Power Algebra for Power Geometry

Power Geometry for Computers

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Lotka-Volterra Equations

V.I. Arnold, Ordinary Differential Equations, Springer-Verlag, 1992.

$$\frac{dx}{dt} = kx - axy,$$
$$\frac{dy}{dt} = -ly + bxy.$$

where t - independent variable, x, y - dependent variables, $k, a, l, b > 0$ - parameters.

The first integral is:

$$y^k e^{-ay} x^l e^{-bx} = C$$

Rewrite equations in polynomial form

$$-\frac{dx}{dt} + kx - axy = 0,$$
$$-\frac{dy}{dt} - ly + bxy = 0.$$

Power Series Substitution

We find solutions of Lotka-Volterra system in form of Puiseux series with finite nonzero principal part

$$x(t) = t^{u_1}(x_0 + \sum_{j=1}^{\infty} x_j t^{j\Delta}),$$
$$y(t) = t^{u_2}(y_0 + \sum_{j=1}^{\infty} y_j t^{j\Delta}),$$

where coefficients $x_0, y_0 \neq 0$, powers $u_1, u_2 < 0$ - rational, $\Delta > 0$ - rational.
After substitution of series to source equations they are transformed to series

$$t^{v_1}(c_{1,0} + \sum_{j=0}^{\infty} c_{1,j} t^{j\Delta}) = 0,$$
$$t^{v_2}(c_{2,0} + \sum_{j=0}^{\infty} c_{2,j} t^{j\Delta}) = 0,$$

coefficients x_0, x_j, y_0, y_j are solutions of equations $c_{1,0}(x_0, y_0) = 0$, $c_{1,j}(x_j, y_j) = 0$, $c_{2,0}(x_0, y_0) = 0$, $c_{2,j}(x_j, y_j) = 0$. Powers v_1, v_2 we consider later.

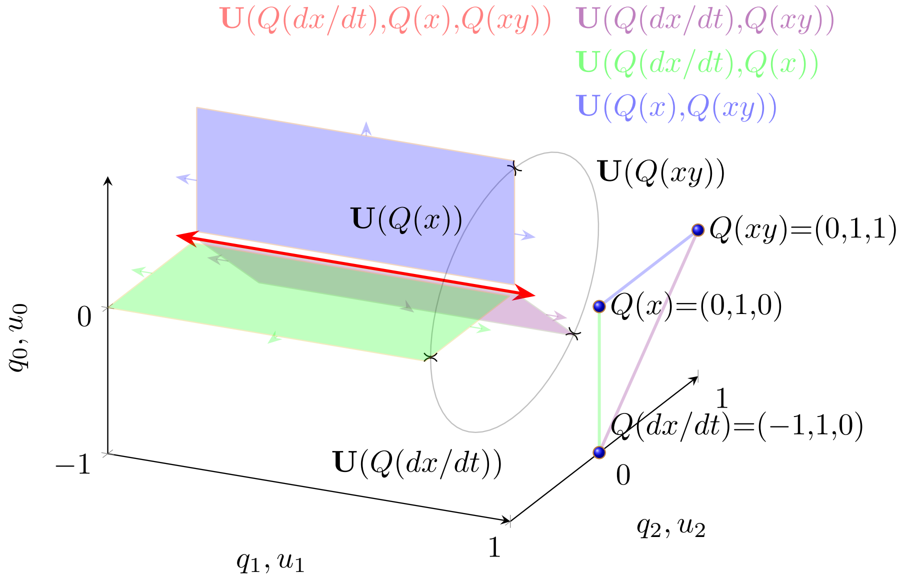
Powers and Coefficients of minimal power Terms

At first we substitute terms $x_0 t^{u_1}, y_0 t^{u_2}$ to Lotka-Volterra system and every monomial of source equations is transformed to power function. Power of that function is the power of the source monomial

$$-u_1 x_0 t^{u_1 - 1} + k x_0 t^{u_1} - a x_0 y_0 t^{u_1 + u_2},$$
$$-u_2 y_0 t^{u_2 - 1} - l y_0 t^{u_2} + b x_0 y_0 t^{u_1 + u_2}.$$

Different sets of power functions with minimal power $v_1 = \min(u_1 - 1, u_1, u_1 + u_2)$, $v_2 = \min(u_2 - 1, u_2, u_1 + u_2)$ corresponds to different u_1, u_2 values.

Newton Polyhedron and Normal Cones



System of Inequalities

Conditions for variables u_i, v_i, $i = 1, 2$ we can write in form of weak and strict inequalities

$$\begin{array}{llllll} 1) & -1 + u_1 \ge v_1 & 4) & -1 + u_2 \ge v_2 & & \\ 2) & u_1 \ge v_1 & 5) & u_2 \ge v_2 & 7) & u_1 < 0 \\ 3) & u_1 + u_2 \ge v_1 & 6) & u_1 + u_2 \ge v_2 & 8) & u_2 < 0 \end{array}$$

and assign a unique number for each inequality. By power geometry terminology of A.D. Bruno vectors of constants and coefficients of variables u_i in left side of inequalities are *vectorial power exponents* of monomials and nequalities 7),8) are *task cone*.

Powers and Coefficients of minimal power Terms

To find solutions for these inequalities we rewrite ones in homogenious form

1) $-1 + u_1 - v_1 \geq 0$
2) $u_1 - v_1 \geq 0$
3) $u_1 + u_2 - v_1 \geq 0$
4) $-1 + u_2 - v_2 \geq 0$
5) $u_2 - v_2 \geq 0$
6) $u_1 + u_2 - v_2 \geq 0$
7) $u_1 < 0$
8) $u_2 < 0$

Then we introduce a vector of projective coordinates $U = (\hat{u}_0, \hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2)$, where $\hat{u}_1 = u_1\hat{u}_0$, $\hat{u}_2 = u_2\hat{u}_0$, $\hat{v}_1 = v_1\hat{u}_0$, $\hat{v}_2 = v_2\hat{u}_0$ and vector $\mathbf{0} = (0, 0, 0, 0, 0)$.

Powers and Coefficients of minimal power Terms

Then we introduce matrix Q of coefficients of homogeneous inequalities and write inequalities in matrix form

$$UQ \leq \mathbf{0}, \quad \text{where } Q = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Columns of matrix Q coincide with inequalities. Appended column is condition $\hat{u}_0 < 0$ for reverse inequality sign. The first row of the matrix Q is constants in the left side of inequalities. Next two rows are coefficients of variables u_1, u_2. Next two rows are coefficients of variables v_1, v_2.

Linear, Convex and Cone Hulls

Lunear Hull of vectors set

$$\sum_{j=1}^{m} \nu_j U_j, \quad \sum_{j=1}^{m} \nu_j = 1$$

Convex Hull of vectors set

$$\sum_{j=1}^{m} \mu_j U_j, \quad \sum_{j=1}^{m} \mu_j = 1, \quad \mu_j > 0$$

Cone Hull of vectors set

$$\sum_{j=1}^{m} \eta_j U_j, \quad \eta_j > 0$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(\ \ 0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(\ \ 0,1,0,1,1)$	0	0	0	–	–	0	0	–	0
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0
$(\ \ 0,0,0,0,1)$	0	0	0	–	–	–	0	0	0
$(-1,0,0,1,1)$	0	–	–	0	–	–	–	0	0
$(\ \ 0,0,0,1,0)$	–	–	–	0	0	0	0	0	0

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(1,-1,-1,-2,-2)$	0	–	0	0	–	0	–	–	–
$(\ \ 0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(\ \ 0,1,0,1,1)$	0	0	0	–	–	0	0	–	0
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0

$$x_0 t^{-2} + k x_0 t^{-1} - a x_0 y_0 t^{-2},$$
$$y_0 t^{-2} - l y_0 t^{-1} + b x_0 y_0 t^{-2}.$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(\ \ 0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(1,-1,-1-\eta,-2-\eta,-2-\eta)$	–	–	0	0	–	0	–	–	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(\ \ 0,1,0,1,1)$	0	0	0	–	–	0	0	–	0
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0

$$x_0 t^{-2} + k x_0 t^{-1} - a x_0 y_0 t^{-2-\eta},$$
$$(1+\eta) y_0 t^{-2-\eta} - l y_0 t^{-1} + b x_0 y_0 t^{-2-\eta}.$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0
$(1,-1,-\mu,-2,-1-\mu)$	0	–	–	0	–	0	–	–	–
$(0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(0,1,0,1,1)$	0	0	0	–	–	0	0	–	0

$$x_0 t^{-2} + k x_0 t^{-1} - a x_0 y_0 t^{-1-\mu},$$
$$\mu y_0 t^{-1-\mu} - l y_0 t^{-\mu} + b x_0 y_0 t^{-1-\mu}.$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(0,1,0,1,1)$	0	0	0	–	–	0	0	–	0
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0
$(1,-1-\eta,-\mu,-2-\eta,-1-\mu-\eta)$	0	–	–	–	–	0	–	–	–
$(0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–

$$(1+\eta)x_0 t^{-2-\eta} + k x_0 t^{-1-\eta} - a x_0 y_0 t^{-1-\mu-\eta},$$
$$\mu y_0 t^{-1-\mu} - l y_0 t^{-\mu} + b x_0 y_0 t^{-1-\mu-\eta}.$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0
$(1,\mu_1-1,\mu_2-1,\mu_1-2,\mu_2-2)$	0	–	–	0	–	–	–	–	–
$(\ \ 0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(\ \ 0,1,0,1,1)$	0	0	0	–	–	0	0	–	0

$$(1-\mu_1)x_0 t^{-2+\mu_1} + k x_0 t^{-1+\mu_1} - a x_0 y_0 t^{-2+\mu_1+\mu_2},$$
$$(1-\mu_2)y_0 t^{-2+\mu_2} - l y_0 t^{-1+\mu_2} + b x_0 y_0 t^{-2+\mu_1+\mu_2}.$$

	$(-1,1,0,-1,0)$	$(0,1,0,-1,0)$	$(0,1,1,-1,0)$	$(-1,0,1,0,-1)$	$(0,0,1,0,-1)$	$(0,1,1,0,-1)$	$(0,-1,0,0,0)$	$(0,0,-1,0,0)$	$(1,0,0,0,0)$
$(-1,1,1,2,2)$	0	–	0	0	–	0	–	–	–
$(\ 0,0,1,1,1)$	–	–	0	0	0	0	0	0	–
$(\ 0,1,0,1,1)$	0	0	0	–	–	0	0	–	0
$(1,-1-\eta_1,-1-\eta_2,$ $-2-\eta_1-\eta_2,-2-\eta_1-\eta_2)$	–	–	0	–	–	0	–	–	–
$(-1,0,1,1,2)$	0	–	0	0	–	–	–	0	–
$(-1,1,0,2,1)$	0	–	–	0	–	0	–	–	0

$$(1+\eta_1)x_0t^{-2-\eta_1}+kx_0t^{-1-\eta_1}-ax_0y_0t^{-2-\eta_1-\eta_2},$$
$$(1+\eta_2)y_0t^{-2-\eta_2}-ly_0t^{-1-\eta_2}+bx_0y_0t^{-2-\eta_1-\eta_2}.$$

Powers and Coefficients of minimal power Terms

For solution $(-1,1,1,2,2)$ powers $u_1 = u_2 = 1/-1 = -1$ and $v_1 = v_2 = 2/-1 = -2$. For these values u_i, v_i

$$-(-1)x_0 t^{-1-1} - a x_0 y_0 t^{-1} = x_0(1 - a y_0)t^{-2} = 0 \Rightarrow y_0 = 1/a,$$
$$-(-1)y_0 t^{-1-1} + b x_0 y_0 t^{-1} = y_0(1 + b x_0)t^{-2} = 0 \Rightarrow x_0 = -1/b$$

Next Terms

To calculate next terms we substitute to source equations first two terms of the solution expansion with already calculated powers and coefficients

$$x = -t^{-1}/b + x_1 t^{-1+\Delta}, \quad y = t^{-1}/a + y_1 t^{-1+\Delta},$$

and reduce similar terms. In result

$$\begin{aligned} &(ay_1/b - x_1\Delta)t^{-2+\Delta} - kt^{-1}/b + kx_1 t^{-1+\Delta} - ax_1 y_1 t^{-2+2\Delta}, \\ &(bx_1/a - y_1\Delta)t^{-2+\Delta} - lt^{-1}/a - ly_1 t^{-1+\Delta} + bx_1 y_1 t^{-2+2\Delta}. \end{aligned}$$

Powers $-2+\Delta < -2+2\Delta$ and $-1 < -1+\Delta$, so we consider terms with powers $-2+\Delta$ and -1 only.

Next Terms

If $0 < \Delta < 1$ then coefficients x_1, y_1 are solutions of the homogenious linear algebraic equations system

$$-\Delta x_1 + (a/b) y_1 = 0,$$
$$(b/a) x_1 - \Delta y_1 = 0,$$

but this system doesn't have solutions if $\Delta \neq \pm 1$.
If $\Delta = 1$ then coefficients x_1, y_1 are solutions of the linear algebraic equations system

$$-x_1 + (a/b) y_1 - k/b = 0,$$
$$(b/a) x_1 - y_1 - l/a = 0,$$

This system has solution $y_1 = (bx_1 - l)/a$, where x_1 is arbitrary coefficient, with condition $k = -l$ that contradict to condition $k, l > 0$. Condition $k = -l$ we call *expandability condition* into Puiseux series. Solution of Lotka-Volterra system is expandable to Puiseux series with not allowed conditions for parameters.

Chasy Equation

$$\frac{d^3y}{t^3} - 2y\frac{d^2y}{t^2} + 3\frac{dy}{dt}^2 = 0$$

$$y = y_0 t^u, \quad u < 0, \quad y_0 \neq 0$$

$$u(u-1)(u-2)y_0 t^{u-3} + u(u+2)y_0^2 t^{2u-2}$$

$$y = y_0/t^2 - 6/t + y_2, \quad y_2 = 0$$

$$y = -6/t + y_1/t^2 + y2/t^3, \quad y_2 = 0$$

$$y = -6/t + y_1, \quad y_1 = 0$$

Euler-Poisson Equations

V.V. Golubev, Lectures on Integration of Equations Motion of a Heavy Rigid Body near Fixed Point, Moscow: GITTL, 1953.

$$A\frac{dp}{dt} = (B-C)qr - Mg(z_0\gamma_2 - y_0\gamma_3),$$
$$B\frac{dq}{dt} = (C-A)rp - Mg(x_0\gamma_3 - z_0\gamma_1),$$
$$C\frac{dr}{dt} = (A-B)pq - Mg(y_0\gamma_1 - x_0\gamma_2),$$
$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3,$$
$$\frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_1,$$
$$\frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2,$$

Parameters of Euler-Poisson Equations

where t - time, A, B, C - principal moments of inertia, which satisfy triangle inequalities

$$A > 0,\ B > 0,\ C > 0,$$
$$A + B \geq C,\ A + C \geq B,\ B + C \geq A,$$

Mg - the body weight, x_0, y_0, z_0 - coordinates of the center of gravity of the rigid body in the body frame, p, q, r - projections of the angular velocity vector onto the body frame axes, $\gamma_1, \gamma_2, \gamma_3$ - direction cosines of the vertical in the body frame.

$$Ap^2 + Bq^2 + Cr^2 - 2Mg(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \mathrm{const},$$
$$Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \mathrm{const},$$
$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.$$

These are energy, momentum and geometry integrals.

We take a system of units where $Mg = 1$.

Known Solutions

$x_0 = y_0 = z_0 = 0$ Euler,
$x_0 = y_0 = 0,\ A = B$ Lagrange,
$y_0 = z_0 = 0,\ A = B = 2C$ S.Kowalevski,
$y_0 = z_0 = 0,\ A = 2C,\ A < B < 3A$ Bobylev-Steklov,
$y_0 = z_0 = 0,\ (A - 2B)(A - 2C) < 0$ Steklov,
$y_0 = z_0 = 0,\ A = 16C(C - B)/(8C - 9B)$ Goryachev,
$y_0 = z_0 = 0,\ B = 4A(2C - A)/(17C - 8A)$ Konosevich-Pozdnyakovich,
$y_0 = z_0 = 0,\ A = 18C(C - B)/(9C - 10B)$ N.Kowalevski, Dokshevich,
$y_0 = z_0 = 0,\ C = 9A(2B - A)/(2(16B - 9A))$ Chaplygin,
$y_0 = z_0 = 0,\ A = B = 4C$ Goryachev-Chaplygin,
$y_0 = 0,\ x_0\sqrt{A(B - C)} = z_0\sqrt{C(A - B)},\ A > B > C$ Hess-Appelrot, Dokchevich,
$z_0 = 0,\ p(t) = q(t) = \gamma_3(t) = 0$ Mlodzievskii,
$y_0 = 0,\ x_0\sqrt{B - C} = z_0\sqrt{A - B},\ A > B > C$ Grioly

I.N. Gashenenko, G.V. Gorr, A.M. Kovalev, Classical Problems of the Dynamics of a Rigid Body, Kiev, Naukova Dumka, 2012.

Power Substitution

We find solutions of Euler-Poisson system in form of Puiseux series with finite nonzero principal part

$$\begin{gathered}
p(t) = t^{u_1}(p_0 + \textstyle\sum_{j=1}^{\infty} p_j t^{j\Delta}), \\
q(t) = t^{u_2}(q_0 + \textstyle\sum_{j=1}^{\infty} q_j t^{j\Delta}), \\
r(t) = t^{u_3}(r_0 + \textstyle\sum_{j=1}^{\infty} r_j t^{j\Delta}), \\
\gamma_1(t) = t^{u_4}(\gamma_{1,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{1,j} t^{j\Delta}), \\
\gamma_2(t) = t^{u_5}(\gamma_{2,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{2,j} t^{j\Delta}), \\
\gamma_3(t) = t^{u_6}(\gamma_{3,0} + \textstyle\sum_{j=1}^{\infty} \gamma_{3,j} t^{j\Delta}),
\end{gathered}$$

where coefficients $p_0, q_0, r_0, \gamma_{1,0}, \gamma_{2,0}, \gamma_{3,0} \neq 0$, powers $u_1, u_2, u_3, u_4, u_5, u_6 < 0$ - rational, $\Delta > 0$ - rational.

Powers and Coefficients of minimal power Terms

Ssubstitute terms $p_0 t^{u_1}$, $q_0 t^{u_2}$, $r_0 t^{u_3}$, $\gamma_{1,0} t^{u_4}$, $\gamma_{2,0} t^{u_5}$, $\gamma_{3,0} t^{u_6}$ to Euler-Poisson system

$$\begin{gathered}
-Au_1 p_0 t^{u_1-1} + (B-C) q_0 r_0 t^{u_2+u_3} + y_0 \gamma_{3,0} t^{u_6} - z_0 \gamma_{2,0} t^{u_5},\\
-Bu_2 q_0 t^{u_2-1} + (C-A) p_0 r_0 t^{u_1+u_3} + z_0 \gamma_{1,0} t^{u_4} - x_0 \gamma_{3,0} t^{u_6},\\
-Cu_3 r_0 t^{u_3-1} + (A-B) p_0 q_0 t^{u_1+u_2} + x_0 \gamma_{2,0} t^{u_5} - y_0 \gamma_{1,0} t^{u_4},\\
-u_4 \gamma_{1,0} t^{u_4-1} + r_0 \gamma_{2,0} t^{u_3+u_5} - q_0 \gamma_{3,0} t^{u_2+u_6},\\
-u_5 \gamma_{2,0} t^{u_5-1} + p_0 \gamma_{3,0} t^{u_1+u_6} - r_0 \gamma_{1,0} t^{u_3+u_4},\\
-u_6 \gamma_{3,0} t^{u_6-1} + q_0 \gamma_{1,0} t^{u_2+u_4} - p_0 \gamma_{2,0} t^{u_1+u_5},\\
Ap_0^2 t^{2u_1} + Bq_0^2 t^{2u_2} + Cr_0^2 t^{2u_3} - 2x_0 \gamma_{1,0} t^{u_4} - 2y_0 \gamma_{2,0} t^{u_5} - 2z_0 \gamma_{3,0} t^{u_6} - ht^0,\\
Ap_0 \gamma_{1,0} t^{u_1+u_4} + Bq_0 \gamma_{2,0} t^{u_2+u_5} + Cr_0 \gamma_{3,0} t^{u_3+u_6} - lt^0,\\
\gamma_{1,0}^2 t^{2u_4} + \gamma_{2,0}^2 t^{2u_5} + \gamma_{3,0}^2 t^{2u_6} - 1t^0.
\end{gathered}$$

S.Kowalevski case

$z_0 = 0,\ x_0, y_0 \neq 0,\ B = A,$
$u_1 = u_2 = u_3 = -1,\ u_4 = u_5 = -2,\ u_6 = -2 + \eta,\ \eta > 0$
$v_1 = v_2 = v_3 = -2,\ v_4 = v_5 = v_6 = -3,\ v_7 = -2,\ v_8 = -3,\ v_9 = -4$

$$\begin{gathered}
Ap_0 + (A - C)q_0 r_0 = 0,\\
Aq_0 + (C - A)p_0 r_0 = 0,\\
Cr_0 + x_0\gamma_{2,0} - y_0\gamma_{1,0} = 0,\\
2\gamma_{1,0} + r_0\gamma_{2,0} = 0,\\
2\gamma_{2,0} - r_0\gamma_{1,0} = 0,\\
q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0,\\
Ap_0^2 + Bq_0^2 + Cr_0^2 - 2x_0\gamma_{1,0} - 2y_0\gamma_{2,0} = 0,\\
Ap_0\gamma_{1,0} + Bq_0\gamma_{2,0} = 0,\\
\gamma_{1,0}^2 + \gamma_{2,0}^2 = 0.
\end{gathered}$$

Solution
$A = 2C,\ r_0 = 2i,\ q_0 = p_0 i,\ \gamma_{1,0} = -2C/(x_0 + y_0 i),\ \gamma_{2,0} = -2Ci/(x_0 + y_0 i)$

Goryachev-Chaplygin case

$y_0 = z_0 = 0,\ x_0 \neq 0,\ B = A,$
$u_1 = u_2 = -1 - \eta_1,\ u_3 = -1,\ u_4 = u_5 = -2,\ u_6 = -2 + \eta_1 + \eta_2,\ \eta_1, \eta_2 > 0$
$v_1 = v_2 = -2 - \eta_1,\ v_3 = -2,\ v_4 = v_5 = -3,\ v_6 = -3 - \eta_1,$
$v_7 = -2(1 - \eta_1),\ v_8 = -3 - \eta_1,\ v_9 = -4$

$$\begin{gathered} Ap_0 + (A - C)q_0 r_0 = 0, \\ Aq_0 + (C - A)p_0 r_0 = 0, \\ Cr_0 + x_0\gamma_{2,0} = 0, \\ 2\gamma_{1,0} + r_0\gamma_{2,0} = 0, \\ 2\gamma_{2,0} - r_0\gamma_{1,0} = 0, \\ q_0\gamma_{1,0} - p_0\gamma_{2,0} = 0, \\ Ap_0^2 + Bq_0^2 = 0, \\ Ap_0\gamma_{1,0} + Bq_0\gamma_{2,0} = 0, \\ \gamma_{1,0}^2 + \gamma_{2,0}^2 = 0. \end{gathered}$$

Solution $A = 2C/(1 - \eta_1),\ r_0 = 2i,\ q_0 = p_0 i,\ \gamma_{1,0} = -2C/x_0,$
$\gamma_{2,0} = -2Ci/x_0,\ \eta_1 = 1/2,\ \eta_2 = 1$

$\mathbf{U} = (-1, -1, -1, -2, -2, -2)$, $y_0 = z_0 = 0$

$$u_1 = -1,\ u_2 = -1,\ u_3 = -1,\ u_4 = -2,\ u_5 = -2,\ u_6 = -2.$$

$$\begin{array}{lll} v_1 = -2, & v_4 = -3, & v_7 = -2, \\ v_2 = -2, & v_5 = -3, & v_8 = -3, \\ v_3 = -2, & v_6 = -3, & v_9 = -4, \end{array}$$

$d = 4/k,\ k = 1, 2, \dots$
$k = 2,\ d = 2,\ (A - 2B)(A - 2C) < 0$ – Steklov Solution
$k > 2,\ d < 2,\ B = A(A - 2C)/((d(d-1) - 4)C + 2A)$
$k = 3,\ d = 4/3,\ C = 9A(2B - A)/(2(16B - 9A))$ – Chaplygin Solution
$k = 8,\ d = 1/2,\ B = 4A(2C - A)/(17C - 8A)$ –
Konosevich-Pozdnyakovich Solution

$\mathbf{U} = (2-3\mu, 2-3\mu, -1, -2, -2, 1-3\mu)$, $y_0 = z_0 = 0$

$$u_1 = 2-3\mu,\ u_2 = 2-3\mu,\ u_3 = -1,\ u_4 = -2,\ u_5 = -2,\ u_6 = 1-3\mu,$$
$$2/3 < \mu < 1.$$

$$\begin{array}{lll} v_1 = 1-3\mu, & v_4 = -2, & v_7 = -2, \\ v_2 = 1-3\mu, & v_5 = -2, & v_8 = -3\mu, \\ v_3 = -2, & v_6 = -3\mu, & v_9 = -4, \end{array}$$

$d = 4/k,\ k = 1, 2, \ldots$
$A = 16C(C-B)/((d(d-2)-8)B + 8C)$
$k = 4,\ d = 1,\ \mu = 5/6,\ A = 16C(C-B)/(8C-9B)$ – Goryachev Solution
$k = 6,\ d = 2/3,\ \mu = 8/9,\ A = 18C(C-B)/(9C-10B)$ – N.Kowalevski Solution

New Cases of Expansibility

$z_0 = 0,\ x_0, y_0 \neq 0,\ B = A$
$z_0 = 0,\ x_0, y_0 \neq 0,\ C = B$