Parameter estimation in ODE models: data interpolation, differential algebra, polynomial system solving

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Implementation:
https://github.com/iliailmer/ParameterEstimation.jl

Plan

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- Toy example to explain approach

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- Outline of approach

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- What can/needs to be proven

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- Software demo

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- What can/needs to be proven
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- Future directions

Toy example

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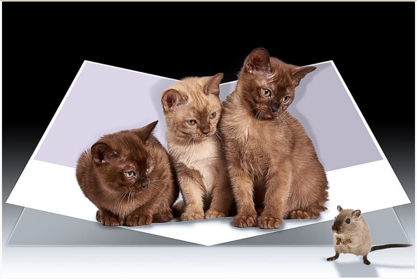
Challenge: what are $y'(0)$ and $y''(0)$?

Example

$$\begin{cases} x_1' = k_1 x_1 - k_2 x_1 x_2 \\ x_2' = -k_3 x_2 + k_4 x_1 x_2 \end{cases}$$



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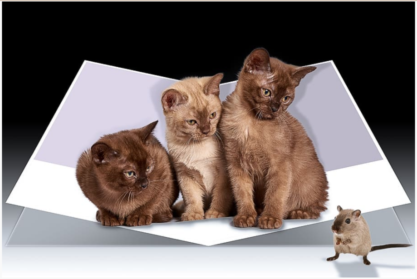

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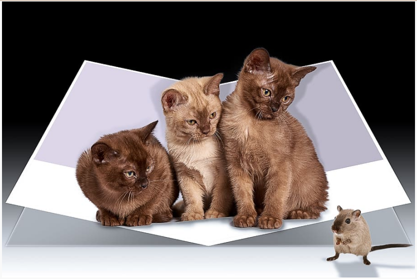

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Which of the parameters can we find from $x_1(t)$?

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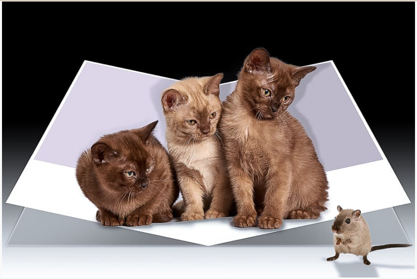

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Can be checked using software:

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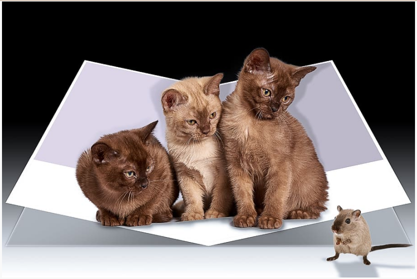

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Which of the parameters can we find from $x_1(t)$?

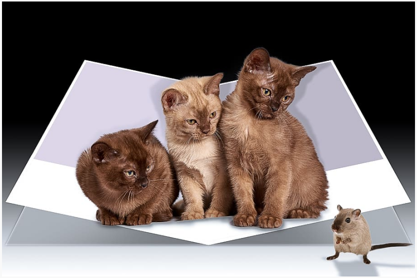
Can be checked using software:

Yes: k_1, k_3, k_4

No: k_2

Example - modified

$$\begin{cases} x_1' = k_1 x_1 - k_2 x_1 x_2 \\ x_2' = -k_2 x_2 + k_3 x_1 x_2 \end{cases}$$



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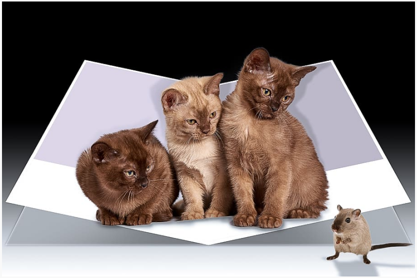

- x_1 - prey
- x_2 - predators

We can now in principle find all parameters

Flowchart

Input:

Model: $\begin{cases} x' &= -\mu x \\ y &= x^2 + x \end{cases}$

Data: $\{(0.000, 2.000), (0.333, 1.563), (0.666, 1.229), (1.000, 0.974)\}$

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↓ 1) Find # of parameter values

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↓ 2) Differentiate

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Differentiated system

$$y = x^2 + x$$
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Numerical polynomial system:

$$2.00 \approx x_0^2 + x_0$$
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Solutions:

$$S_1 = \begin{cases} \mu &= 0.499 \\ x_0 &= 1.000 \end{cases}, S_2 = \begin{cases} \mu &= 0.249 \\ x_0 &= -2.000 \end{cases}$$

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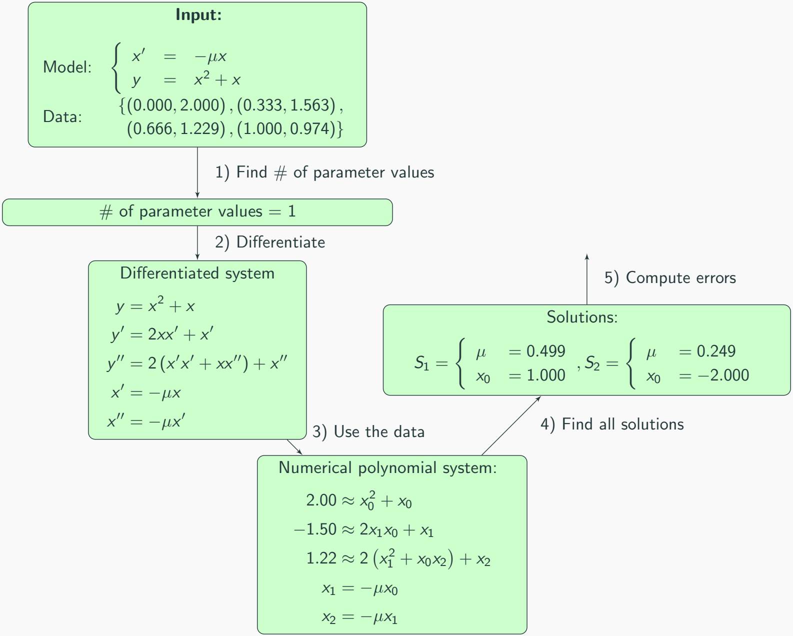
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Errors:

$$S_1 \Rightarrow 6.87 \cdot 10^{-4},$$
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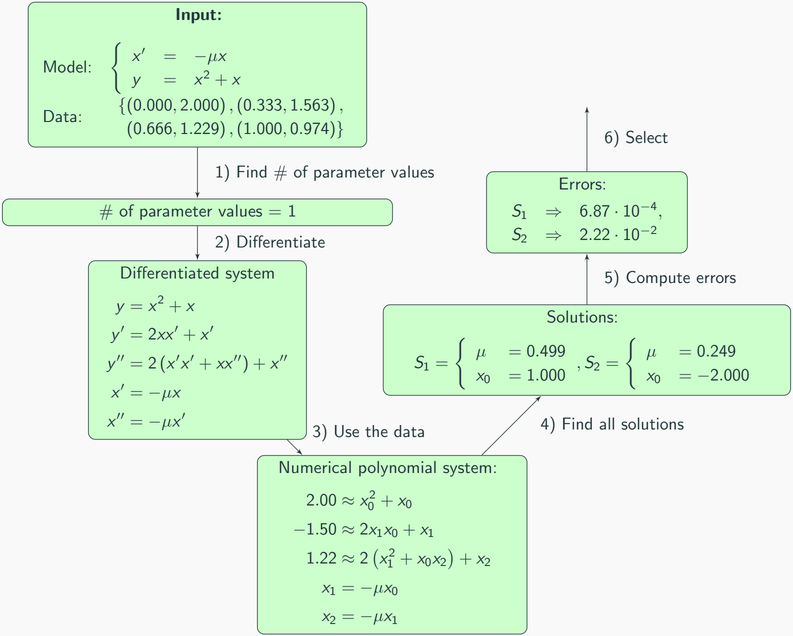
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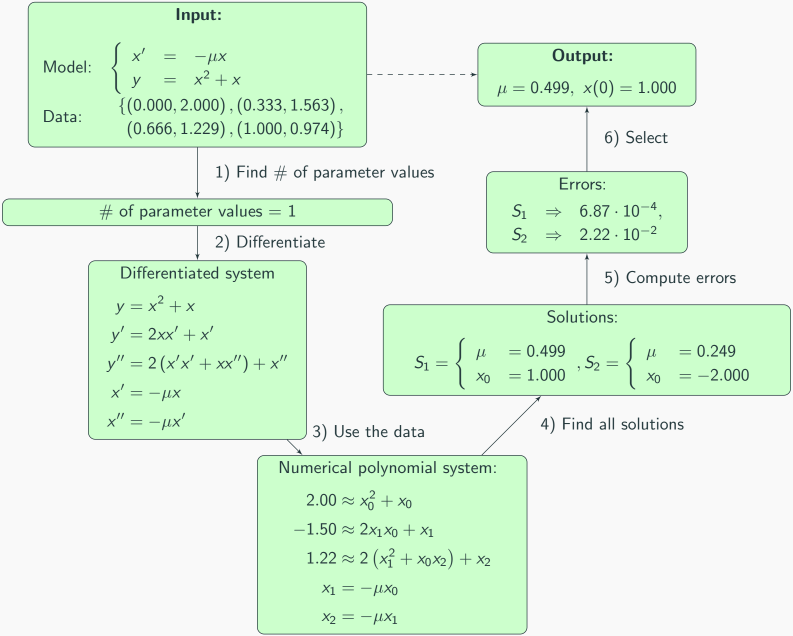

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Using software in Julia

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```julia
using ParameterEstimation
using ModelingToolkit

# Input:
# -- Differential model
@parameters mu
@variables t x(t) y(t)
D = Differential(t)
@named Sigma = ODESystem([D(x) ~ -mu * x],
                         t, [x], [mu])
outs = [y ~ x^2 + x]

# -- Data
data = Dict(
  "t"      => [0.000, 0.333, 0.666, 1.000],
  x^2 + x => [2.000, 1.563, 1.229, 0.974])

# Run
res = estimate(Sigma, outs, data);
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The software returns this result:

```
# Output:
Parameter(s)        : mu   = 0.499
Initial Condition(s): x(t) = 1.000
```

Performance

Mean of Relative Errors in %

Software		IQM			SciML			AMIGO2			ParameterEstimation.jl
Search Range		[0,1]	[0,2]	[0,3]	[0,1]	[0,2]	[0,3]	[0,1]	[0,2]	[0,3]	Any
Models	Harmonic, eq. (2)	66.3	68.7	101.9	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	Van der Pol, eq. (4)	7.5	0.0	9.6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	FitzHugh-Nagumo, eq. (6)	82.4	131.1	210.3	8.3	15.9	20.4	0.0	0.0	0.2	0.0
	HIV, eq. (8)	85.6	104.2	144.7	14.8	36.9	102.9	18.5	43.2	28.4	0.0
	Mammillary 3, eq. (10)	76.1	104.3	128.2	11.9	21.5	24.7	0.0	0.0	0.0	0.0
	Lotka-Volterra, eq. (12)	72.2	75.4	75.4	54.7	76.9	111.7	13.6	112.6	29.7	0.0
	Crauste, eq. (14)	99.6	125.2	179.7	47.3	72.9	226.4	2.6	105.7	51.7	0.0
	Biohydrogenation, eq. (16)	93.7	184.0	130.3	77.7	151.1	306.6	17.8	13.5	28.3	0.0
	Mammillary 4, eq. (18)	94.1	109.6	105.3	66.2	67.7	118.8	29.8	56.7	59.0	0.1
	SEIR, eq. (20)	132.3	230.8	338.9	18.4	60.6	118.1	25.9	40.7	50.8	0.0

HIV model

HIV infection dynamics during interaction with immune system in various treatments.

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HIV infection dynamics during interaction with immune system in various treatments.

$$\begin{cases} \dot{x} = \lambda - d\,x - \beta\,x\,v \\ \dot{y} = \beta\,x\,v - a\,y \\ \dot{v} = k\,y - u\,v \\ \dot{w} = c\,x\,y\,w - c\,q\,y\,w - b\,w \\ \dot{z} = c\,q\,y\,w - h\,z \end{cases}$$

$$\begin{cases} y_1 = w,\ \ y_2 = z \\ y_3 = x,\ \ y_4 = y + v \end{cases}$$

What can be proven?

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Solutions:

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$\mu = 0.499,\ x(0) = 1.000$

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- Improve quality of derivative estimates

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