On valuations of meromorphic solutions of arbitrary-order linear difference systems with polynomial coefficients

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$k \subset \mathbb{C}, \quad k[x], \quad k(x), \quad \mathrm{Mat}_m(k[x]), \quad \mathrm{Mat}_m(k(x))$

We consider systems of the form

$$A_r(x)y(x+r) + \cdots + A_1(x)y(x+1) + A_0(x)y(x) = b(x), \qquad (1)$$

where

- $A_0(x), A_1(x), \ldots, A_r(x) \in \mathrm{Mat}_m(k[x])$ with the assumption that the leading and trailing matrices $A_r(x), A_0(x)$ are nonzero,
- $b(x) = (b_1(x), b_2(x), \ldots, b_m(x))^T \in k[x]^m$ is the right-hand side of the system,
  $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions.

The number r is called the order of the system.

Let the homogeneous system S' be obtained by dropping the right-hand side of the original system S of the form (1). We assume that equations of S' are independent over $k[x, \phi]$, where ϕ is the shift operator:

$$\phi(y(x)) = y(x+1).$$

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One can consider analytical and, in particular, meromorphic solutions of the system (1). For a meromorphic function $f(x)$ and $\alpha \in \mathbb{C}$, the valuation $\mathrm{val}_{x-\alpha} f(x)$ is defined as the lowest degree of $x - \alpha$ for which the Laurent series expansion of $f(x)$ about the point α has a nonzero coefficient (by convention, $\mathrm{val}_{x-\alpha} 0 = \infty$).

For two meromorphic functions the following relations hold:

$$\begin{aligned} \mathrm{val}_{x-\alpha}(f(x)g(x)) &= \mathrm{val}_{x-\alpha} f(x) + \mathrm{val}_{x-\alpha} g(x), \\ \mathrm{val}_{x-\alpha}(f(x)+g(x)) &\geq \min\{\mathrm{val}_{x-\alpha} f(x),\ \mathrm{val}_{x-\alpha} g(x)\}. \end{aligned} \tag{2}$$

For a vector $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ consisting of meromorphic functions, $\mathrm{val}_{x-\alpha} y(x)$ is defined to be $\min_{i=1}^{m} \mathrm{val}_{x-\alpha} y_i(x)$.

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Remark on valuations of rational functions

The set of monic irreducible polynomials from $k[x]$ is denoted as $\mathrm{Irr}(k[x])$. If $p(x) \in \mathrm{Irr}(k[x])$ and $f(x) \in k[x]$, then $\mathrm{val}_{p(x)} f(x)$ is defined to be the greatest $n \in \mathbb{N}$ such that $p^n(x) | f(x)$, and

$$\mathrm{val}_{p(x)} F(x) = \mathrm{val}_{p(x)} f(x) - \mathrm{val}_{p(x)} g(x) \tag{3}$$

for $F(x) = \frac{f(x)}{g(x)}$, $f(x), g(x) \in k[x]$.

If $F(x) \in k(x)$, $p(x) \in \mathrm{Irr}(k[x])$, $\alpha \in \bar{k}$ and $p(\alpha) = 0$, then evidently

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There is a significant difference between the solution spaces of linear ordinary differential and linear ordinary difference systems: the solutions of the latter may be multiplied not only by constants, but also by functions with the period equal to 1.

Along with a meromorphic solution $y(x)$ the system has also, for example, solutions $(\sin 2\pi(x+\beta))y(x)$ and $(\sin 2\pi(x+\beta))^{-1}y(x)$ for any $\beta \in \mathbb{C}$.

The singularities of solutions of a differential system similar to (1) with equations which are independent over $k[x, \frac{d}{dx}]$ constitute a finite set.

The situation is different for difference systems and even for scalar difference equations with polynomial coefficients — it is enough to mention the gamma function, which satisfies the scalar equation $y(x+1) - xy(x) = 0$.

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We take an interest in two problems.

The **first** problem is the computation of a lower bound for

$$\mathrm{val}_{x-\alpha} y(x), \quad \alpha \in \mathbb{C}. \tag{5}$$

It is assumed that a common lower bound for the valuations

$$\mathrm{val}_{x-\alpha} y(x+n), \quad n = N, N+1, \dots, N+r-1 \tag{6}$$

is given for some integer N.
(Note that $\mathrm{val}_{x-\alpha} y(x+n) = \mathrm{val}_{x-\alpha-n} y(x)$.)

The **second** problem, a refinement of the first one, is the computation of separate lower bounds for each

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We solve the problems by means of **embracing systems**.

For any system S of the form (1) one can construct an *l-embracing* system $\bar{S}$

$$\bar{A}_r(x)y(x+r)+\cdots+\bar{A}_1(x)y(x+1)+\bar{A}_0(x)y(x)=\bar{b}(x), \tag{9}$$

with the leading matrix $\bar{A}_r(x)$ being invertible in $\mathrm{Mat}_m(k(x))$, and with the set of solutions containing all the solutions of the system S.

Similarly, one can construct a *t-embracing* system $\bar{\bar{S}}$

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The construction of the embracing systems can be performed with the algorithms EG or EG$'$; the algorithm EG$'$ is an improved version of the algorithm EG.

Let $y(x)$ be a meromomorphic solution of (1) and $\alpha \in \mathbb{C}$. Then in view of the existence of the l- and t-embracing systems, the value $\mathrm{val}_{x-\alpha} y(x+n)$ is bounded from below when n runs through $\mathbb{Z}$:

Let $p(\alpha) = 0$ for $p(x) \in \mathrm{Irr}(k[x])$. Let $\bar{A}_r(x)$ be the leading matrix of an l-embracing system and $\bar{\bar{A}}_0(x)$ be the trailing matrix of a t-embracing system for (1). Let

$$V(x) = \det \bar{A}_r^{-1}(x-r), \quad W(x) = \det \bar{\bar{A}}_0^{-1}(x)$$

and $N_0, N_1 \in \mathbb{Z}$ be such that $p(x) \nmid V(x+n)W(x+n)$ for all $n \geq N_0$ and all $n \leq N_1$.
Then there exist $\lambda, \mu \in \mathbb{Z}$ such that

$$\min_{n=n_0}^{n_0+r-1} \mathrm{val}_{x-\alpha} y(x+n) = \lambda \text{ for all integer } n_0 \geq N_0,$$

and

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The first problem of computing lower bounds: computing a lower bound on $\mathrm{val}_{x-\alpha}y(x)$

Theorem. *The following inequality holds for any mutual disposition of the point α and the roots of the polynomials $W(x), V(x)$:*

$$\mathrm{val}_{x-\alpha}y(x) \geq \max\left\{\lambda - \sum_{n\in\mathbb{N}} \mathrm{val}_{p(x)}V(x-n),\right.$$
$$\left.\mu - \sum_{n\in\mathbb{N}} \mathrm{val}_{p(x)}W(x+n)\right\} \qquad (11)$$

(the sums in the right-hand side of the inequality are finite).

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The second problem of computing lower bounds: computing lower bounds on

$$\mathrm{val}_{x-\alpha} y_i(x), \quad i = 1, 2, \ldots, m, \tag{12}$$

assuming that for some non-negative integer n_0 lower bounds on the valuations

$$\mathrm{val}_{x-\alpha} y_i(x-n)$$

are given (separately for $n = n_0, n_0+1, \ldots, n_0+r-1$ and $i = 1, 2, \ldots, m$), or that for some non-negative integer n_1 lower bounds on the valuations

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A2$_M$: operating on matrices with entries from $k(x)$

If $M(x) \in \mathrm{Mat}_m(k(x))$ and $1 \le i \le m$ then the minimum of valuations of the i-th row entries of a matrix $M(x)$ will be denoted by $\mathrm{val}^{(i)}_{x-\alpha} M(x)$.

The algorithm of M. van Hoeij for finding denominator bounds for rational solutions of a system of the form

$$y(x+1) = A(x)y(x) \tag{13}$$

where $A(x) \in \mathrm{Mat}_m(k(x))$ is an invertible matrix, is based on the following observation.

If a meromorphic solution $y(x)$ is such that $\mathrm{val}_{x-\alpha} y(x - n_0) \ge 0$ for a non-negative integer n_0, then for any $1 \le i \le m$ we have

$$\mathrm{val}_{x-\alpha} y_i(x) \ge \mathrm{val}^{(i)}_{x-\alpha} \left(A(x-1)A(x-2)\dots A(x-n_0)\right). \tag{14}$$

Similarly, if $\mathrm{val}_{x-\alpha} y(x + n_1) \ge 0$ for a non-negative integer n_1, then for any $1 \le i \le m$ we have

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Using l- and t-embracing systems, the algorithm of M. van Hoeij can be generalized to estimate the valuations of components of meromorphic solutions of a system of the form (1). But the computational complexity of such an approach would be high since the entries of the matrix product "swell" quickly when the number of factors grows.

Below we describe an algorithm which is applicable to arbitrary-order systems of the form (1), and is based on the so-called tropical operations on matrices with entries from $\mathbb{Z} \cup \{\infty\}$, rather than on the costly operations on matrices with entries from $k(x)$.

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A2$_T$: tropical calculations

We will consider the set $\mathbb{Z}^\circ = \mathbb{Z} \cup \{\infty\}$ with operations

$$a \odot b = a + b, \quad a \oplus b = \min\{a, b\}, \tag{16}$$

which replace the usual operations of multiplication $\cdot$ and addition $+$. The neutral element for $\odot$ is 0, and ∞ plays the analogous role for $\oplus$. Both operations are associative, and $\odot$ is distributive over $\oplus$.

The operations (16) can be extended to matrices and vectors with entries from $\mathbb{Z}^\circ$.

Let $p(x) \in \mathrm{Irr}(k[x])$ be fixed. For an arbitrary function $f(x) \in k(x)$ we consider the double-sided sequence

$$f^\circ(n) = \mathrm{val}_{p(x)} f(x+n), \quad n = 0, \pm 1, \pm 2, \ldots \tag{17}$$

of elements of $\mathbb{Z}^\circ$. Similarly, for an arbitrary matrix $A(x) \in \mathrm{Mat}_m(k(x))$ we consider the matrix $A^\circ(n)$ whose entries are sequences of the mentioned form. The same for rational function vectors.

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The resulting l- and t-embracing systems for the original system (1) might be rewritten in the form

$$y(x) = B_1(x)y(x-1) + \cdots + B_r(x)y(x-r) + \varphi(x), \tag{18}$$

$$y(x) = C_1(x)y(x+1) + \cdots + C_r(x)y(x+r) + \psi(x). \tag{19}$$

Theorem. *Let the components of a vector* $v(n) = (v_1(n), v_2(n), \ldots, v_m(n))^T$ *be sequences of elements of* $\mathbb{Z}^\circ$. *Let* $y(x)$ *be a meromorphic solution of a system of the form (18), and* $\mathrm{val}_{x-\alpha} y_i(x+n) \geq v_i(n)$ *for* $n = n_0, n_0 - 1, \ldots, n_0 - r + 1$, *where* $n_0 \in \mathbb{Z}$ *is such that if* $n > n_0$ *then the equation*

$$v(n) = B_1^\circ(n) \odot v(n-1) \oplus \cdots \oplus B_r^\circ(n) \odot v(n-r) \oplus \varphi^\circ(n) \qquad (20)$$

holds. Then $\mathrm{val}_{x-\alpha} y_i(x+n) \geq v_i(n)$ *for all* $n > n_0$.

Similarly, let $w(n) = (w_1(n), w_2(n), \ldots, w_m(n))^T$ *be a vector of sequences of elements of* $\mathbb{Z}^\circ$. *Let* $y(x)$ *be a meromorphic solution of a system of the form (19), and let* $\mathrm{val}_{x-\alpha} y_i(x+n) \geq w_i(n)$ *for* $n = n_1, n_1 + 1, \ldots, n_1 + r - 1$, *where* $n_1 \in \mathbb{Z}$ *is such that if* $n < n_1$ *then the equation*

$$w(n) = C_1^\circ(n) \odot w(n+1) \oplus \cdots \oplus C_r^\circ(n) \odot w(n+r) \oplus \psi^\circ(n) \qquad (21)$$

holds. Then $\mathrm{val}_{x-\alpha} y_i(x+n) \geq w_i(n)$ *for all* $n < n_1$.

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Our implementation: experiments

We use the first-order system given on the help page of the procedure LinearFunctionalSystems[RationalSolutions] in Maple:

$$
\begin{aligned}
\{(x+3)(x+6)(x+1)(x+5)xy_1(x+1)-(x-1)(x+2)(x+3)(x+6)(x+1)y_1(x)-& \\
-x(x^6+11x^5+41x^4+65x^3+50x^2-36)y_2(x)+6(x+2)(x+3)(x+6)(x+1)xy_4(x) &= 0, \\
(x+6)(x+2)y_2(x+1)-x^2y_2(x) &= 0, \\
(x+6)(x+1)(x+5)xy_3(x+1)+(x+6)(x+1)(x-1)y_1(x)-& \\
-x(x^5+7x^4+11x^3+4x^2-5x+6)y_2(x)-& \\
-y_3(x)(x+6)(x+1)(x+5)x+3(x+6)(x+1)x(x+3)y_4(x) &= 0, \\
(x+6)y_4(x+1)+x^2y_2(x)-(x+6)y_4(x) &= 0\}
\end{aligned}
\tag{22}
$$

For this system:
$W(x) = (x-1)(x+2)(x+3)(x+6)(x+1)(x+5)x^2$,
$V(x) = (x+1)(x+2)(x+5)x(x+4)(x-1)$.

Let $\lambda = 0, \mu = 0$ for the solutions to be found. $A1$ gives for example:

$$\begin{aligned} \mathrm{val}_{x-4} y(x) &\geq 0, \\ \mathrm{val}_{x-1} y(x) &\geq -1, \\ \mathrm{val}_{x+4} y(x) &\geq -2, \\ \mathrm{val}_{x+8} y(x) &\geq 0. \end{aligned} \tag{23}$$

If $\lambda = 0, \mu = -1$ instead then

$$\begin{aligned} \mathrm{val}_{x-4} y(x) &\geq 0, \\ \mathrm{val}_{x-1} y(x) &\geq -1, \\ \mathrm{val}_{x+4} y(x) &\geq -3, \\ \mathrm{val}_{x+8} y(x) &\geq -1. \end{aligned} \tag{24}$$

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If we know that $\mathrm{val}_{x+4} y_i(x+10) \geq 0$, $\mathrm{val}_{x+4} y_i(x-10) \geq 0$ for $i = 1, \ldots, 4$ then $A2_T$ gives

$$\begin{aligned}
\mathrm{val}_{x+4} y_1(x) &\geq -2, \\
\mathrm{val}_{x+4} y_2(x) &\geq -1, \\
\mathrm{val}_{x+4} y_3(x) &\geq -2, \\
\mathrm{val}_{x+4} y_4(x) &\geq -1.
\end{aligned} \tag{25}$$

$A2_M$ gives the same result. It is more accurate than the bound with $A1$, which gives $\mathrm{val}_{x+4} y_i(x) \geq -2$ for each i.
But $A1$ took 0.093 sec., $A2_T$ took 0.405 sec., and $A2_M$ took 0.967 sec.

The system under consideration has rational solutions:

$$
\begin{aligned}
y_1(x) &= \frac{4(-7108272c_2+c_1)}{(x-1)(x+2)(x+3)(x+4)},\\
y_2(x) &= 0,\\
y_3(x) &= \frac{(5x^5c_2+50x^4c_2+175x^3c_2+250x^2c_2-35541240xc_2+5xc_1-28433088c_2+4c_1)}{5x(x+1)(x+2)(x+3)(x+4)},\\
y_4(x) &= 0.
\end{aligned}
$$

For this solution

$$
\begin{aligned}
\operatorname{val}_{x+4} y_1(x) &= -1,\\
\operatorname{val}_{x+4} y_2(x) &= \infty,\\
\operatorname{val}_{x+4} y_3(x) &= -1,\\
\operatorname{val}_{x+4} y_4(x) &= \infty.
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$$\begin{aligned} y_1(x) &= \frac{4(-7108272c_2+c_1)}{(x-1)(x+2)(x+3)(x+4)}, \\ y_2(x) &= 0, \\ y_3(x) &= \frac{(5x^5c_2+50x^4c_2+175x^3c_2+250x^2c_2-35541240xc_2+5xc_1-28433088c_2+4c_1)}{5x(x+1)(x+2)(x+3)(x+4)}, \\ y_4(x) &= 0. \end{aligned}$$

For this solution

$$\begin{aligned} \mathrm{val}_{x+4}\, y_1(x) &= -1, \\ \mathrm{val}_{x+4}\, y_2(x) &= \infty, \\ \mathrm{val}_{x+4}\, y_3(x) &= -1, \\ \mathrm{val}_{x+4}\, y_4(x) &= \infty. \end{aligned} \tag{26}$$

It is interesting that when the problem of bounding the valuation is an auxiliary task for solving another problem, it can happen that the bounds obtained with $A2_T$ (or $A2_M$), although more accurate than the bounds obtained with $A1$, save no computation time, or even lead to additional costs when used on the further steps in solving the main problem.

For example, our experiments which use the valuation bounding as an auxiliary task for computing rational solutions of systems of the form (1), show that this phenomenon occurs for the system (22). This is related to the fact that the more accurate bounds for denominators of the desired rational solutions, obtained by means of $A2_T$ (or $A2_M$), in this case yield a system whose polynomial solutions take longer to find on the next step than those of the system resulting by using the less accurate bounds obtained with $A1$.

Nevertheless, in most cases the more accurate bounds lead to shorter overall running times, which is why efficient computation of more accurate bounds is of practical value for this problem as well.

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Nevertheless, in most cases the more accurate bounds lead to shorter overall running times, which is why efficient computation of more accurate bounds is of practical value for this problem as well.

Let us consider an example of a system of higher order. We modify system (22) by shifting some of the equations ($x \to x+1$ in the first and second equations, $x \to x+4$ in the fourth equation).

$$
\begin{aligned}
\{(x+4)(x+7)(x+2)(x+6)(x+1)y_1(x+2) - x(x+3)(x+4)(x+7)(x+2)y_1(x+1) - &\\
-(x+1)(132+520x+x^6+17x^5+111x^4+359x^3+616x^2)y_2(x+1) + &\\
+6(x+3)(x+4)(x+7)(x+2)(x+1)y_4(x+1) &= 0,\\
(x+7)(x+3)y_2(x+2) - (x+1)^2 y_2(x+1) &= 0,\\
(x+6)(x+1)(x+5)xy_3(x+1) + (x+6)(x+1)(x-1)y_1(x) - &\\
-x(x^5+7x^4+11x^3+4x^2-5x+6)y_2(x) - y_3(x)(x+6)(x+1)(x+5)x + &\\
+3(x+6)(x+1)x(x+3)y_4(x) &= 0,\\
(x+10)y_4(x+5) + (x+4)^2 y_2(x+4) - (x+10)y_4(x+4) &= 0\}
\end{aligned}
$$

For this system:

$W(x) = (x-1)(x+3)(x+1)(x+5)(x+6)(x+2)x^2$,

$V(x) = (x+2)x(x+4)(x-1)(x+5)(x+1)$.

Let $\lambda = 0, \mu = 0$ for the solutions to be found. $A1$ and $A2_T$ give the same results as above (i.e., $A2_T$ gives a more accurate bound than $A1$).

But $A1$ took 0.125 sec., while $A2_T$ took 0.515 sec. ($A2_M$ is not applicable in this case).

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Consider the system

$$\begin{aligned}\{y_1(x+1) - (x+2)y_2(x+1) - (x+1)y_1(x) + (x+1)y_2(x) &= 0,\\ y_1(x+1) + (x+2)y_2(x+1) - (x+1)y_1(x) - (x+1)y_2(x) &= 0\}\end{aligned}$$

Its solution:

$$\begin{aligned}y_1(x) &= c_1\Gamma(x+1),\\ y_2(x) &= \frac{c_2}{x+1}.\end{aligned}$$

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y_1(x+1) + (x+2)y_2(x+1) - (x+1)y_1(x) - (x+1)y_2(x) &= 0\}
\end{aligned}
$$

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Let $\lambda = 0, \mu = 0$ for the solutions to be found. $A1$ gives, for example:

$$\begin{aligned} \mathrm{val}_{x-2} y(x) &\geq 0, \\ \mathrm{val}_{x+1} y(x) &\geq -1, \\ \mathrm{val}_{x+2} y(x) &\geq 0. \end{aligned} \tag{27}$$

If $\lambda = 0, \mu = -1$ instead then

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If we know that $\mathrm{val}_{x-2} y_1(x+10) \geq 0$, $\mathrm{val}_{x-2} y_2(x+10) \geq 0$, $\mathrm{val}_{x-2} y_1(x-10) \geq -1$, $\mathrm{val}_{x-2} y_2(x-10) \geq 0$. Then $A2_T$ gives

$$\begin{array}{rcl} \mathrm{val}_{x-2} y_1(x) & \geq & 0, \\ \mathrm{val}_{x-2} y_2(x) & \geq & 0. \end{array}$$

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First-order systems containing five equations, whose coefficients are polynomials with random integer roots from $[-9, 9]$ were generated; the bounds in two fixed points with the same a-priori known bounds are computed for each of the systems.

The generation approach leads to non-zero valuation bounds for almost all generated system, e.g. for $\lambda = 0$, $\mu = 0$.

The total time taken by each of the algorithms to compute bounds in two fixed points for 10 generated systems:

	Total time
$A1$	3.719
$A2_T$	20.061
$A2_M$	790.020

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The systems of order $r > 1$ containing five equations, whose coefficients are polynomials with random integer roots from $[-9, 9]$ were generated; the bounds in two fixed points with the same a-priori known bounds are computed for each of the systems (the same as the previous experiment, but for the systems of higher order).

When the order of the systems generated in this way grows, the chance to obtain non-zero valuation bounds decreases; for non-zero bounds, $A2_T$ turns out to produce more accurate results than $A1$.

The total time taken by each of the algorithms to compute bounds in two fixed points for 10 generated systems of each of the orders $r = 2, 6, 10$:

	r=2	r=6	r=10
$A1$	8.766	12.155	13.954
$A2_T$	56.294	181.676	288.379

Computation time of $A2_T$ on systems of order r grows linearly with r, provided that all the other size-related parameters are fixed.

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When the order of the systems generated in this way grows, the chance to obtain non-zero valuation bounds decreases; for non-zero bounds, $A2_T$ turns out to produce more accurate results than $A1$.

The total time taken by each of the algorithms to compute bounds in two fixed points for 10 generated systems of each of the orders $r = 2, 6, 10$:

	r=2	r=6	r=10
$A1$	8.766	12.155	13.954
$A2_T$	56.294	181.676	288.379

Computation time of $A2_T$ on systems of order r grows linearly with r, provided that all the other size-related parameters are fixed.

The systems of order $r > 1$ containing five equations, whose coefficients are polynomials with random integer roots from $[-9, 9]$ were generated; the bounds in two fixed points with the same a-priori known bounds are computed for each of the systems (the same as the previous experiment, but for the systems of higher order).

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