On m-Interlacing Solutions of Linear Difference Equations*

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Abstract. We consider linear homogeneous difference equations with rational-function coefficients. The search for solutions in the form of the m-interlacing ($1 \le m \le \operatorname{ord} L$, where L is a given operator) of finite sums of hypergeometric sequences, plays an important role in the Hendriks–Singer algorithm for constructing all Liouvillian solutions of $L(y) = 0$. We show that Hendriks–Singer's procedure for finding solutions in the form of such m-interlacing can be simplified. We also show that the space of solutions of $L(y) = 0$ spanned by the solutions of the form of the m-interlacing of hypergeometric sequences possesses a cyclic permutation property. In addition, we describe adjustments of our implementation of the Hendriks–Singer algorithm to utilize the presented results.

1 Introduction

In [5] the definition of Liouvillian sequence was given, and the fact that the Galois group of a linear homogeneous difference equation with rational-function coefficients is solvable iff the equation has a fundamental system of Liouvillian solutions was proven (the definition of the Galois group of an equation of this type was given earlier in [10]). Let $\mathcal{C}$ be an algebraically closed subfield of the field $\mathbb{C}$ of complex numbers. In [5] two sequences $u, v : \mathbb{N} \to \mathcal{C}$ are supposed to be equal iff $u_n = v_n$ for all integer n large enough, i.e., factually the germs of sequences are considered. The ring of the germs of sequences is denoted by $\mathcal{S}$. As it is done in [5], we will frequently identify a sequence with its equivalent class in $\mathcal{S}$. We will use the symbol $\overline{\forall} n$ as "for all integer n large enough". Denote $k = \mathcal{C}(x)$. This field can be embedded in $\mathcal{S}$: since the germs of sequences are considered we can map $f \in k$, e.g., to the sequence u such that $u_n = 0$ if n is a pole of f and $f(n)$ otherwise.

The map $\phi : \mathcal{S} \to \mathcal{S}$ defined by $\phi(u_0, u_1, u_2, \ldots) = (u_1, u_2, u_3, \ldots)$ is an automorphism of $\mathcal{S}$ (with $\phi(f(x)) = f(x + 1)$ for $f(x) \in k$). We say that a sequence $u = \langle u_n \rangle$ satisfies the equation $L(y) = 0$ with

$$L = \phi^d + a_{d-1}(x)\phi^{d-1} + \ldots + a_1(x)\phi + a_0(x), \tag{1}$$

$a_1(x), a_2(x), \ldots, a_{d-1}(x) \in k,\ a_0(x) \in k \setminus \{0\}$, if the sequence

$$\langle u_{n+d} + a_{d-1}(n)u_{n+d-1} + \ldots + a_1(n)u_{n+1} + a_0(n)u_n \rangle$$

is equal to zero sequence, i.e., if

$$u_{n+d} + a_{d-1}(n)u_{n+d-1} + \ldots + a_1(n)u_{n+1} + a_0(n)u_n = 0,\ \ \overline{\forall} n.$$

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For short, we will talk about solutions of L instead of solutions of the equation $L(y) = 0$.

The definition of Liouvillian sequence (we will discuss this definition in Section 5.1) uses the notion of the *interlacing* of sequences: for sequences $b^{(0)} = \langle b_n^{(0)} \rangle, b^{(1)} = \langle b_n^{(1)} \rangle, \dots, b^{(m-1)} = \langle b_n^{(m-1)} \rangle$, $m \geq 1$, their interlacing is the sequence $u = \langle u_n \rangle$ defined by

$$u_n = \begin{cases} b^{(0)}_{\frac{n}{m}}, & \text{if } n \equiv 0 \pmod{m}, \\ b^{(1)}_{\frac{n-1}{m}}, & \text{if } n \equiv 1 \pmod{m}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ b^{(m-1)}_{\frac{n-m+1}{m}}, & \text{if } n \equiv m-1 \pmod{m}. \end{cases} \tag{2}$$

For example, the interlacing of two sequences

$$b^{(0)}: \quad b_0^{(0)}, b_1^{(0)}, b_2^{(0)}, \dots,$$

$$b^{(1)}: \quad b_0^{(1)}, b_1^{(1)}, b_2^{(1)}, \dots$$

is the sequence $u_0, u_1, u_2, \dots$ of the form

$$b_0^{(0)}, b_0^{(1)}, b_1^{(0)}, b_1^{(1)}, b_2^{(0)}, b_2^{(1)}, \dots$$

A non-zero sequence g is *hypergeometric* if it satisfies a first order operator

$$\phi - h(x), \quad h(x) \in k,$$

the rational function $h(x)$ is the *certificate* of g. By definition, zero sequence is also hypergeometric with zero certificate.

The interlacing of m sequences, $m \geq 1$, which have the form of finite sums of hypergeometric sequences will be called an *m-interlacing*.

The $\mathcal{C}$-linear spaces of all sequences that satisfy L and, resp., of all m-interlacings, that satisfy L will be denoted by $V(L)$ and, resp., $V_m(L)$, $m \geq 1$. The Hendriks–Singer algorithm (HS) for constructing a basis for the $\mathcal{C}$-linear space of Liouvillian solutions of L is based on two facts proven in [5]:

(a) If L has a Liouvillian solution then for some integer m, $1 \leq m \leq \operatorname{ord} L$, the operator L has a solution in the form of an m-interlacing.

(b) For any integer m, $1 \leq m \leq \operatorname{ord} L$, one can construct algorithmically an operator $H \in k[\phi]$ such that $V(H) = V_m(H) = V_m(L)$. (It is possible, of course, that $\operatorname{ord} H = \dim V_m(H) = 0$.)

The central part of HS is constructing for a given m the operator H mentioned in (b), and a basis for $V_m(H)$. This procedure (a part of HS) will be denoted by mHS.

In Section 2, a simplification of the procedure mHS by removing some unnecessary actions is described. (The authors of the paper [5] notice that they ignore effectiveness questions and just try to present their algorithm in an understandable form — see Remarks on p. 251 of [5].)

In Section 3, we briefly consider the special cases when $\mathcal{C}$ is not algebraically closed, and the case of an irreducible L (the Cha – van Hoeij algorithm [2]).

In Section 4, we prove some properties of the space $V_m(L)$.

In Section 5, an implementation of a simplified version of mHS and a modified version of the search for all Liouvillian solutions is described.

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2 Search for m-interlacing Solutions of L for a Fixed m

2.1 The Hendriks–Singer Procedure for Finding m-interlacing Solutions

Let L be of the form (1), an integer m such that $1 \leq m \leq d$ be fixed, and $\tau : k \to k$ be the automorphism defined by $x \mapsto mx$. The procedure described in [5] by Hendriks and Singer for finding all solutions of L which have the form of m-interlacings is as follows.

mHS$_1$: Constructing a polynomial $P \in k[Z]$ of smallest degree such that the operator $P(\phi^m)$ is right divisible by L in $k[\phi]$.

mHS$_2$: Constructing polynomials $P_0, P_1, \ldots, P_{m-1} \in k[Z]$ such that if L has a solution in the form of an m-interlacing of some sequences $l^{(0)} = \langle l_n^{(0)} \rangle, l^{(1)} = \langle l_n^{(1)} \rangle, \ldots, l^{(m-1)} = \langle l_n^{(m-1)} \rangle$, then $P_i(\phi)(l^{(i)}) = 0$:

$$P_i = \tau \phi^i P, \quad i = 0, 1, \ldots, m-1.$$

mHS$_3$: Constructing finite sets $\mathcal{G}_i \subset k^*$, $i = 0, 1, \ldots, m-1$, such that $V_1(P_i(\phi)) \subset V(\operatorname{LCLM}_{h \in \mathcal{G}_i}(\phi - h))$ (one can use algorithms from [8], [6], and [3] for this).

mHS$_4$: Constructing an operator L_m such that $V_m(L) \subset V_m(L_m) = V(L_m)$:

$$L_m = \operatorname{LCLM}_{h \in \mathcal{H}}(\phi^m - h), \tag{3}$$

where

$$\mathcal{H} = \bigcup_{0 \leq i \leq m-1} \{\phi^{-i} \tau^{-1}(h) \,|\, h \in \mathcal{G}_i\}. \tag{4}$$

mHS$_5$: Constructing the operator $H = \operatorname{GCRD}(L, L_m)$ and a basis for $V(H)$ such that each element of this basis is the m-interlacing of hypergeometric sequences.

2.2 A Simplification of the Hendriks–Singer Procedure

The procedure mHS can be simplified by removing some unnecessary actions.

Theorem 1 *Let $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{m-1}$ be as in* mHS$_3$*, and $\mathcal{G} = \tau^{-1}\mathcal{G}_0$. In this case*

(i) $\mathcal{G}_i = \tau \phi^i \tau^{-1} \mathcal{G}_0 = \tau \phi^i \mathcal{G}$, $i = 0, 1, \ldots, m-1$;
(ii) one can use $\mathcal{G}$ instead of $\mathcal{H}$ in the right-hand side of (3).

Proof: (i) For P as in mHS$_1$, and $P_0, P_1, \ldots, P_{m-1}$ as in mHS$_2$ we have

$$P_i = \tau \phi^i P = \tau \phi^i \tau^{-1}(\tau P) = \tau \phi^i \tau^{-1} P_0.$$

The proof follows from the definition of $\mathcal{G}$. (Notice that $\tau \phi^i \tau^{-1}$ is defined by $x \mapsto x + \frac{i}{m}$, and $\tau \phi^i$ is defined by $x \mapsto mx + i$.)
(ii) We have $\mathcal{G}_i = \tau \phi^i \mathcal{G}$. Therefore, if $h \in \mathcal{G}_i$, then $\phi^{-i}\tau^{-1}(h) \in \mathcal{G}$. □

As a consequence of this theorem we obtain a simplified version of mHS which we denote by mHS$'$:

mHS$'_1$: The same as mHS$_1$.

mHS$'_2$: Constructing the polynomial $P_0 = \tau P$.

mHS_3': Constructing $\mathcal{G} = \tau^{-1}\mathcal{G}_0$, where the finite set $\mathcal{G}_0 \subset k^*$ is such that $V_1(P_0(\phi)) \subset V(\operatorname*{LCLM}_{h\in\mathcal{G}_0}(\phi - h))$.

mHS_4': Set $L_m = \operatorname*{LCLM}_{h\in\mathcal{G}}(\phi^m - h)$.

mHS_5': The same as mHS_5.

The cost of mHS_1', mHS_5' is the same as the cost of mHS_1, mHS_5. The cost of mHS_2', mHS_3', mHS_4' is m times less than the cost of mHS_2, mHS_3, mHS_4.

Example 1 *Let*

$$L = \phi^5 - \frac{2}{x+5}\phi^3 + \frac{x-1}{x+5}\phi^2 - \frac{2}{x+5}, \tag{5}$$

$m = 2$. *Then*

$$\begin{aligned}
&P(Z) = (x+10)(x+7)Z^5 - (4x+30)Z^4 + 4Z^3 - (x+4)(x+1)Z^2 + (4x+6)Z - 4,\\
&P_0(\phi) = (2x+10)(2x+7)\phi^5 - (8x+30)\phi^4 + 4\phi^3 - (2x+4)(2x+1)\phi^2 + (8x+6)\phi - 4,\\
&\mathcal{G}_0 = \left\{\tfrac{1}{x+1}, \tfrac{2}{2x-1}\right\},\\
&\mathcal{G} = \left\{\tfrac{2}{x+2}, \tfrac{2}{x-1}\right\},\\
&L_2 = \phi^4 - \tfrac{(4x+6)}{(x+4)(x+1)}\phi^2 + \tfrac{(4x+16)}{(x+1)},\\
&H = \phi^2 + \tfrac{(12x+12)}{(x+2)(x^3-x-8)}\phi - \tfrac{(2x^3+6x^2+4x-16)}{(x+2)(x^3-x-8)}.
\end{aligned}$$

A basis for $V(H) = V_2(L)$ *consists of two following sequences*

the 2*-interlacing of the sequences* $\left\langle \frac{1}{\Gamma(n-1/2)} \right\rangle$ *and* $\left\langle \frac{1}{\Gamma(n+3/2)} \right\rangle$,
the 2*-interlacing of the sequences* $\left\langle \frac{1}{\Gamma(n+1)} \right\rangle$ *and* $\left\langle \frac{1}{\Gamma(n)} \right\rangle$.

Notice that once the set $\mathcal{G}$ is constructed the operators L_m and H are not needed for constructing a basis for $V_m(L)$. This would simplify mHS'. But the operator H is used by the Hendriks–Singer algorithm for a recursion to construct all Liouvillian solutions of L (Section 5.3).

3 Some Special Cases

3.1 When $\mathcal{C}$ is Not Algebraically Closed

Suppose that $\mathcal{C}$ is not algebraically closed. Then L may have hypergeometric solutions whose certificates belong to $\bar{\mathcal{C}}(x)$ but not to $k = \mathcal{C}(x)$. However, the following statement holds (has been proven in [7]):

Let $L \in k[\phi]$ *and each of the sets* $\mathcal{G}_i$, $i = 0, 1, \ldots, m$, *constructed at the step* mHS_3 *contains all belonging to* $\bar{\mathcal{C}}(x)$ *certificates of hypergeometric solutions of* $P_i(\phi)$. *Then the operator* H *computed at the step* mHS_5 *belongs to* $k[\phi]$.

As a consequence we have that if $L \in k[\phi]$, and and the step mHS_4 we use some algorithm A for finding all certificates belonging to $\bar{\mathcal{C}}(x)$, then we obtain $H \in k[\phi]$. The operator L is right-divisible by H, and we have $L = \tilde{L}H$, $\tilde{L} \in k[\phi]$. Even if the algorithm A is applicable only to operators from $k[\phi]$ then we always can apply this algorithm to $\tilde{L}$. The same is correct if we use mHS' instead of mHS for constructing H since we construct the same H in both cases. This fact might be quite important for finding all Liouvillian solutions of L, if the corresponding implementation of an algorithm for finding hypergeometric solution is not applicable to operators with the coefficients from $\bar{\mathcal{C}}(x)$.

3.2 When L Is Irreducible

An algorithm which does not compute hypergeometric solutions of $P_0(\phi)$ was proposed in [2] for the case of an irreducible operator L. The idea of this algorithm is based on the notion of *gauge equivalence* of operators. Two operators $L, \tilde{L} \in k[\phi]$ are gauge equivalent if $\operatorname{ord} L = \operatorname{ord} \tilde{L}$ and there exists an operator T, $\operatorname{ord} T < \operatorname{ord} L$, such that $V(L) = T(V(\tilde{L}))$. If T exists then there also exists an "inverse" T' such that $V(\tilde{L}) = \tilde{T}(V(L))$. An irreducible L is gauge equivalent to an operator of the form (6) iff L has Liouvillian solutions. When L is irreducible, getting (6) is equivalent to computing Liouvillian solutions ([5]). It was shown in [2] that this approach is very productive for irreducible L of order 2 or 3. However the gauge equivalence is not sufficient to get (6) for L in the general case. The full factorization has a high complexity. In addition L may have a Liouvillian solution although $L = KM$ with an irreducible operator M which has no Liouvillian solution. It means one factorization $L = KM$ may not be sufficient for finding Liouvillian solution using factorization and paper [5]; it can happen that other factorizations $L = K'M'$ are needed to be searched for. So algorithms that are directly applicable in the general case are of definite value.

4 Some Properties of the Space $V_m(L)$

4.1 The Dimension of the Space $V_m(L)$

Lemma 1 *Any operator A of the form*

$$\phi^d - a(x), \ \ a(x) \in k, \tag{6}$$

can be written as

$$A = \mathrm{LCLM}(Q_1, Q_2, \ldots, Q_l) \tag{7}$$

for some irreducible operators $Q_1, Q_2, \ldots, Q_l$ of the same order ρ, $l\rho = d$.

Proof: By Corollary 4.4 from [5] A has an irreducible right factor Q of the form $\phi^s - b(x)$, $b(x) \in k$, s divides d. Let R_j be the operator $\phi - e^{\frac{2\pi j i}{d}}$, and the sequences $z^{(j)} = \langle z_n^{(j)} \rangle$ be such that $z_n^{(j)} = e^{\frac{2\pi j i}{d} n}$, $j = 1, 2, \ldots, d$ (recall that $k = \mathcal{C}(x)$ and the ground field $\mathcal{C}$ is an algebraically closed subfield of $\mathbb{C}$). Set Q_j to be equal to the symmetric product of Q and R_j. The operator Q_j is monic irreducible and of same order as Q, $j = 1, 2, \ldots, d$. Then $A = \mathrm{LCLM}(Q_1, Q_2, \ldots, Q_d)$ holds since there exists $y \in V(Q)$ such that $y_n \neq 0$ for all n, and therefore the sequences $z^{(j)}y$, $j = 1, 2, \ldots, d$, are linearly independent elements of $V(A)$. Notice that some of operators $Q_1, Q_2, \ldots, Q_d$ can be equal. We get (7) with pairwise different irreducible $Q_1, Q_2, \ldots, Q_l$ after removing duplicates.[4] □

Lemma 2 *Let L be of the form (1), $V_j(L) = 0$ for $j = 0, 1, \ldots, m-1$ and $V_m(L) \neq 0$. Let H be an operator such that $V(H) = V_m(L)$. Then H can be written as*

$$H = \mathrm{LCLM}(S_1, S_2, \ldots, S_t) \tag{8}$$

for some irreducible $S_1, S_2, \ldots, S_t$ of order m.

Proof: Follows from the construction of the operator H (see mHS_4, mHS_5), Lemma 1 and the fact that if H has a right factor of order $s < m$ then $V_j(L) \neq 0$ for some j such that $1 \leq j \leq s < m$. □

[4] This proof is by M. van Hoeij (a private communication).

Theorem 2 *Let L be of the form (1), $V_j(L) = 0$ for $j = 0, 1, \ldots, m-1$ and $V_m(L) \neq 0$. Then m divides* $\dim V_m(L)$.

Proof: Follows from Lemma 2. □

Example 2 *The operator*

$$L = \phi^5 - \phi^4 - (x+1)(x+3)\phi + x(x+2) \tag{9}$$

has no hypergeometric (i.e. 1-interlacing) solutions, so $V_1(L) = 0$. But it has 2-interlacing solutions, a basis for $V_2(L)$ consists of four following sequences

the 2-interlacing of the sequences $\langle 0 \rangle$ and $\left\langle (-2)^{n+1/2}\Gamma(n+1/2) \right\rangle$,
the 2-interlacing of the sequences $\langle 0 \rangle$ and $\left\langle 2^{n+1/2}\Gamma(n+1/2) \right\rangle$,
the 2-interlacing of the sequences $\langle (-2)^n\Gamma(n) \rangle$ and $\langle 0 \rangle$,
the 2-interlacing of the sequences $\langle 2^n\Gamma(n) \rangle$ and $\langle 0 \rangle$.

As by Theorem 2, $m = 2$ divides $\dim V_2(L) = 4$.

4.2 Structure of m-interlacing Solutions

Let sequences $\langle f_n \rangle, \langle f_n^{(0)} \rangle, \langle f_n^{(1)} \rangle, \ldots, \langle f_n^{(m-1)} \rangle$ be such that

$$f_n = \begin{cases} f_n^{(0)}, & \text{if } n \equiv 0 \pmod m, \\ f_n^{(1)}, & \text{if } n \equiv 1 \pmod m, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ f_n^{(m-1)}, & \text{if } n \equiv m-1 \pmod m, \end{cases} \tag{10}$$

$\overline{\forall} n$. Then we will use the notation

$$f = \left\langle f_n^{(0)}, f_n^{(1)}, \ \ldots \ , f_n^{(m-1)} \right\rangle,$$

e.g., we will write

$$\left\langle b^{(0)}_{\frac{n}{m}}, b^{(1)}_{\frac{n-1}{m}}, \ \ldots \ , b^{(m-1)}_{\frac{n-m+1}{m}} \right\rangle$$

for the sequence defined by (2) for all integer n large enough.

If $h(x) \in k$ is the certificate of a hypergeometric sequence $\langle g_n \rangle$ then the *source* of the sequence $\langle g_n \rangle$ is a meromorphic function $G(x)$ such that

- $G(x)$ is defined for all x with large enough $\operatorname{Re} x$, and $G(x+1) - h(x)G(x) = 0$,
- $g_n = G(n)$, $\overline{\forall} n$.

Remark. A priory the function $G(x)$ is not defined uniquely but up to a factor in the form of a 1-periodic holomorphic function. We suppose that one of such functions which does not vanish for $x \in \frac{1}{m}\mathbb{Z}$ is fixed, and, therefore, we get the source uniquely defined by the certificate $h(x)$.

Let

$$\langle g_n^{(1)} \rangle, \langle g_n^{(2)} \rangle, \ldots, \langle g_n^{(s)} \rangle \tag{11}$$

be hypergeometric sequences whose certificates are in $\mathcal{G}$ (see mSH$'_3$), and

$$G_1(x), G_2(x), \ldots, G_s(x) \tag{12}$$

be the sources of the hypergeometric sequences (11). By Theorem 1(i) any element of $V_m(L)$ can be represented in the form

$$\left\langle \sum_{j=1}^{s} c_{0,j}\, U_j(n), \ \sum_{j=1}^{s} c_{1,j}\, U_j(n), \ \ldots \ , \sum_{j=1}^{s} c_{m-1,j}\, U_j(n) \right\rangle, \tag{13}$$

with

$$U_j(x) = G_j\left(\frac{x}{m}\right), \quad j = 1, 2, \ldots, s, \tag{14}$$

and with some concrete complex constants

$$c_{i,j}, \quad i = 0, 1, \ldots, m-1, \; j = 1, 2, \ldots, s. \tag{15}$$

Note that in representation (13), all components of any element of $V_m(L)$ have identical structure, and each of the steps mHS_5 and mHS'_5 constructs a basis for the space of suitable constants (15).

Example 3 *The sequences belonging to the basis constructed in Example 1 can be presented as*

$$\left\langle \frac{c_1}{\Gamma(\frac{n}{2}-\frac{1}{2})} + \frac{c_2}{\Gamma(\frac{n}{2}+1)}, \frac{c_2}{\Gamma(\frac{n}{2}-\frac{1}{2})} + \frac{c_1}{\Gamma(\frac{n}{2}+1)} \right\rangle,$$

with $(1,0),(0,1)$ *as* (c_1,c_2)*, and the sequences belonging to the basis constructed in Example 2 can be presented as*

$$\left\langle c_1\,(-2)^{n/2}\,\Gamma(n/2) + c_2\,2^{n/2}\,\Gamma(n/2),\, c_3\,(-2)^{n/2}\,\Gamma(n/2) + c_4\,2^{n/2}\,\Gamma(n/2) \right\rangle$$

with $(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)$ *as* (c_1,c_2,c_3,c_4)*.*

We suppose that one unique value α *of* $2^{1/2}$ *and, resp., one unique value* β *of* $(-2)^{1/2}$ *are selected. Then* $2^{n/2} = \alpha^n$, $(-2)^{n/2} = \beta^n$*. This is in agreement with the remark to definition of the source of a hypergeometric sequence.*

The following proposition enables one to apply an operator of $k[\phi]$ to an m-interlacing of sequences.

Proposition 1 *If* $\langle f_n^{(0)}\rangle, \langle f_n^{(1)}\rangle, \ldots, \langle f_n^{(m-1)}\rangle$ *are arbitrary sequences, then*

$$\phi\left(\left\langle f_n^{(0)}, f_n^{(1)}, \ldots, f_n^{(m-1)} \right\rangle\right) = \left\langle f_{n+1}^{(1)}, f_{n+1}^{(2)}, \ldots, f_{n+1}^{(m-1)}, f_{n+1}^{(0)} \right\rangle.$$

Proof: A direct check. □

4.3 Cyclic Permuted Solutions

Some of functions (12) can be *similar*, i.e., such that $G_i(x)/G_j(x) \in k$ for some indexes $i \neq j$ (the corresponding hypergeometric sequences are also called similar).

Lemma 3 *Let* $h = \langle h_n \rangle$ *be a hypergeometric sequence with the source* $G(x)$*. Let* $f = \langle f_n \rangle$ *be the* m*-interlacing of sequences* $\langle S_0(n)h_n\rangle, \langle S_1(n)h_n\rangle, \ldots, \langle S_{m-1}(n)h_n\rangle$ *with* $S_0(x), S_1(x), \ldots, S_{m-1}(x) \in k$*. Then for some* $R_0(x), R_1(x), \ldots, R_m(x) \in k$ *and* $U(x) = G\left(\frac{x}{m}\right)$

$$f = \langle\, R_0(n)\,U(n),\; R_1(n)\,U(n-1),\; \ldots,\; R_{m-1}(n)\,U(n-m+1) \,\rangle, \tag{16}$$

and

$$\begin{aligned}\phi(f) = \langle\, R_1(n+1)\,U(n),\; R_2(n+1)\,U(n-1),\; \ldots \\ \ldots,\, R_m(n+1)\,U(n-m+1) \,\rangle.\end{aligned} \tag{17}$$

Proof: Indeed, by definition of f

$$f = \left\langle S_0\left(\frac{n}{m}\right) G\left(\frac{n}{m}\right),\ S_1\left(\frac{n-1}{m}\right) G\left(\frac{n-1}{m}\right),\ \ldots \right.$$
$$\left. \ldots,\ S_{m-1}\left(\frac{n-m+1}{m}\right) G\left(\frac{n-m+1}{m}\right) \right\rangle.$$

This proves (16). By Proposition 1 applying ϕ to a sequence of the form (16) gives

$$\langle\, R_1(n+1)\, U(n),\ R_2(n+1)\, U(n-1),\ \ldots$$
$$\ldots\ , R_{m-1}(n+1)\, U(n-m+2),\ R_0(n+1)\, U(n+1)\,\rangle\,.$$

But $U(n+1) = U((n-m+1)+m) = S(n)U(n-m+1)$ with a rational function $S(x)$. Setting $R_{m+1}(x) = S(x-1)R_0(x)$ we get (17). □

Lemma 4 *Any element of $V_m(L)$ can be represented as a sum of solutions such that each of these solutions has the form of an m-interlacing of similar hypergeometric sequences.*

Proof: We can present any element of $V_m(L)$ as a sum of the m-interlacings of similar hypergeometric sequences such that the components of different summands are not similar. Application of L to the m-interlacing of similar hypergeometric sequences gives again the m-interlacing of similar hypergeometric sequences whose components are similar to components of the original m-interlacing. The claimed follows. □

Theorem 3 *Let L have a solution (13). Then L has the cyclic permuted solution*

$$\left\langle \sum_{j=1}^{s} c_{1,j}\, U_j(n),\ \sum_{j=1}^{s} c_{2,j}\, U_j(n),\ \ldots,\ \sum_{j=1}^{s} c_{m-1,j}\, U_j(n),\ \sum_{j=1}^{s} c_{0,j}\, U_j(n) \right\rangle. \tag{18}$$

Proof: By Lemmas 3, 4 it is sufficient to consider the case (16) of the m-interlacing of similar hypergeometric sequences. We have to prove that if (16) is a solution of L then

$$\langle\, R_1(n)\, U(n-1),\ R_2(n)\, U(n-2),\ \ldots$$
$$\ldots\ , R_{m-1}(n)\, U(n-m+1),\ R_0(n)\, U(n)\rangle \tag{19}$$

is also a solution of L. By the second part of Lemma 3 the result of applying L to (19) has the form

$$\langle\, S_0(n)\, U(n-1),\ S_1(n)\, U(n-2),\ \ldots\ , S_{m-1}(n)\, U(n-m)\,\rangle\,, \tag{20}$$

$S_0(x), S_1(x), \ldots, S_{m-1}(x) \in k$.

We introduce the operation $'$:

$$\left\langle f_n^{(0)}, f_n^{(1)}, \ldots, f_n^{(m-1)} \right\rangle' = \left\langle f_{n+1}^{(0)}, f_{n+1}^{(1)}, \ldots, f_{n+1}^{(m-1)} \right\rangle$$

(in the case $m = 1$ this operation coincides with ϕ). For the operator (1) we set $L' = \phi^d + a_{d-1}(x+1)\phi^{d-1} + \ldots + a_1(x+1)\phi + a_0(x+1)$. It is easy to see that $L'(f') = (L(f))'$ for any m-interlacing.

Using Proposition 1 we have

$$L\,(\langle\, R_0(n)\, U(n),\ R_1(n)\, U(n-1),\ \ldots\ , R_{m-1}(n)\, U(n-m+1)\,\rangle) =$$
$$= L\phi^{-1}\,(\langle\, R_1(n+1)\, U(n),\ R_2(n+1)\, U(n-1),\ \ldots$$

$$
\begin{aligned}
&\quad \ldots, R_{m-1}(n+1)\,U(n-m),\ R_0(n+1)\,U(n+1)\,\rangle) = \\
&= \phi^{-1} L'\,(\langle\, R_1(n+1)\,U(n),\ R_2(n+1)\,U(n-1),\ \ldots \\
&\quad \ldots, R_{m-1}(n+1)\,U(n-m),\ R_0(n+1)\,U(n+1)\,\rangle) = \\
&= \phi^{-1} L'\,(\langle\, R_1(n)\,U(n-1),\ R_2(n)\,U(n-2),\ \ldots \\
&\quad \ldots, R_{m-1}(n)\,U(n-m-1),\ R_0(n)\,U(n)\,\rangle') = \\
&= \phi^{-1}((L\,(\langle\, R_1(n)\,U(n-1),\ R_2(n)\,U(n-2),\ \ldots \\
&\quad \ldots, R_{m-1}(n)\,U(n-m-1),\ R_0(n)\,U(n)\,\rangle))') = \\
&= \phi^{-1}\,(\langle\, S_0(n+1)\,U(n),\ S_1(n+1)\,U(n-1),\ \ldots \\
&\quad \ldots, S_{m-1}(n+1)\,U(n-m+1)\,\rangle) = \\
&= (\langle\, S_{m-1}(n)\,U(n-m),\ S_0(n)\,U(n-1),\ \ldots, S_{m-2}(n)\,U(n-m+1)\,\rangle)\,.
\end{aligned}
$$

Since (16) is a solution of L we have

$$S_i(n) = 0 \quad \text{when} \quad n \equiv i+1 \pmod{m}, \quad i = 0, 1, \ldots m-1.$$

But this implies that $S_i(x)$, $i = 0, 1, \ldots, m-1$, are equal to zero identically. Therefore, (19) is a solution of L. $\square$

Example 4 *The sequences belonging to the basis constructed in Example 1 can be presented as*

$$\left\langle \frac{1}{\Gamma(\frac{n}{2}-\frac{1}{2})}, \frac{1}{\Gamma(\frac{n}{2}+1)} \right\rangle, \left\langle \frac{1}{\Gamma(\frac{n}{2}+1)}, \frac{1}{\Gamma(\frac{n}{2}-\frac{1}{2})} \right\rangle.$$

5 Implementation

The improvements proposed in Section 2 are implemented as a modification of the MAPLE implementation [7] of the original Hendriks–Singer algorithm for finding Liouvillian solutions. The implementation is done as `LiouvillianSolution` function that extends the MAPLE package `LREtools` containing various functions for solving linear recurrence equations. To the best of the authors' knowledge it is the only one existing full implementations (at least, in MAPLE) of this algorithm.

The paper [1] presents a modification of the Hendriks–Singer algorithm HS and describes its implementation. However, the implementation described in that paper is not full since it solves the second-order equations only.

5.1 Liouvillian Solutions

By [5] the ring $\mathcal{L}$ of Liouvillian sequences is the smallest subring of $\mathcal{S}$ such that
$k \subset \mathcal{L}$,
$u \in \mathcal{L}$ iff $\phi(u) \in \mathcal{L}$,
$u \in k$ implies that $v \in \mathcal{L}$ if $\phi(v) = uv$,
$u \in \mathcal{L}$ implies that $v \in \mathcal{L}$ if $\phi(v) = u + v$,
$u^{(0)}, u^{(1)}, \ldots, u^{(m-1)} \in \mathcal{L}$ implies that the interlacing of these sequences belongs to $\mathcal{L}$.

Note that the definition of Liouvillian sequences may be given in a different way, but the defined object is still the same. For example, by [9] (also used in [7]) the ring $\mathcal{L}$ of Liouvillian sequences is the smallest subring of $\mathcal{S}$ that contains the set of all hypergeometric elements from $\mathcal{S}$ and closed with respect to ϕ, ϕ^{-1}, Σ (the summation), and the interlacing.

The space of all Liouvillian solutions of L will be denoted by $V_{\mathcal{L}}(L)$. We also will use notations $V(L), V_m(L)$ introduced before.

5.2 Finding the Operator H and a Basis for the Space $V_m(H)$ for a Fixed m

The implementation [7] was adjusted to utilize the procedure mHS′ described in Section 2.2. In the initial implementation [7], the procedure mHS was an internal integral part of the function `LiouvillianSolution` to find all Liouvillian solutions. The output option `output=interlacing[m]` is added to `LiouvillianSolution` to provide users with a possibility to search for the space $V_m(L)$ for a given m. Note that m can be omitted in the option, and in this case, the function will find itself the smallest m for which $V_m(L) \neq 0$ (if such integer m exist). This is exactly what is needed as a first step for finding all Liouvillian solutions by HS, i.e., to construct the space $V_{\mathcal{L}}(L)$. Our implementation represents m-interlacing solutions in the form (13) using MAPLE's structure `piecewise` in the sense of the expression (10).

Example 5 *Consider equation (5) from Example 1:*

```
> rec := (n+5)*y(n+5)-2*y(n+3)+(n-1)*y(n+2)-2*y(n):
```

There is no 1-interacing solution:

```
> sol1 := LiouvillianSolution(rec, y(n), {},
output=interlacing[1]);
```

$$sol1 := FAIL$$

There are 2-interacing solutions as was presented in Example 1, and 2 *is the smallest m for which there are m-interlacing solutions:*

```
> sol2 := LiouvillianSolution(rec, y(n), {}, output=interlacing);
```

$$sol2 := \begin{cases} \dfrac{\_C_1}{\Gamma(\frac{n}{2}+1)} + \dfrac{\_C_2}{\Gamma(\frac{n}{2}-\frac{1}{2})} & \mathrm{irem}(n,\,2)=1 \\ \dfrac{\_C_2}{\Gamma(\frac{n}{2}+1)} + \dfrac{\_C_1}{\Gamma(\frac{n}{2}-\frac{1}{2})} & \mathit{otherwise} \end{cases}$$

Now try to find 4-interlacing solutions:

```
> sol4 := LiouvillianSolution(rec, y(n), {},
output=interlacing[4]);
```

$$sol4 := \begin{cases} \dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\_C_1}{\Gamma(\frac{n}{4}+\frac{1}{4})\,\Gamma(\frac{n}{4}-\frac{1}{4})} + \dfrac{1}{4}\,\dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\sqrt{2}\,\_C_2}{\Gamma(\frac{n}{4}+\frac{1}{2})\,\Gamma(\frac{n}{4}+1)} & \mathrm{irem}(n,\,4)=1 \\ \dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\_C_2}{\Gamma(\frac{n}{4}+\frac{1}{4})\,\Gamma(\frac{n}{4}-\frac{1}{4})} + \dfrac{1}{4}\,\dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\sqrt{2}\,\_C_1}{\Gamma(\frac{n}{4}+\frac{1}{2})\,\Gamma(\frac{n}{4}+1)} & \mathrm{irem}(n,\,4)=2 \\ \dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\_C_1}{\Gamma(\frac{n}{4}+\frac{1}{4})\,\Gamma(\frac{n}{4}-\frac{1}{4})} + \dfrac{1}{4}\,\dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\sqrt{2}\,\_C_2}{\Gamma(\frac{n}{4}+\frac{1}{2})\,\Gamma(\frac{n}{4}+1)} & \mathrm{irem}(n,\,4)=3 \\ \dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\_C_2}{\Gamma(\frac{n}{4}+\frac{1}{4})\,\Gamma(\frac{n}{4}-\frac{1}{4})} + \dfrac{1}{4}\,\dfrac{(\frac{1}{4})^{(\frac{n}{4})}\,\sqrt{2}\,\_C_1}{\Gamma(\frac{n}{4}+\frac{1}{2})\,\Gamma(\frac{n}{4}+1)} & \mathit{otherwise} \end{cases}$$

They exist. But in this case, they actually correspond to the 2-interlacing solutions up to arbitrary constants transformation, which can be checked directly.

There is also an implementation consideration which does not relate to the algorithm efficiency but to the efficiency of the implementation in MAPLE. As follows from Theorem 1(ii) the elements of the union in the right-hand side of (4) are equal for all i. But if `LCLM` function is applied to $\mathcal{H}$ directly without removing duplicated elements it works very ineffective. Since it is just a peculiarity of the MAPLE implementation rather than algorithm's feature, in order to check only the gain from the algorithm simplification itself, the trick of removing duplicates in $\mathcal{H}$ before application of `LCLM` was added into our mHS implementation.

Example 6 *Consider the following equation:*

```
> rec := m^2*y(n+4+m)-((n+4)^2-m^2)*y(n+4)+(n-10)*m^2*y(n+1+m)-
> (n-10)*((n+1)^2-m^2)*y(n+1)-m^2*y(n+m)+(n^2-m^2)*y(n);
```

$$rec := m^2\,\mathrm{y}(n+4+m) - ((n+4)^2 - m^2)\,\mathrm{y}(n+4) + (n-10)\,m^2\,\mathrm{y}(n+1+m)$$
$$-\,(n-10)\,((n+1)^2 - m^2)\,\mathrm{y}(n+1) - m^2\,\mathrm{y}(n+m) + (n^2-m^2)\,\mathrm{y}(n)$$

The equation has m-interlacing solutions with the components which differ by constant factors. For example, let $m = 3$.

```
> m := 3: LiouvillianSolution(rec, y(n), {},
> output=interlacing[m]);
```

$$\begin{cases} \Gamma(\frac{n}{3}+1)\,\Gamma(-1+\frac{n}{3})\,\_C_1 & \mathrm{irem}(n,\,3)=1 \\ \Gamma(\frac{n}{3}+1)\,\Gamma(-1+\frac{n}{3})\,\_C_2 & \mathrm{irem}(n,\,3)=2 \\ \Gamma(\frac{n}{3}+1)\,\Gamma(-1+\frac{n}{3})\,\_C_3 & \mathit{otherwise} \end{cases}$$

To check the performance changes we use $m = 10$.

```
> m := 10:
```

Let us find 10-interlacing solutions (not printed, but it has the same structure as above for the case $m = 3$) and check the time needed to compute the result.

```
> st := time(): LiouvillianSolution(rec, y(n), {},
> output=interlacing[m]): time()-st;

m=10:

 ---Finding P took 0.6 seconds

 ---Constructing L_10 took 3.3 seconds

 ---Computing H took 0.0 seconds

 ---Constructing basis took 0.1 seconds
```

$$4.063$$

```
> st := time(): LiouvillianSolution_old(rec, y(n), {},
> output=interlacing[m]): time()-st;

m=10:

 ---Finding P took 0.6 seconds

 ---Constructing L_10 took 34.9 seconds

 ---Computing H took 0.0 seconds

 ---Constructing basis took 0.1 seconds
```

35.562

The old version (mHS) took more time than the simplified one (mHS') to check existence of 10-interlacing solution. But if we missed LCLM trick the old version is even worse. Note that unchanged parts in both versions took the same time.

```
> st := time(): LiouvillianSolution_old_no_trick(rec, y(n), {},
> output=interlacing[m]): time()-st;

m=10:

 ---Finding P took 0.6 seconds

 ---Constructing L_10 took 74.5 seconds

 ---Computing H took 0.0 seconds

 ---Constructing basis took 0.1 seconds
```

75.125

5.3 Finding All Liouvillian Solutions

Liouvillian solutions of general form are constructed recursively by HS. The recursive application of mHS or mHS$'$ leads to a factorization $L = RH_t \dots H_2H_1$ where the operator R is such that $V_m(R) = 0$ for all integer $m \geq 1$, and where each of the operators H_i satisfies $V(H_i) = V_{m_i}(H_i) \neq 0$ for an integer $m_i \geq 1$. For any $i = 1, 2, \dots, t$ a basis B_i for $V_{m_i}(H_i)$ has to be constructed. Once a basis B_i for $V(H_i) = V_{m_i}(H_i)$ is constructed, $i = 1, 2, \dots, t$, algorithm HS constructs a basis B of $V(H_t \dots H_2H_1) = V_{\mathcal{L}}(H_t \dots H_2H_1) = V_{\mathcal{L}}(L)$, using the difference version of the method of variation of parameters ([4]). To do this HS solves $t-1$ linear algebraic systems whose determinants are shifted Casoratians which correspond to the bases $B_1, B_2, \dots, B_{t-1}$ for solutions spaces of the operators $H_1, H_2, \dots, H_{t-1}$.

Recall that the bases $B_1, B_2, \dots, B_t$ and the basis B consist of the elements of the ring $\mathcal{S}$ of the germs of sequences. The problem of defining integer n_0 such that any of the germs from B is a sequence (in the usual meaning) that satisfies L for all $n \geq n_0$ looks like quite actual. We will describe below two rules following which a suitable n_0 can be computed. Our implementation provides users with such n_0 in addition to B.

We will suppose that all operators under consideration are of the form

$$\phi^s + r_{s-1}(x)\phi^{s-1} + \dots + r_0(x), \tag{21}$$

where $r_0(x), r_1(x), \dots, r_{s-1}(x) \in k$. For an integer l we set $\mathbb{N}_l = \{n \in \mathbb{N}, n \geq l\}$. The mentioned rules are as follows.

1) If a rational function $h(x)$ is defined and does not vanish on $\mathbb{N}_l$ then a hypergeometric sequence with certificate $h(x)$ is defined and does not vanish on $\mathbb{N}_l$. Note that the Casoratian of a basis for the solutions space of (21) also represents a hypergeometric sequence with $h(x) = (-1)^s r_0(x)$ ([4]).

2) If operators $L, \tilde{L}, H$ are such that $L = \tilde{L}H$ and we use the procedure described in [5, Lemma 5.4.1] for constructing a basis for $V(H)$ in the form of a finite set of linear combinations of some computed sequences, then elements of the basis sequences are defined on $\mathbb{N}_l$ if initial computed sequences are defined on $\mathbb{N}_l$ and all coefficients of $L, \tilde{L}, H$ are defined on $\mathbb{N}_l$ as well.

Our implementation computes the Casoratian as a hypergeometric sequence using its certificate. The computed result may differ from the Casoratian by a constant factor. It still leads to computing correct basis elements since the elements are also defined up to an arbitrary constant non-zero factor.

Example 7 *Consider again equation (5) from Example 1:*

```
> rec := (n+5)*y(n+5)-2*y(n+3)+(n-1)*y(n+2)-2*y(n):
```

Find all Liouvillian solutions. We use the implicit output form, since the explicit forms are too huge. The computation time is also printed:

```
> st := time(): LiouvillianSolution(rec, y(n), {},
output=implicit, usepiecewise=true); time()-st;
```

$$\Bigg[\_C_1 \left(\left(\sum_{i1=2}^{n-1} \left(-\frac{B_{12}(i1+1)\, B_{21}(i1)}{\mathrm{D}_1(i1)}\right)\right) B_{11}(n) + \left(\sum_{i1=2}^{n-1} \frac{B_{11}(i1+1)\, B_{21}(i1)}{\mathrm{D}_1(i1)}\right) B_{12}(n)\right) +$$

$$\_C_2 \left(\left(\sum_{i1=2}^{n-1} \left(-\frac{B_{12}(i1+1)\, B_{22}(i1)}{\mathrm{D}_1(i1)}\right)\right) B_{11}(n) + \left(\sum_{i1=2}^{n-1} \frac{B_{11}(i1+1)\, B_{22}(i1)}{\mathrm{D}_1(i1)}\right) B_{12}(n)\right) +$$

$$\_C_3 \left(\left(\sum_{i1=2}^{n-1} \left(-\frac{B_{12}(i1+1)\, B_{23}(i1)}{\mathrm{D}_1(i1)}\right)\right) B_{11}(n) + \left(\sum_{i1=2}^{n-1} \frac{B_{11}(i1+1)\, B_{23}(i1)}{\mathrm{D}_1(i1)}\right) B_{12}(n)\right) + \_C_4\, B_{11}(n) + \_C_5\, B_{12}(n),$$

$$\Bigg[B_{21}(n) = \frac{(-1)^n\,(6+3\,n+n^2)}{(n+2)\,(n^3-n-8)},$$

$$B_{22}(n) = \frac{(\frac{1}{2}-\frac{1}{2}\,I\,\sqrt{3})^n\,(-3-\sqrt{3}\,I+n\,\sqrt{3}\,I+n^2)}{(n+2)\,(n^3-n-8)},$$

$$B_{23}(n) = -\frac{(\frac{1}{2}+\frac{1}{2}\,I\,\sqrt{3})^n\,(3-\sqrt{3}\,I+n\,\sqrt{3}\,I-n^2)}{(n+2)\,(n^3-n-8)},$$

$$B_{11}(n) = \begin{cases} \dfrac{1}{\Gamma(\frac{n}{2}+1)} & \mathrm{irem}(n,\,2)=1 \\ \dfrac{1}{\Gamma(\frac{n}{2}-\frac{1}{2})} & \textit{otherwise} \end{cases},$$

$$B_{12}(n) = \begin{cases} \dfrac{1}{\Gamma(\frac{n}{2}-\frac{1}{2})} & \mathrm{irem}(n,\,2)=1 \\ \dfrac{1}{\Gamma(\frac{n}{2}+1)} & \textit{otherwise} \end{cases},$$

$$\mathrm{D}_1(n) = \frac{(-2)^{(n+1)}\,((n+1)^3-n-9)}{\Gamma(n+3)}\Bigg],\ 2 \le n\Bigg]$$

4.125

The solutions basis is formed from a 2*-interlacing basis of* 2 *elements and a 1-interlacing basis of 3 elements. The corresponding shifted Casoratian* $D_1(n)$ *is also presented. The expression is applicable for* $n \geq 2$*.*

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