On full rank differential systems with power series coefficients

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Abstract

We consider the following problem: given a linear ordinary differential system of arbitrary order with formal power series coefficients, decide whether the system has non-zero Laurent series solutions, and find all such solutions if they exist (in a truncated form preserving the space dimension). If the series coefficients of the original systems are represented algorithmically then these problems are algorithmically undecidable. However, it turns out that they are decidable in the case when we know in advance that a given system is of full rank.

We define the width of a given full rank system S with formal power series coefficients as the smallest non-negative integer w such that any l-truncation of S with $l \geqslant w$ is a full rank system. We prove that the value w exists for any full rank system and can be found algorithmically.

We propose corresponding algorithms and their Maple implementation, and report some experiments.

Keywords: differential system, series coefficients, Laurent series solutions, system width

1. Introduction and definitions

Let K be a field of characteristic 0. We denote by $K[[x]]$ the ring of formal power series with coefficients in K and $K((x)) = K[[x]][x^{-1}]$ its quotient field; the elements of $K((x))$ are Laurent series. For a nonzero element $a(x) = \sum a_i x^i$ of $K((x))$ the *valuation* $\mathrm{val}_x a(x)$ is defined by $\mathrm{val}_x a(x) = \min\{i \,:\, a_i \neq 0\}$. By convention $\mathrm{val}_x 0 = \infty$. For $l \in \mathbb{Z} \cup \{-\infty\}$, the *$l$-truncation* $a^{\langle l \rangle}(x)$ is obtained by vanishing all the coefficients of the terms of degree larger than l in the series

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$a(x)$ (if $l = -\infty$ then obviously $a^{\langle l \rangle}(x) = 0$). If $M(x)$ is a matrix or a vector with entries belonging to $K((x))$ then we define $\mathrm{val}_x M(x)$ as the minimum of the valuations of the entries of $M(x)$. We define $M^{\langle l \rangle}(x)$ as the matrix or vector whose entries are the l-truncations of the corresponding entries of $M(x)$. The notation A^T is used for the transpose of a matrix (vector) A. The ring of square matrices of order m with entries belonging to a ring R is denoted by $\mathrm{Mat}_m(R)$.

We write θ for $x\frac{d}{dx}$ and consider differential systems of the form

$$A_r(x)\theta^r y + A_{r-1}(x)\theta^{r-1} y + \cdots + A_0(x)y = 0 \tag{1}$$

where $y = (y_1, y_2, \ldots, y_m)^T$ is a column vector of unknown "functions" (e.g., in the form of Laurent series) of x. For the coefficient matrices

$$A_0(x), A_1(x), \ldots, A_r(x) \tag{2}$$

we have $A_i(x) \in \mathrm{Mat}_m(K[[x]])$, $i = 0, 1, \ldots, r$, and $A_r(x)$ is non-zero. The number r is the *order* of the system. We suppose that the entries of matrices (2) are represented *algorithmically*: for any entry $a(x)$ an algorithm Λ_a such that $a(x) = \sum_{i=0}^{\infty} \Lambda_a(i)x^i$ is given (thus, due to the classical results of A. Turing (Turing (1936)) we are not able, in general, to recognize whether a given series is equal to zero or not). We suppose also that the system is of *full rank*, i.e., its equations are linearly independent over $K((x))[\theta]$. We suppose finally that at least one power series coefficient in each equation of the system has a non-zero constant term, i.e., is of valuation 0.

For S be a system of the form (1), we define V_S as the space of Laurent series solutions of S, and $V_S^{\langle l \rangle}$ as the space whose elements are the l-truncations of the corresponding elements of V_S (thus $V_S^{\langle l \rangle}$ consists of Laurent polynomials). It will be shown that there exists $l_0 \in \mathbb{Z} \cup \{-\infty\}$ such that the l_0-truncation mapping

$$V_S \to V_S^{\langle l_0 \rangle} \tag{3}$$

is bijective (if $V_S = \{0\}$ then, e.g., $-\infty$ can be taken as l_0).

For a given system S of the form (1) of full rank we are concerned with three problems. The first two of them are as follows:

P1. Compute $l_0 \in \mathbb{Z} \cup \{-\infty\}$ such that the l_0-truncation mapping (3) is bijective (i.e., it preserves the solution space dimension).

P2. Let l_0 be as in **P1** and $l \geqslant l_0$, construct a basis for $V_S^{\langle l \rangle}$.

The third problem will be formulated below in this section.

Note that we are not able to check algorithmically whether or not a given system is of full rank (this follows from (Abramov et al., 2011, Prop. 2)). However if we know *in advance* that a given system of the form (1) is of full rank then our algorithms completely solve the problems **P1**, **P2**. The analogous problems for the case when we know in advance that the leading matrix $A_r(x)$ of (1) is invertible in $\mathrm{Mat}_m(K((x)))$ (this supposition is stronger than the supposition that the system is of full rank) were solved in Abramov et al. (2011).

Let a system S be of the form (1), $l \in \mathbb{Z} \cup \{-\infty\}$, and define the l-truncation $S^{\langle l \rangle}$ as the system

$$A_r^{\langle l \rangle}(x)\theta^r y + A_{r-1}^{\langle l \rangle}(x)\theta^{r-1} y + \cdots + A_0^{\langle l \rangle}(x) y = 0.$$

If S is of full rank then the minimal integer w such that $S^{\langle l \rangle}$ is of full rank for all $l \geqslant w$ is called the *width* of S. The third problem we propose an algorithm to solve, is

P3. Compute the width of S.

We prove existence of the width for an arbitrary full rank system (Theorem 2). It is not true that if $S^{\langle l \rangle}$ is of full rank then $S^{\langle l+1 \rangle}$ is of full rank (see Example 1). So it is not enough to look for the first l such that $S^{\langle l \rangle}$ is of full rank.

Our algorithms for solving **P1**, **P2** and **P3** are based on "reduction + shift" steps applied to the induced recurrence system, which will be defined in Section 2. A system S of the form (1) has a Laurent series solution $y(x) = z(v)x^v + z(v+1)x^{v+1} + \dots$ iff the sequences of vector coefficients $z(n)$ of $y(x)$ satisfies the induced recurrent system. In the case when power series entries of the coefficient matrices (2) are infinite, the induced recurrent system is of infinite order.

The system representation form (1) is natural and convenient for presentation of our algorithms. At the same time, for algorithms correction proving we will also use in Section 3 the operator form, considering the ring $\mathrm{Mat}_m(K((x)))[\theta]$ together with the isomorphic matrix ring $\mathrm{Mat}_m(K((x))[\theta])$. We prove Theorem 1 that the sequence of "reduction + shift" steps always terminates and gives a recurrent system with the leading matrix invertible in $\mathrm{Mat}_m(K(n))$. The determinant of this matrix can be considered as a kind of indicial equation of the original differential system.

Some supplementary properties of full rank operators from $\mathrm{Mat}_m(K[[x]])[\theta]$ and the corresponding induced recurrent operators are formulated in Theorem 3.

Since we are not able to work directly with recurrent systems of infinite order, a lazy computation strategy is used in our algorithms and their implementation (Sections 6, 8).

2. Induced recurrent systems

2.1. Sequences of coefficients of Laurent series solutions

Let E denote the shift operator: $Ez(n) = z(n+1)$ for any sequence $z(n)$. The mapping

$$x \to E^{-1}, \quad x^{-1} \to E, \quad \theta \to n \tag{4}$$

transforms an original differential system S into the *induced* recurrent system which has the form

$$B_0(n)z(n) + B_{-1}(n)E^{-1}z(n) + \cdots = 0 \tag{5}$$

or $B_0(n)z(n) + B_{-1}(n)z(n-1) + \cdots = 0$, where

- $z(n) = (z_1(n), \dots, z_m(n))^T$ is a column vector of unknown sequences such that $z_i(n) = 0$ for all negative integers n with $|n|$ large enough, $i = 1, 2, \dots, m$.

- $B_0(n), B_{-1}(n), \dots \in \mathrm{Mat}_m(K[n])$, each of polynomial entries of these matrices is of degree less than or equal to r.

- $B_0(n)$ is a non-zero matrix, it is called the *leading* matrix of the system (5).

The mapping (4) produces the ring isomorphism

$$\mathcal{M}: \ \mathrm{Mat}_m(K((x)))[\theta] \to \mathrm{Mat}_m(K[n])((E^{-1})) \tag{6}$$

(the proof is analogous to the proof given in (Abramov et al., 2000, Sect. 5) for the case of scalar operators with polynomial coefficients). It is evident that the original differential system S is of full rank iff the induced system (5) is of full rank (i.e., the equations of (5) are independent over $K[n][[E^{-1}]]$). A system S of form (1) has a Laurent series solution $y(x) = z(v)x^v + z(v+1)x^{v+1} + \dots$ iff the double-sided sequences

$$\dots, 0, 0, z(v), z(v+1), \dots \tag{7}$$

of vector coefficients of $y(x)$ satisfies the induced recurrent system of form (5):

$$\begin{array}{l} B_0(v)z(v) = 0, \\ B_0(v+1)z(v+1) + B_{-1}(v+1)z(v) = 0, \\ B_0(v+2)z(v+2) + B_{-1}(v+2)z(v+1) + B_{-2}(v+2)z(v) = 0, \\ \dots \end{array}$$

(the proof is similar to the proof for the scalar case which is given in Abramov et al. (2000).)

If the leading matrix $B_0(n)$ is invertible in $\mathrm{Mat}_m(K(n))$ then its determinant can be considered as a kind of indicial polynomial of the original differential system S (the set of the roots of $\det B_0(n)$ is finite and contains the set of all possible valuations of Laurent series solutions of S). However in many cases this matrix is not invertible even when the leading matrix $A_r(x)$ of S is invertible in $\mathrm{Mat}_m(K((x)))$.

In the next sections we develop an algorithm that transforms an induced recurrent system of form (5) into a system having an invertible leading matrix.

2.2. Sequence of "reduction + shift" steps

Together with transforming the induced recurrent system we will transform the vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_m)$ with positive integer components. Initially $\gamma = (r, r, \dots, r)$.

A "reduction + shift" step of transformation of the recurrent system is the following:

Use any available method to check whether the rows of the leading matrix are linearly dependent over $K(n)$, and if they are, find the coefficients

$$p_1(n), p_2(n), \ldots, p_m(n) \in K[n] \tag{8}$$

of a dependence. Set

$$\mu = \max_{\substack{0 \leqslant j \leqslant m \\ p_j(n) \neq 0}} (\gamma_j + \deg p_j(n)). \tag{9}$$

Let i be such that

$$0 \leqslant i \leqslant m, \quad p_i(n) \neq 0, \quad \gamma_i + \deg p_i(n) = \mu. \tag{10}$$

Replace the i-th equation of the induced recurrent system by the linear combination of all its equations with the coefficients $p_1(n), p_2(n), \ldots, p_m(n)$. As a result, the i-th row of the leading matrix becomes zero. (This substep is called a reduction.*)*

Then apply the operator E to the i-th equation of the system which is the result of the reduction substep. (This substep is called a shift.*)*

Finally, increase γ_i by $\deg p_i(n)$, i.e. assign $\gamma_i := \mu$.

Observe that when we continue the process consisting of sequential "reduction + shift" steps we never get the zero equation in the recurrent system since the equations of the original system are linearly independent over $K[n][[E^{-1}]]$.

In Section 3.3 it will be proved that after a finite number of "reduction + shift" steps we get a system with a leading matrix whose rows are linearly independent over $K(n)$. This is the termination condition of our algorithm for transforming the induced recurrent system into a system with an invertible leading matrix.

The described process (the sequence of "reduction + shift" steps) is a version of EG-eliminations ((Abramov, 1999; Abramov & Bronstein, 2001, 2002; Abramov et al., 2003)). The new version is intended for infinite recurrent systems of the form described in Section 2.1.

In the new version the following trick (proposed originally in Abramov & Bronstein (2001)) can be used. Searching for coefficients (8) of a linear dependence of the rows of the leading matrix is equivalent to solving a homogeneous system of linear algebraic equations with polynomial coefficients. If we obtain s linearly independent solutions of the linear algebraic system then it is possible to use all of them for reductions, which yields s zero rows in the leading matrix. To do that, we first represent the s dependencies as rows of an $s \times m$ matrix $D(n)$, and use the first row of $D(n)$ to zero the i-th row of the leading matrix, and apply the operator E to the i-th equation. We then transform $D(n)$ by eliminating the i-th element in its rows having the numbers $2, 3, \ldots, s$, using the i-th element of the first row as pivot. After this elimination, each remaining row of $D(n)$ contains the coefficients of a linear dependence of the rows

$1, \ldots, i-1, i+1, \ldots, m$ of the leading matrix. So we may perform s "reduction + shift" steps. New values of μ and i are computed in each of such steps in accordance with (9) and (10); the value of γ_i is changed as well. The order in which we use the rows $D(n)$ is in fact arbitrary, so different heuristic strategies can be used to slow down the growth of degrees of system coefficients.

2.3. Linear constraints

The reduction substep can generate a set of linear constraints because of multiplications of the transformed equations by polynomials having integer roots. Suppose that we replace the i-th equation of the system by the linear combination of all equations with the coefficients $p_1(n), p_2(n), \ldots, p_m(n)$, and n_0 is an integer root of $p_i(n)$. If $y(x) = \sum_{n=v}^{\infty} z(n)x^n$, $v \leqslant n_0$, is a solution of the original differential system then we get the constraint

$$[B_0(n_0)]_{i,*}\, z(n_0) + [B_{-1}(n_0)]_{i,*}\, z(n_0-1) + \cdots + [B_{-n_0+v}(n_0)]_{i,*}\, z(v) = 0, \quad (11)$$

where the notation

$$[M]_{i,*}\,, \quad 1 \leqslant i \leqslant m,$$

is used for the $(1 \times m)$-matrix which is the i-th row of an $(m \times m)$-matrix M.

3. Operators related to systems

3.1. Row frontal matrix of differential operators

Denote the ring $\mathrm{Mat}_m(K[[x]])[\theta]$ by $\mathcal{D}_m$. System (1) can be written as $L(y) = 0$ where

$$L = A_r(x)\theta^r + A_{r-1}(x)\theta^{r-1} + \cdots + A_0(x) \in \mathcal{D}_m, \quad (12)$$

the matrix $A_r(x)$ supposed to be nonzero. We say that the operator L is of full rank if system (1) is. Note that the operator L can be also represented in the matrix form, i.e., in the form

$$\begin{pmatrix} L_{11} & \ldots & L_{1m} \\ \ldots & \ldots & \ldots \\ L_{m1} & \ldots & L_{mm} \end{pmatrix}, \quad (13)$$

where $L_{ij} \in K[[x]][\theta]$, $i, j = 1, 2, \ldots, m$, and $\max_{i,j} \operatorname{ord} L_{ij} = r$. The operator L is of full rank iff the rows of (13) are linearly independent over $K[[x]][\theta]$.

Let an operator $L \in \mathcal{D}_m$ (not necessary of full rank) be of the form (12). If $1 \leqslant i \leqslant m$ then define $\alpha_i(L)$ as the maximal integer k, $0 \leqslant k \leqslant r$, such that $[A_k(x)]_{i,*}$ is a nonzero row.

The matrix $M(x) \in \mathrm{Mat}_m(K[[x]])$ such that $[M(x)]_{i,*} = [A_{\alpha_i(L)}]_{i,*}$, $i = 1, 2, \ldots, m$, is the *row frontal matrix* of L. The vector $(\alpha_1(L), \alpha_2(L), \ldots, \alpha_m(L))$ is the *row order vector* of L. If $M(x)$ is the row frontal matrix of L then we set

$$\nu(L) = \mathrm{val}_x \det M(x)$$

($\nu(L) = \infty$ when $M(x)$ is not invertible). The row frontal matrix of any $L \in \mathcal{D}_m$ belongs to $\mathrm{Mat}_m(K[[x]])$, and thus $\nu(L) \geqslant 0$.

It is easy to check that if the row frontal matrix of an operator belonging to $\mathcal{D}_m$ is invertible then that operator is of full rank. An operator $L \in \mathcal{D}_m$ with an invertible row frontal matrix is a *row reduced* operator.

The ring $\mathrm{Mat}_m(K[n])[[E^{-1}]]$ of recurrent operators will be denoted by $\mathcal{E}_m$. Thus (6) is the isomorphism $\mathcal{M} : \mathcal{D}_m \to \mathcal{E}_m$. Using this isomorphism we can rewrite system (5) as $R(z) = 0$ with $R = \mathcal{M}(L)$,

$$R = B_0(n) + B_1(n)E^{-1} + B_1(n)E^{-2} + \ldots \tag{14}$$

In Sections 3.2, 3.3 we will prove that the process consisting of sequential "reduction + shift" steps terminates. A sketch of the proof is the following. The "reduction + shift" steps produce the sequence of operators

$$R^{(0)} = R,\ R^{(1)}, R^{(2)},\ \ldots\ \in \mathcal{E}_m \tag{15}$$

related to the appearing recurrent systems (constructing operators (15) we have to follow formulas (9), (10) for selecting the i-th row on the reduction substep). We consider in addition the sequence of differential operators

$$L^{(0)} = L = \mathcal{M}^{-1}(R),\ \ L^{(1)} = \mathcal{M}^{-1}(R^{(1)}),\ \ L^{(2)} = \mathcal{M}^{-1}(R^{(2)}),\ \ \ldots \tag{16}$$

Supposing that the leading matrix of L is invertible we prove that any $L^{(j)}$, $j = 1, 2, \ldots$, has the invertible row frontal matrix. For $L^{(j)}$ the current value of the vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is such that $\alpha_t(L^{(j)}) = \gamma_t$, $t = 1, 2, \ldots, m$, and $\nu(L^{(0)}) > \nu(L^{(1)}) > \nu(L^{(2)}) > \ldots$, that implies the termination. If the leading matrix of L is not invertible then we show that there exists $N \in \mathcal{D}_m$ such that the leading matrix of LN is invertible, and $\nu(L^{(0)}N) > \nu(L^{(1)}N) > \nu(L^{(2)}N) > \ldots$ This implies the termination for an arbitrary full rank operator L.

Observe finally that the reduction substep described in Section 2.2 is the left multiplication R by the matrix (zero order operator)

$$U(n) = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ p_1(n) & \ldots & p_{i-1}(n) & p_i(n) & p_{i+1}(n) & \ldots & \ldots & p_m(n) \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}, \tag{17}$$

while the shift substep is the left multiplication by the operator (we write it in the matrix form)

$$
\begin{array}{c c}
 & \\
 & \\
i & \\
 & \\
\end{array}
\begin{pmatrix}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
 & & & E & & & & \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{pmatrix}. \tag{18}
$$

3.2. x-Cancelation transformation

In the following Lemmas 1, 2 and Propositions 1, 2 we suppose that $G \in \mathcal{D}_m$ is a full rank row reduced operator.

Lemma 1. *Let $(\gamma_1, \gamma_2, \ldots, \gamma_m)$ be the row order vector of G, i.e.,*

$$(\gamma_1, \gamma_2, \ldots, \gamma_m) = (\alpha_1(G), \alpha_2(G), \ldots, \alpha_m(G)).$$

Let polynomials $p_1(n), p_2(n), \ldots, p_m(n)$ and the number i be such that $\gamma_i + \deg p_i(n) = \mu$, where $\mu = \max_{\substack{1 \leqslant j \leqslant m \\ p_j(n) \neq 0}} (\gamma_j + \deg p_j(n))$. Let $U(n)$ be as in (17) and $P = \mathcal{M}^{-1}U(n)$. Then the row frontal matrix of PG is invertible, and $\nu(PG) = \nu(G)$.

Proof. Since $P = \mathcal{M}^{-1}U(n)$, the matrix form (13) of the operator P is

$$
\begin{pmatrix}
1 & & & & & & & \\
 & \ddots & & & & & & \\
 & & 1 & & & & & \\
p_1(\theta) & \ldots & p_{i-1}(\theta) & p_i(\theta) & p_{i+1}(\theta) & \ldots & \ldots & p_m(\theta) \\
 & & & & 1 & & & \\
 & & & & & \ddots & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{pmatrix}.
$$

If M is the row frontal matrix of G then the row frontal matrix of PG is equal to CM where

$$C = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ c_1 & \dots & c_{i-1} & c_i & c_{i+1} & \dots & \dots & c_m \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix},$$

with $c_k = \operatorname{lc} p_k(n)$ if $\gamma_k + \deg p_k(n) = \mu$ and $c_k = 0$ otherwise, $k = 1, 2, \dots, m$. Thus $c_i \neq 0$, $\det CM = c_i \det M$. The equality $\operatorname{val}_x \det CM = \operatorname{val}_x M$ follows. $\square$

Lemma 2. *Let Q be obtained from (18) by applying $\mathcal{M}^{-1}$, i.e., Q is the diagonal $(m \times m)$-matrix such that its diagonal entry in the i-th row is equal to $\frac{1}{x}$ and all other diagonal entries are equal to 1. Let $QG \in \mathcal{D}_m$. Then the row frontal matrix of QG is invertible, and $\nu(QG) = \nu(G) - 1$.*

Proof. If M is the row frontal matrix of G then $QM \in \operatorname{Mat}_m(K[[x]])$ since $QG \in \mathcal{D}_m$. Thus QM is the row frontal matrix of QG, and $\operatorname{val}_x \det QM = \operatorname{val}_x \det M - 1$. $\square$

Let an operator P and a matrix Q be as in Lemmas 1, 2, and the operator $G^{(1)} = QPG$ belongs to $\mathcal{D}_m$. The transformation G into $G^{(1)}$ will be called the *$x-$cancelation.*

When we perform step-by-step the $x-$cancelation transformations (possibly with different i's) we get by Lemmas 1, 2 operators $G^{(0)} = G, G^{(1)}, G^{(2)}, \dots \in \mathcal{D}_m$ such that $\nu(G^{(0)}) > \nu(G^{(1)}) > \nu(G^{(2)}) > \dots$ Since the value $\nu(\cdot)$ is non-negative for any operator from $\mathcal{D}_m$, we have

Proposition 1. *The $x-$cancelation transformation can be applied to G only a finite number of times (the sequence $G^{(0)}, G^{(1)}, G^{(2)}, \dots$ cannot be infinite).*

Of course, not any full rank operator has an invertible row frontal matrix. The following proposition will be useful for us in the sequel:

Proposition 2. *Let L be a full rank operator of the form (12). Then there exists $N \in \mathcal{D}_m$ such that the leading matrix of LN is invertible and LN is of same order as L (i.e. of order r).*

Proof. It follows from, e.g., Abramov & Khmelnov (2011), Abramov & Khmelnov (2012), Barkatou et al. (2013), that for any operator $\tilde{L} \in \mathcal{D}_m$ of full rank there exists $F \in \mathcal{D}_m$ such that $F\tilde{L}$ is of same order as the original operator and the leading matrix of the product is invertible. To prove the *existence* of such a factor we can ignore the question on algorithmic recognizing whether a matrix entry is equal to zero. Therefore for the adjoint operator L^* for L (see Remark 1 below) there exists $F \in \mathcal{D}_m$ such that FL^* has an invertible leading matrix and the order of FL^* is equal to the order of L^*. In this case $(FL^*)^*$ has an invertible leading matrix (which is the transposed matrix for the leading matrix of FL^*) and its order is equal to the order of L. The operator $(FL^*)^*$ is equal to LF^*, and we can set $N = F^*$. □

Remark 1. *For a scalar operator $L = a_k(x)\theta^k + \cdots + a_1(x)\theta + a_0(x)$ the adjoint operator L^* is defined as usual by $L^* = (\theta^*)^k a_k(x) + \cdots + \theta^* a_1(x) + a_0(x)$, where $\theta^* = \left(x\frac{d}{dx}\right)^* = -\frac{d}{dx}x = -(\theta+1)$. If an operator (12) is represented in the form (13), i.e., is represented as the matrix (L_{ij}) with $L_{ij} \in K[[x]][\theta]$, $i, j = 1, 2, \ldots, m$, then $L^* = (L^*_{ji})$. The equalities $(L^*)^* = L$, $(L_1L_2)^* = L_2^*L_1^*$ are evident. It follows from (Cohn, 1971, Chap. 8.1, Thm. 1.1) that L is of full rank iff L^* is of full rank.*

3.3. Induced recurrent systems transformation

Theorem 1. *The process of "reduction + shift" steps terminates.*

Proof. The "reduction + shift" steps produce the sequence (15) of operators related to the recurrent systems that appear progressively. In addition to operators (16) we consider

$$G^{(0)} = L^{(0)}N, \;\; G^{(1)} = L^{(1)}N, \;\; G^{(2)} = L^{(2)}N, \;\; \ldots \;\; , \tag{19}$$

where N is as in Proposition 2.

When we transform $R^{(j)}$ into $R^{(j+1)}$ we select i as it is prescribed by the reduction substep. Denote by $Q^{(j)}, P^{(j)}$ the operators which we use when transform $R^{(j)}$ into $R^{(j+1)}$. We have

$$G^{(j+1)} = \left(Q^{(j)}P^{(j)}L^{(j)}\right)N = Q^{(j)}P^{(j)}\left(L^{(j)}N\right), \tag{20}$$

thus $G^{(j+1)} = Q^{(j)}P^{(j)}G^{(j)}$. By induction on j it can be proved that $\alpha_s(G^{(j)}) = \gamma_s^{(j)}$, $j = 0, 1, \ldots$, where $\gamma_s^{(j)}$ is the values of γ_s, $s = 1, 2, \ldots, m$, computed for $R^{(j)}$ as described in Section 2.2. The hypothesis of Lemma 1 is satisfied due to (10) for any j when we consider $G^{(j)}$ as G. The matrix $Q^{(j)}$ satisfies the hypothesis of Lemma 2 since $Q^{(j)}P^{(j)}L^{(j)} \in \mathcal{D}_m$ and thus $Q^{(j)}\left(P^{(j)}L^{(j)}N\right) \in \mathcal{D}_m$ (we consider again $G^{(j)}$ as G). Therefore the transformation of $G^{(j)}$ into $G^{(j)}$ is the $x-$cancelation transformation. By Proposition 1 the sequence (19) must be finite. This implies that the sequence (15) must be finite as well. □

This process leads to a system of the form

$$\bar{B}_0(n)z(n) + \bar{B}_{-1}(n)z(n-1) + \cdots = 0 \tag{21}$$

and correspondingly to an operator

$$\bar{B}_0(n) + \bar{B}_{-1}(n)E^{-1} + \cdots \tag{22}$$

with the invertible leading matrix $\bar{B}_0(n)$. Any solution of the form (7) of the original system (5) is a solution of (21). Recall that the described process of receiving system (21) generates in addition a finite set $\mathcal{C}$ of linear constraints defined in Section 2.3. If system (21) is considered together with $\mathcal{C}$ then the Laurent solution space of this extended system coincides with the Laurent solution space of (5).

The algorithmic search for the solution space (algorithms for solving the problems **P1**, **P2** posed in Section 1) will be discussed in Section 6.

3.4. On reduced forms of operators

It has been noted that we use a generalization (a new version) of EG-eliminations. The notion of row frontal matrix plays an important role in the given justification of this version. This notion and the notion of row order vector were introduced in Beckermann et al. (2006) (however in that paper instead of the term "row frontal matrix" the term "leading coefficient matrix" was used; in our paper another matrix is called "leading"). In Beckermann et al. (2006) algorithm Row-Reduction has been proposed. This algorithm transforms a given operator or system to the row reduced form. If an operator or system is of full rank then in the row reduced form its row frontal matrix is invertible. In contrast to Row-Reduction algorithm, EG-eliminations produces additionally a finite set of linear constraints. In the recurrent system case the linear constraints filter out all extra sequential solutions. Besides this, the trick mentioned in the final paragraph of Section 2.2 speeds up EG-eliminations significantly.

We emphasize that the termination of EG-eliminations and Row-Reduction algorithm were earlier proved only for finite order systems, while in this paper we deal with recurrent systems of form (6) of infinite order.

4. Width of differential systems of full rank

Recall that if a system S is of full rank then the minimal integer w such that $S^{\langle l \rangle}$ is of full rank for all $l \geqslant w$ is called the *width* of S. This notion was introduced in Section 1.

Theorem 2. *Let a system S of the form (1) be of full rank. Then there exists a non-negative integer s such that any truncation $S^{\langle l \rangle}$ of S, $l \geqslant s$, is of full rank.*

Proof. Let (5) be the induced recurrent system. By Theorem 1 the process described in Section 2.2 terminates, and we get a system with the invertible leading matrix. Only a finite number of coefficient matrices of (5) are involved in the obtained invertible leading matrix. Let those matrices be $B_0(n), B_{-1}(n), \dots, B_{-s}(n)$, $s \geqslant 0$. For $l \geqslant s$ the matrices

$$B_0(n), B_{-1}(n), \dots, B_{-s}(n), B_{-s-1}(n), \dots, B_{-l}(n)$$

of (5) are defined by $S^{\langle l \rangle}$. Thus s is the desired number. □

The minimal s which possesses the property formulated in Theorem 2 is the width w of S. It follows from Theorem 2 that the width is defined for any system of full rank.

Remark 2. *A similar way to prove Theorem 2 is to reformulate Theorem 1: Let $R = \mathcal{M}L$ where L is a full rank operator from $\mathcal{D}_m$. Then there exists $J \in \mathrm{Mat}_m(K[n])[E]$ such that the leading matrix $\bar{B}_0(n)$ of JR is invertible, and J can be constructed algorithmically. Thus $s = \operatorname{ord} J$ (a finite number!) is an upper bound for the width of L.*

Remark 3. *Theorem 2 can be considered as a generalization of the evident property of linear algebraic systems of the form*

$$A(x)y(x) = 0, \quad A(x) \in \mathrm{Mat}_m(K[[x]]) \tag{23}$$

of full rank, i.e. with $\det A(x) \neq 0$: there exists an integer w such that $\det A^{\langle l \rangle}(x) \neq 0$ for all $l \geqslant w$ (obviously the minimal w does not exceed $\mathrm{val}_x \det A(x)$.) Factually this is the case of differential operators of order 0.

Example 1. *It is possible that for a system S of the form (1) and some positive l the system $S^{\langle l \rangle}$ is of full rank while $S^{\langle l+1 \rangle}$ is not:*

$$A(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & x & x^2 + x^3 \\ 1 & 1 & x \end{pmatrix},$$

$M_i(x) \in \mathrm{Mat}_3(K[[x]]), \mathrm{val}_x M_i(x) \geqslant 4, i = 0, 1$ *(in other words, $M_i(x) = O(x^4)$, $i = 0, 1$).*

Let S be the system $A_1(x)\theta y + A_0(x)y = 0$, $A_i(x) = A(x) + M_i(x)$, $i = 0, 1$. Then the system $S^{\langle l \rangle}$ is of full rank iff $l = 1, 3, 4\ \ldots$ (2 is not in this list; thus the width of S is 3). Observe that θy_1 has the coefficient 1 in each equation of $S^{\langle l \rangle}$, $l = 1, 2, \ldots$, and $B_0(n)$ in (5) has no zero row.

Note that this example can be generalized: if d is a positive integer, and

$$A(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & x^d & x^{2d} + x^{2d+1} \\ 1 & 1 & x^d \end{pmatrix},$$

then the l-truncated system is of full rank iff

$$l = d, d+1, \ldots,\ 2d-1,\ 2d+1,\ 2d+2,\ \ldots$$

($2d$ is not in this list; thus the width of S is $2d+1$).

5. "Normalization" of operators of full rank by operator multiplication from the left

As we have mentioned in Remark 2, if an operator $L \in \mathcal{D}_m$ is of full rank and $R = \mathcal{M}L$ then there exists $J \in \mathrm{Mat}_m(K[n])[E]$ such that the leading matrix of the operator JR is invertible (the operator J can be constructed algorithmically, see the last paragraph of Section 3.1). Therefore the operator $\mathcal{M}^{-1}(JR)$ can be also constructed algorithmically. However, the leading matrix of the latter differential operator is not, in general, invertible.

The facts proved in Section 3 allow additionally to prove some statements concerning transformations of both L and its induced recurrent operator into operators with invertible row frontal or leading matrices.

Theorem 3. *Let $L \in \mathcal{D}_m$ be a full rank operator of order r. Then*

(i) If the row frontal matrix of L is invertible then there exists $H \in \mathcal{D}_m$ such that HL has an invertible row frontal matrix and $\mathcal{M}(HL)$ has an invertible leading matrix. If the row order vector of L (see Section 3.1) is known in advance then such an operator H can be constructed algorithmically.

(ii) There exists an operator $\tilde{H} \in \mathcal{D}_m$ such that the leading matrices of both $\tilde{H}L$ and $\mathcal{M}(\tilde{H}L)$ are invertible. If it is known in advance that the leading matrix of L is invertible then such an operator $\tilde{H} \in \mathrm{Mat}_m(K[x])[\theta]$ (i.e., having polynomial coefficients) can be constructed algorithmically.

Proof. (i) Let $(\alpha_1(L), \alpha_2(L), \ldots, \alpha_m(L))$ be the row order vector of L. Then the sequence of "reduction + shift" steps will terminate yielding the wanted result if we set initially

$$\gamma = (\alpha_1(L),\ \alpha_2(L),\ \ldots,\ \alpha_m(L)) \tag{24}$$

instead of $\gamma = (r, r, \ldots, r)$. The proof is similar to the proof of Theorem 1, but now we do not need the operator N. If we know the row order vector of L in advance then we can construct such an operator H algorithmically (by applying the sequence of "reduction + shift" steps to $\mathcal{M}L$ and using $\mathcal{M}^{-1}$ when the process terminated).

(ii) We have mentioned in the proof of Proposition 2 that there exists an operator F such that the operator FL has an invertible leading matrix (or, the same, the invertible row frontal matrix with row order vector $(r, r, \ldots, r)$). By (i) there exists $H \in \mathrm{Mat}_m(K[x])[\theta]$ such that HFL has the invertible row frontal matrix while $\mathcal{M}(HFL)$ has the invertible leading matrix. Let r', $(\alpha'_1, \alpha'_2, \ldots, \alpha'_m)$ be the order and, resp., the row order vector of HFL. We can left multiply the operator HFL by the operator whose matrix form representation is the diagonal matrix $\mathrm{diag}(\theta^{r'-\alpha'_1}, \theta^{r'-\alpha'_2}, \ldots, \theta^{r'-\alpha'_m})$ (the corresponding recurrent operator $\mathcal{M}(HFL)$ will be left multiplied by the diagonal matrix $\mathrm{diag}(n^{r'-\alpha'_1}, n^{r'-\alpha'_2}, \ldots, n^{r'-\alpha'_m})$). This yields a differential operator with invertible leading matrix. The induced recurrent operator has invertible leading matrix as well.

If the leading matrix of L is invertible then we do not need the left multiplying by F and the described transformations can be done algorithmically. $\square$

Remark 4. *Suppose that the entries of matrix coefficients in a full rank operator L are not power series represented algorithmically but, e.g., polynomials represented in the usual way. In this case the frontal matrix shape can be evidently* *determined, and the operators H, $\tilde{H}$, HL, $\tilde{H}L$ certainly can be constructed algorithmically.*

In our next example we use the matrix form representation (13) for differential and difference operators. This form is also used in our implementation (see Section 8).

Example 2. *Let d be a positive integer,*

$$L = \begin{pmatrix} \theta^3 & 0 & 0 \\ \theta^2 & x^d\theta^2 & (x^{2d}+x^{2d+1})\theta^2 \\ \theta & \theta & x^d\theta \end{pmatrix}. \tag{25}$$

The leading matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of the operator L is not invertible, while its row frontal matrix is (it coincides with the matrix $A(x)$ from Example 1). The induced recurrent operator is

$$R^{(0)} = R = \begin{pmatrix} n^3 & 0 & 0 \\ n^2 & (n-d)^2E^{-d} & (n-2d)^2E^{-2d} + (n-2d-1)^2E^{-2d-1} \\ n & n & (n-d)E^{-d} \end{pmatrix}.$$

The leading matrix of $R^{(0)}$ is given by

$$B_0(n) = \begin{pmatrix} n^3 & 0 & 0 \\ n^2 & 0 & 0 \\ n & n & 0 \end{pmatrix}.$$

It is not invertible. We now apply the "reduction + shift" process to R.

Since the row frontal matrix of L is invertible, we can following (24) set

$$\gamma = (\gamma_1, \gamma_2, \gamma_3) = (\alpha_1(L), \alpha_2(L), \alpha_3(L)) = (3, 2, 1).$$

The rows of $B_0(n)$ are linearly dependent, and $p(n) = (p_1(n), p_2(n), p_3(n)) = (-1, n, 0)$ is an element of the left kernel of $B_0(n)$. Then

$$\mu = \max(\gamma_1 + \deg p_1, \gamma_2 + \deg p_2) = 1.$$

Since $\gamma_2 + \deg p_2 = 1$ we can take $i = 2$ and replace the 2-nd row of R by $-[R^{(0)}]_{1,*} + n[R^{(0)}]_{2,*}$. This yields the new recurrent operator:

$$R^{(0)'} = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & n(n-d)^2E^{-d} & n(n-2d)^2E^{-2d} + n(n-2d-1)^2E^{-2d-1} \\ n & n & (n-d)E^{-d} \end{pmatrix}.$$

Now we have to multiply on the left the 2-nd row of $R^{(0)'}$ by E^d to get:

$$R^{(1)} = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & (n+d)n^2 & (n+d)(n-d)^2E^{-d} + (n+d)(n-d-1)^2E^{-d-1} \\ n & n & (n-d)E^{-d} \end{pmatrix}.$$

The leading matrix of $R^{(1)}$ is

$$B_0^{(1)}(n) = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & (n+d)n^2 & 0 \\ n & n & 0 \end{pmatrix}.$$

We update γ to its new value $\gamma = (3,3,1)$ and compute $p(n) = (n+d, n, -(n+d)n^2)$ as an element of the left kernel of $B_0^{(1)}(n)$. The new value of $\mu = \max(\gamma_1 + \deg p_1, \gamma_2 + \deg p_2)$ is 4. Since $\mu = \gamma_3 + \deg p_3 = 4$ we can take $i = 3$ and replace $[R^{(1)}]_{3,*}$ by $(n+d)[R^{(1)}]_{1,*} + n[R^{(1)}]_{2,*} - (n+d)n^2[R^{(1)}]_{3,*}$ to get the new recurrent operator:

$$R^{(1)'} = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & (n+d)n^2 & (n+d)(n-d)^2E^{-d} + (n+d)(n-d-1)^2E^{-d-1} \\ 0 & 0 & -n(n^2-d^2)dE^{-d} + n(n+d)(n-d-1)^2E^{-d-1} \end{pmatrix}.$$

Multiplying on the left $[R^{(1)'}]_{3,*}$ by E^d we obtain the operator

$$R^{(2)} = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & (n+d)n^2 & (n+d)(n-d)^2E^{-d} + (n+d)(n-d-1)^2E^{-d-1} \\ 0 & 0 & -(n+d)((n+d)^2-d^2)d + (n+d)(n+2d)(n-1)^2E^{-1} \end{pmatrix}$$

whose leading matrix is given by

$$B_0^{(2)}(n) = \begin{pmatrix} n^3 & 0 & 0 \\ 0 & (n+d)n^2 & 0 \\ 0 & 0 & -(n+d)(n+2d)nd \end{pmatrix}.$$

We have $\det B_0^{(2)}(n) = -dn^6(n+d)^2(n+2d)$, which is not identically zero as a polynomial in n since d is positive.

Now we can compute the differential operator:

$$\mathcal{M}^{-1}(R^{(2)}) = \begin{pmatrix} \theta^3 & 0 & 0 \\ 0 & (\theta+d)\theta^2 & x^d(\theta+2d)\theta^2 + x^{d+1}(\theta+2d+1)\theta^2 \\ 0 & 0 & -(\theta+d)((\theta+d)^2-d^2)d + x(\theta+d+1)(\theta+2d+1)\theta^2 \end{pmatrix}.$$

It has, as expected (in accordance with Theorem 3(i)), an invertible row frontal matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & x \end{pmatrix},$$

but its leading matrix is not invertible. The operator H mentioned in Theorem 3(i) is equal to $\mathcal{M}^{-1}F$, where

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E^d \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ n+d & n & -(n+d)n^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & E^d & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -1 & n & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

6. A lazy computation approach

The basis of the algorithms for **P1**, **P2**, **P3** is the procedure to perform the "reduction + shift" steps. The technical problem is that the system (5) to be transformed is infinite and hence a lazy computation needs to be used. For the "reduction + shift" steps, the lazy computation is based on storing of the sequence of already performed reductions and shifts. This sequence will be called the **RS**-*sequence*, its elements are the pairs

$$[\mathbf{R}, \mathbf{S}],$$

where

- $\mathbf{R}$ is the list of reduction parameters $[i, p, N]$, where $1 \leqslant i \leqslant m$ is the index of the replaced row, $p = (p_1(n), p_2(n), \ldots, p_m(n)) \in K[n]^m$ is the vector of the dependency coefficients, N is the set of the indices n_0 of linear constraints in the form (11).

- $\mathbf{S}$ is the list of shifted rows; the number of shifts in the rows is counted for each row, when a row is replaced in a reduction substep then the resulting row gets the shift counter equal to the maximum of the shift counters of the rows with non-zero coefficient of dependency.

We start with the problem to compute an invertible leading matrix of the transformed recurrent system. The computation is initiated with an empty **RS**-sequence and the induced recurrence $B_0(n)z(n) + \cdots + B_{-t}(n)z(n-t) = 0$ for t-truncation of the given system with $t = 1$. Then the "reduction + shift" steps are performed; **RS**-sequence is updated. If a next shift substep on a leading matrix depends on the initial matrix $B_{-(t+1)}(n)$ (i.e. the shift counter of the shifting row is t) then:

- $t \leftarrow t + 1$,

- the initial induced recurrence is extended accordingly,

- the "reduction + shift" steps are reproduced with **RS**-sequence in the extended induced recurrence.

The "reduction + shift" process is then continued with analogous extensions to be repeated if needed again.

Since the "reduction + shift" process is finite the described lazy computation will terminate. In accordance with Theorem 1, the procedure allows to construct the invertible leading matrices $\bar{B}_0(n)$. Using lazy computation of this type we can similarly compute any number of the matrix $\bar{B}_0(n), \bar{B}_{-1}(n), \ldots$ of system (21).

In order to compute linear constraints some intermediate results of the transformation are needed. That is why the **RS**-sequence represents the step-by-step transformation rather than, e.g., the final transformation matrix or the transformation matrix for each step.

Observe that the lazy computation on the row level is a possible way of additional optimization. For example, on the current step of the process we have a zero row in the leading matrix $B_0(n)$ and need to shift the equation. It means that we need the row of the matrix $B_{-1}(n)$. In order to get it we need to perform the reduction taken from **RS**-sequence in the matrix $B_{-1}(n)$ as well. In this moment it might turn out that some of the rows of the matrix which are involved in the reduction are not yet evaluated and we need to evaluate them as well (which might in turn requires to evaluate the rows in previous matrix $B_{-2}(n)$ if the rows of $B_{-1}(n)$ were shifted in accordance to **RS**-sequence), and so on. Since at each moment **RS**-sequence is finite we eventually evaluate all the elements needed to perform the next "reduction + shift" step. For using this approach we need to store additionally the current state of the recurrent system for each of the element of **RS**-sequence (i.e. for each of the "reduction + shift" steps). Each of the recurrent system is represented as the list of the equations. Each of the equations is represented by its evaluated initial part, with the lengths of the initial parts being potentially different for different equations. The initial parts of the equations are extended when needed on the next steps of the transformation. The similar technique was used in Abramov et al. (2005) for computing the initial parts of the involved series in the course of intermediate calculations.

7. Problems P1, P2, P3

If the equation $\det \bar{B}_0(n) = 0$ has no integer roots then the original differential system has no non-zero Laurent solution. Then $-\infty$ and $\{0\}$ are solutions of **P1** and **P2**. Otherwise, let e^*, e_* be the maximal and the minimal integer roots of this equation (it is possible that $e^* = e_*$). Then $l_0 = e^*$ is evidently a solution of **P1**.

Let $\mathcal{C}$ be the set of the linear constraints of the form (11) with $v = e_*$, discarding all the constraints in which $n_0 < e_*$. Let $n_{\mathcal{C}}$ be the maximal of the corresponding values of n_0 for all the constraints from the set $\mathcal{C}$ (note that N of **R** in **RS**-sequence is used to compute $n_{\mathcal{C}}$ using the lazy calculation). If $l \geqslant \max\{l_0, n_{\mathcal{C}}\}$ then we can find a basis for the space $V_S^{\langle l \rangle}$ of the truncated Laurent series solutions of the original differential system. Indeed, consider the linear algebraic system which consists of equations

$$\begin{array}{l}
\bar{B}_0(e_*)z(e_*) = 0, \\
\bar{B}_0(e_*+1)z(e_*+1) + \bar{B}_{-1}(e_*+1)z(e_*) = 0, \\
\bar{B}_0(e_*+2)z(e_*+2) + \bar{B}_{-1}(e_*+2)z(e_*+1) + \bar{B}_{-2}(e_*+2)z(e_*) = 0, \\
\qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots \\
\bar{B}_0(l)z(l) + \bar{B}_{-1}(l)z(l-1) + \bar{B}_{-2}(l)z(l-2) + \ \ldots \ + \bar{B}_{e_*-l}(l)z(e_*) = 0,
\end{array} \tag{26}$$

and of all constraints from $\mathcal{C}$ (the constraints add some supplementary equations to the algebraic system and enable us to remove "parasitic" solutions from the solution space which might appear due to reduction substeps of EG-eliminations).

Having a basis for the solution space of the obtained system we can construct a basis for the space $V_S^{\langle l \rangle}$ of the truncated Laurent series solutions of the original differential system. This solves the problem **P2**. If a given l is such that $l_0 \leqslant l < \max\{l_0, n_{\mathcal{C}}\}$ then we can first solve **P2** for $\tilde{l} = \max\{l_0, n_{\mathcal{C}}\}$ and then l-truncate the elements of the basis that was found.

Remark 5. *Let s be the non-negative number which was defined in the proof of Theorem 2 (in other words s is the maximal shift counter of the rows after the invertible leading matrix $\bar{B}_0(n)$ is computed), and $l \geqslant \max\{l_0, n_{\mathcal{C}}\}$. To find the matrices $\bar{B}_0(n), \bar{B}_{-1}(n), \dots, \bar{B}_{e_*-l}(n)$ which are used in system (26) it is sufficient to know the matrices $B_0(n), B_{-1}(n), \dots, B_{-l_1}(n)$ involved into the induced recurrence (5), where*

$$l_1 = l + s - e_*. \tag{27}$$

Therefore the solution of **P2** *for the original system S is the same as the analogous solution for $S^{\langle l_1 \rangle}$.*

A way for receiving an upper bound s for the width of the original differential system is given in the proof of Theorem 2 (this s has been used in Remark 5). The exact solution of **P3** might be computed by gradually refining the upper bound using the finite version of the EG-eliminations.

In our next example we use the matrix form representation (13) for differential and difference operators. This form is also used in our implementation (see Section 8).

Example 3. *Going back to Example 2 we can make two observations. First, the course of the computation of, say, the leading matrix $\bar{B}_0(n)$ would be the same if we add to entries of the operator matrix from the right-hand side of (25) some scalar operators of the form $x^{2d+2}(u_3(x)\theta^3 + u_2(x)\theta^2 + u_1(x)\theta + u_0(x))$ with power series $u_0(x), u_1(x), u_2(x), u_3(x)$.*

Second, we ignored in Example 2 the appearing of linear constraints. However we can easily compute, e.g., $n_{\mathcal{C}}$. When we transform R into $R^{(1)}$ we take $i = 2$ and multiply $[R]_{2,*}$ by n, transforming $R^{(1)}$ into $R^{(2)}$ we take $i = 3$ and multiply $[R^{(1)}]_{2,*}$ by $(n+d)n^2$. Thus $n_{\mathcal{C}} = 0$. The integer roots $0, -d, -2d$ of the determinant of the final leading matrix $\bar{B}_0(n) = B_0^{(0)}(n)$ form a super-set for the set of the valuations of Laurent series solutions of the operator L. Thus $e_* = -2d, e^* = 0$. So $l_0 = 0$. We see also that $s = 2d$. Using (27) we derive that $l_1 = \max\{n_{\mathcal{C}}, l\} + s - e_* = l + 2d$. Therefore the solution of **P2** *for S which has the form $L(y) = 0$ with L as in (25) coincides with the analogous solution for $S^{\langle l+2d \rangle}$.*

It is easy to check that in this concrete example we would get the same value of l_1 if we use the standard initial value $\gamma = (3,3,3)$ instead of $\gamma = (3,2,1)$ (the latter initial value of γ was used in Example 2).

8. Implementation

The algorithms are implemented in Maple (**?**) in the framework of the package `EG` (Abramov & Khmelnov, 2013). Its code and examples of using the package are available from http://www.ccas.ru/ca/doku.php/eg. The implementation is partially based on the implementation of a version of EG-eliminations for the finite case described in Abramov et al. (2003) (in the package `EG` the difference version of EG-eliminations is named EG_σ to be consistently named with the similar algorithm EG_δ for the differential case (Abramov & Khmelnov, 2013)).

Given a differential system to be solved, the procedure `LaurentSolution` computes its Laurent solution up to the l-th degree, i.e., solves the problem **P2**. The system is to be given in the matrix form, with the matrix entry with indices i,j being a function of an integer argument (e.g. k) which computes the coefficient of x^k in the operator L_{ij}. The coefficient is to be computed as a polynomial in θ. The functions may be defined as special procedures with any algorithm to compute the coefficients. In simpler cases, Maple functions `if` or `piecewise` may be used (as in Example 4 below). Additionally to the system, the procedure takes three additional parameters: θ as the name of the operator $x\frac{d}{dx}$ used in the system, x as the name of the variable, and l.

The approach described in Remark 5 is used to implement the computation of $V_S^{\langle l\rangle}$ to solve **P2**. In the course of solving the problem **P2** `LaurentSolution` also computes l_0, i.e. solves the problem **P2**, and computes s, i.e. finds the upper bound of the solution of the problem **P3** which then may be refined using `EG_sigma` procedure if needed.

Example 4. *Let L be the operator*

$$\begin{pmatrix} \theta^3 & 0 & 0 \\ \theta^2 & x^{k_0}\theta^2 & (x^{2k_0}+x^{2k_0+1})\theta^2 \\ \theta & \theta & x^{k_0}\theta \end{pmatrix} +$$

$$+\begin{pmatrix} \sum\limits_{k=2k_0+2}^{\infty} x^k(k\theta+k^2) & \sum\limits_{k=2k_0+2}^{\infty} x^k 3k\theta & \sum\limits_{k=2k_0+2}^{\infty} x^k(2\theta+k) \\ \sum\limits_{k=2k_0+2}^{\infty} x^k(k\theta^2+3) & \sum\limits_{k=2k_0+2}^{\infty} x^k(\theta+k) & \sum\limits_{k=2k_0+2}^{\infty} x^k(k\theta+k) \\ \sum\limits_{k=2k_0+2}^{\infty} x^k(k+1) & \sum\limits_{k=2k_0+2}^{\infty} x^k k\theta^2 & \sum\limits_{k=2k_0+2}^{\infty} x^k(k\theta^2+3k) \end{pmatrix}.$$

As discussed in Example 3, the operator L is the operator (25) with additional terms of order greater than $2k_0+2$. Let $k_0=2$.

The system to be solved is represented in Maple as the following:

```
> k0:=2:L:=Matrix([[k->piecewise(k=0,theta^3,k<2*k0+2,0,k*theta+k^2),
  k->piecewise(k<2*k0+2,0,3*k*theta),k->piecewise(k<2*k0+2,0,2*theta+k)],
  [k->piecewise(k=0,theta^2,k<2*k0+2,0,k*theta^2+3),k->piecewise(k=k0,
  theta^2,k<2*k0+2,0,theta+k),k->piecewise(k=2*k0,theta^2,k=2*k0+1,theta^2,
  k<2*k0+2,0,k*theta+k)],[k->piecewise(k=0,theta,k<2*k0+2,0,k+1),k->piecewise
  (k=0,theta,k<2*k0+2,0,theta^2*k),k->piecewise(k=k0,theta,k<2*k0+2,0,
  theta^2+3*k)]]);
```

$$L := \Big[\Big[k \to piecewise\big(k=0, \theta^3, k < 2\,k0 + 2, 0, k\,\theta + k^2\big), k \to piecewise\big(k < 2\,k0 + 2, 0, 3\,k\,\theta\big), k \to piecewise\big(k < 2\,k0 + 2, 0, 2\,\theta + k\big)\Big],$$
$$\Big[k \to piecewise\big(k=0, \theta^2, k < 2\,k0 + 2, 0, k\,\theta^2 + 3\big), k \to piecewise\big(k = k0, \theta^2, k < 2\,k0 + 2, 0, \theta + k\big), k \to piecewise\big(k = 2\,k0, \theta^2, k = 2\,k0 + 1, \theta^2, k < 2\,k0 + 2, 0, k\,\theta + k\big)\Big],$$
$$\Big[k \to piecewise\big(k=0, \theta, k < 2\,k0 + 2, 0, k + 1\big), k \to piecewise\big(k=0, \theta, k < 2\,k0 + 2, 0, k\,\theta^2\big), k \to piecewise\big(k = k0, \theta, k < 2\,k0 + 2, 0, \theta^2 + 3\,k\big)\Big]\Big]$$

Let us check the matrix coefficients of x^0, x^1, x^2, x^3, x^4, x^5 and x^k with $k \geqslant 6$ in the operator L:

```
> [map(apply,L,0),map(apply,L,1),map(apply,L,2),map(apply,L,3),map(apply,L,
  4),map(apply,L,5),map(apply,L,k) assuming (k>=6)];
```

$$\left[\begin{bmatrix} \theta^3 & 0 & 0 \\ \theta^2 & 0 & 0 \\ \theta & \theta & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta^2 & 0 \\ 0 & 0 & \theta \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta^2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta^2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} k\,\theta + k^2 & 3\,k\,\theta & 2\,\theta + k \\ k\,\theta^2 + 3 & \theta + k & k\,\theta + k \\ k + 1 & k\,\theta^2 & \theta^2 + 3\,k \end{bmatrix}\right]$$

Let us solve **P2** *for $l = 1$, i.e. compute $V_S^{\langle 1 \rangle}$ using* `LaurentSolution` *(the procedure is in the mode to print out some useful intermediate results):*

```
> st:=time():LaurentSolution(L, theta, x,1);time()-st;
```

Trace of computation

RS-sequence:

$$\left[\left[[\ ], \left[\left[2, \left[\begin{matrix}-1 & n & 0\end{matrix}\right], \{0\}\right]\right]\right], \left[[2,2], \left[\left[3, \left[\begin{matrix}-n-2 & -n & (n+2)\,n^2\end{matrix}\right], \{-2, 0\}\right]\right]\right], [[3,3], [\ ]]\right]$$

The invertible leading matrix:

$$\left[\begin{matrix} n^3 & 0 & 0 \\ 0 & (n+2)\,n^2 & 0 \\ 0 & 0 & 2\,(n+2)^3 - 8\,n - 16 \end{matrix}\right]$$

Roots of its determinant:

$-4, -2, 0$

$l_0 = 0, s = 4, l_1 = 9$

Constraints:

table([0 = {[0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 216, 54, 8, 343, 70, 9, 512, 88, 10, 729, 108], [0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 16, 0, 0, 25, 219, 0, -30, 346, 0, -42, 515, 0, -56, 732, 0, -72]}, -2 = {[-2, -2, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 384, 82, 8, 567, 102, 9, 800, 124, 10, 1089, 148]}])

Solution before applying constraints:

$$\left[\_c_1 + \mathrm{O}(x^2),\ \frac{3547}{312}\,\frac{\_c_3}{x^2} + \frac{8\,\_c_3}{x} + \_c_2 - \frac{157}{6}\,x\,\_c_3 + \mathrm{O}(x^2),\ \frac{\_c_3}{x^4} - \frac{8}{3}\,\frac{\_c_3}{x^3} + \frac{41}{4}\,\frac{\_c_3}{x^2} - \frac{157}{6}\,\frac{\_c_3}{x} + \_c_4 + \frac{445181}{3120}\,x\,\_c_3 + \mathrm{O}(x^2)\right]$$

End of computation

$$\left[\_c_1 + \mathrm{O}(x^2),\ \_c_2 + \mathrm{O}(x^2),\ \_c_3 + \mathrm{O}(x^2)\right]$$

0.171

Let us solve **P2** *for* $l = 3$, *i.e. compute* $V_S^{\langle 3\rangle}$ *using* `LaurentSolution` *(again the procedure is in the mode to print out some useful intermediate results):*

```
> st:=time():LaurentSolution(L, theta, x,3);time()-st;
```

Trace of computation

RS-sequence:

$$\left[\left[[\ ], \left[\left[2, \left[\begin{matrix}-1 & n & 0\end{matrix}\right], \{0\}\right]\right]\right], \left[[2,2], \left[\left[3, \left[\begin{matrix}-n-2 & -n & (n+2)\,n^2\end{matrix}\right], \{-2, 0\}\right]\right]\right], [[3,3], [\ ]]\right]$$

The invertible leading matrix:

$$\left[\begin{matrix} n^3 & 0 & 0 \\ 0 & (n+2)\,n^2 & 0 \\ 0 & 0 & 2\,(n+2)^3 - 8\,n - 16 \end{matrix}\right]$$

Roots of its determinant:

$-4, -2, 0$

$l_0 = 0, s = 4, l_1 = 11$

Constraints:

table([0 = {[0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 216, 54, 8, 343, 70, 9, 512, 88, 10, 729, 108, 11, 1000, 130, 12, 1331, 154], [0, 0, 0, 0, 0, 0, 0, 4, 0, 0, 0, 0, 0, 0, 16, 0, 0, 25, 219, 0, -30, 346, 0, -42, 515, 0, -56, 732, 0, -72, 1003, 0, -90, 1334, 0, -110]}, -2 = {[-2, -2, 0, 0, 0, 0, 0, 0, -4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 7, 384, 82, 8, 567, 102, 9, 800, 124, 10, 1089, 148, 11, 1440, 174, 12, 1859, 202]}])

Solution before applying constraints:

$$\Big[\_c_1 + \frac{1}{4}x^2\,\_c_3 + \frac{1}{27}x^3\,\_c_3 + \mathrm{O}(x^4),\ \frac{3547}{312}\frac{\_c_3}{x^2} + \frac{8\,\_c_3}{x} + \_c_2 - \frac{157}{6}x\,\_c_3 - \frac{69}{4}x^2\,\_c_3 - \frac{1008983}{28080}x^3\,\_c_3 + \mathrm{O}(x^4),\ \frac{\_c_3}{x^4} - \frac{8}{3}\frac{\_c_3}{x^3} + \frac{41}{4}\frac{\_c_3}{x^2} - \frac{157}{6}\frac{\_c_3}{x} + \_c_4 + \frac{445181}{3120}x\,\_c_3 + x^2\left(\frac{5}{4}\_c_4 - \frac{14989337}{37440}\_c_3 - \frac{3}{4}\_c_1 + \frac{3}{2}\_c_2\right) + x^3\left(\frac{13}{6}\_c_2 + \frac{11}{6}\_c_4 - \frac{33750113}{393120}\_c_3 - \frac{7}{6}\_c_1\right) + \mathrm{O}(x^4)\Big]$$

End of computation

$$\Big[\_c_1 + \mathrm{O}(x^4),\ \_c_2 + \mathrm{O}(x^4),\ \_c_3 + x^2\left(\frac{5}{4}\_c_3 - \frac{3}{4}\_c_1 + \frac{3}{2}\_c_2\right) + x^3\left(\frac{13}{6}\_c_2 + \frac{11}{6}\_c_3 - \frac{7}{6}\_c_1\right) + \mathrm{O}(x^4)\Big]$$

0.172

The following solutions of **P2** *has been found:*

$$V_S^{\langle 1\rangle} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

$$V_S^{\langle 3\rangle} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 + (\frac{5}{4}c_3 - \frac{3}{4}c_1 + \frac{3}{2}c_2)x^2 + (\frac{13}{6}c_2 + \frac{11}{6}c_3 - \frac{7}{6}c_1)x^3 \end{pmatrix}.$$

In the course of computation, **P1** *was solved ($l_0 = 0$), and an upper bound for the solution of* **P3** *($s = 4$) was found as well.*

The "parasitic" solutions were discarded with the linear constraints, and that two lower roots (-4,-2) of the determinant of the invertible leading matrix did not correspond to the valuations of Laurent solutions in this example, only the maximal root 0 did.

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