functions are defined via `piecewise`, but might be also defined as special procedures with any complicated algorithm to compute the coefficients.

The second block of the session above shows the matrix coefficients of x^0, x^1, x^2, x^3, x^4, x^5 and x^k with $k \geq 6$ in the operator L.

Solve **P2** for $l = 1$ – compute $V_S^{\langle 1 \rangle}$:

```
> LaurentSolution(L, theta, x,1);
```

Trace of computation

RS-sequence:

$$[[[\ ], [[2, [-1\ n\ 0], \{0\}]]], [[2,2], [[3, [-n-2\ \ -n\ \ (n+2)\,n^2], \{-2, 0\}]]], [[3,3], [\ ]]]$$

The invertible leading matrix:

$$\begin{bmatrix} n^3 & 0 & 0 \\ 0 & (n+2)\,n^2 & 0 \\ 0 & 0 & 2\,(n+2)^3 - 8\,n - 16 \end{bmatrix}$$

Roots of its determinant:

$$-4, -2, 0$$

$$l_0 = 0, s = 4, l_1 = 9$$

Constraints:

$$table([0 = \{[0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 7, 216, 54, 8, 343, 70, 9, 512, 88, 10, 729, 108], [0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 16, 0, 0, 25, 219, 0, -30, 346, 0, -42, 515, 0, -56, 732, 0, -72]\}, -2 = \{[-2, -2, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 0, 7, 384, 82, 8, 567, 102, 9, 800, 124, 10, 1089, 148]\}])$$

Solution before applying constraints:

$$\left[\_c_1 + O(x^2), \frac{3547}{312}\frac{\_c_3}{x^2} + \frac{8\,\_c_3}{x} + \_c_2 - \frac{157}{6}\,x\,\_c_3 + O(x^2), \frac{\_c_3}{x^4} - \frac{8}{3}\frac{\_c_3}{x^3} + \frac{41}{4}\frac{\_c_3}{x^2} - \frac{157}{6}\frac{\_c_3}{x} + \_c_4 + \frac{445181}{3120}\,x\,\_c_3 + O(x^2)\right]$$

End of computation

$$\left[\_c_1 + O(x^2), \_c_2 + O(x^2), \_c_3 + O(x^2)\right]$$

Solve **P2** for $l = 3$ – compute $V_S^{\langle 3 \rangle}$:

```
> LaurentSolution(L, theta, x,3);
```

Trace of computation

RS-sequence:

$$\left[\left[[\,], \left[\left[2, \left[\begin{array}{ccc} -1 & n & 0 \end{array}\right], \{0\}\right]\right]\right], \left[[2,2], \left[\left[3, \left[\begin{array}{ccc} -n-2 & -n & (n+2)\,n^2 \end{array}\right], \{-2, 0\}\right]\right]\right], [[3,3], [\,]]\right]$$

The invertible leading matrix:

$$\left[\begin{array}{ccc} n^3 & 0 & 0 \\ 0 & (n+2)\,n^2 & 0 \\ 0 & 0 & 2\,(n+2)^3 - 8\,n - 16 \end{array}\right]$$

Roots of its determinant:

$$-4, -2, 0$$

$$l_0 = 0, s = 4, l_1 = 11$$

Constraints:

$table([0 = \{[0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 216, 54, 8, 343, 70, 9, 512, 88, 10, 729, 108, 11, 1000, 130, 12, 1331, 154], [0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 16, 0, 0, 25, 219, 0, -30, 346, 0, -42, 515, 0, -56, 732, 0, -72, 1003, 0, -90, 1334, 0, -110]\}, -2 = \{[-2, -2, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 384, 82, 8, 567, 102, 9, 800, 124, 10, 1089, 148, 11, 1440, 174, 12, 1859, 202]\}])$

Solution before applying constraints:

$$\left[\_c_1 + \frac{1}{4}x^2\,\_c_3 + \frac{1}{27}x^3\,\_c_3 + O(x^4), \frac{3547}{312}\frac{\_c_3}{x^2} + \frac{8\,\_c_3}{x} + \_c_2 - \frac{157}{6}x\,\_c_3 - \frac{69}{4}x^2\,\_c_3 - \frac{1008983}{28080}x^3\,\_c_3 + O(x^4), \frac{\_c_3}{x^4} - \frac{8}{3}\frac{\_c_3}{x^3} + \frac{41}{4}\frac{\_c_3}{x^2} - \frac{157}{6}\frac{\_c_3}{x} + \_c_4 + \frac{445181}{3120}x\,\_c_3 + x^2\left(\frac{5}{4}\_c_4 - \frac{14989337}{37440}\_c_3 - \frac{3}{4}\_c_1 + \frac{3}{2}\_c_2\right) + x^3\left(\frac{13}{6}\_c_2 + \frac{11}{6}\_c_4 - \frac{33750113}{393120}\_c_3 - \frac{7}{6}\_c_1\right) + O(x^4)\right]$$

End of computation

$$\left[\_c_1 + O(x^4), \_c_2 + O(x^4), \_c_3 + x^2\left(\frac{5}{4}\_c_3 - \frac{3}{4}\_c_1 + \frac{3}{2}\_c_2\right) + x^3\left(\frac{13}{6}\_c_2 + \frac{11}{6}\_c_3 - \frac{7}{6}\_c_1\right) + O(x^4)\right]$$

The following solutions of **P2** were found:

$$V_S^{\langle 1\rangle} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$V_S^{\langle 3\rangle} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 + (\frac{5}{4}c_3 - \frac{3}{4}c_1 + \frac{3}{2}c_2)x^2 + (\frac{13}{6}c_2 + \frac{11}{6}c_3 - \frac{7}{6}c_1)x^3 \end{pmatrix}.$$

In the course of computation **P1** was solved ($l_0 = 0$), and an upper bound for the solution of **P3** ($s = 4$) was found as well.

"Parasitic" solutions were discarded with the linear constraints. Two lower roots (-4,-2) of the determinant of the invertible leading matrix did not correspond to the valuations of Laurent solutions in this example, only the maximal root (0) did.