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Rational Normal Forms and Minimal Decompositions of Hypergeometric Terms

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We describe a multiplicative normal form for rational functions which exhibits the shift structure of the factors, and investigate its properties. On the basis of this form we propose an algorithm which, given a rational function R, extracts a rational part F from the product of consecutive values of R: $\prod_{k=n_0}^{n-1} R(k) = F(n) \prod_{k=n_0}^{n-1} V(k)$ where the numerator and denominator of the rational function V have minimal possible degrees. This gives a minimal multiplicative representation of the hypergeometric term $\prod_{k=n_0}^{n-1} R(k)$. We also present an algorithm which, given a hypergeometric term $T(n)$, constructs hypergeometric terms $T_1(n)$ and $T_2(n)$ such that $T(n) = \Delta T_1(n) + T_2(n)$ and $T_2(n)$ is minimal in some sense. This solves the additive decomposition problem for indefinite sums of hypergeometric terms: $\Delta T_1(n)$ is the "summable part", and $T_2(n)$ the "non-summable part" of $T(n)$. In other words, we get a minimal additive decomposition of the hypergeometric term $T(n)$.

1. Introduction

Multiplicative normal forms for rational functions which exhibit the shift structure of the factors are useful tools in the investigation of problems of summation and solution of difference equations in closed form. In Section 2 we represent a rational function $R(x)$ in the form

$$R(x) = V(x)\frac{F(x+1)}{F(x)} \tag{1}$$

where $V(x) = r(x)/s(x)$ and $F(x)$ are rational functions such that the polynomials $r(x)$ and $s(x+k)$ are relatively prime for all $k \in \mathbb{Z}$. We call such a representation a *rational normal form* (RNF) of R. Although a rational function can have several RNF's, the degrees of the numerator and denominator of V in (1) are uniquely defined.

Using the concept of RNF, we solve two decomposition problems for univariate hypergeometric terms. (For definitions, see the last paragraph of this section.) First, recall the well-known decomposition problems for indefinite integrals (Hermite, 1872; Ostrogradsky, 1845) and indefinite sums (Abramov, 1975, 1995; Paule, 1995; Pirastu and Strehl, 1995) of rational functions. Suppose for simplicity that a rational function R has no poles at non-negative arguments. Then it is possible to construct the representations

$$\int_0^x R(t)\,dt = F(x) + \int_0^x H(t)\,dt, \qquad \sum_{k=0}^{n-1} R(k) = S(n) + \sum_{k=0}^{n-1} T(k),$$

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where F, H and S, T are rational functions such that H, T have denominators of minimal possible degrees. In Section 3 we show how to obtain a *minimal multiplicative representation* of a hypergeometric term $T(n)$, i.e. how to find rational functions V and F such that $T(n) = F(n)\prod_{k=n_0}^{n-1} V(k)$ and the numerator and denominator of V are both of minimal possible degrees.

In Section 4 we describe an algorithm which solves the *minimal additive decomposition problem* for hypergeometric terms. Recall that the well-known Gosper's algorithm (Gosper, 1978) solves the *problem of indefinite hypergeometric summation*: given a hypergeometric term $T(n)$, find another hypergeometric term $T_1(n)$ such that

$$T(n) = \Delta T_1(n), \tag{2}$$

provided that such a term exists. If no hypergeometric term $T_1(n)$ satisfies (2), we can ask for *two* hypergeometric terms $T_1(n)$ and $T_2(n)$ such that

$$T(n) = \Delta T_1(n) + T_2(n) \tag{3}$$

and $T_2(n)$ is minimal in some sense. Given $T(n) = U(n)\prod_{k=n_0}^{n-1} D(k)$ with D having the numerator and denominator of minimal possible degrees, we describe how to find $T_1(n)$ and $T_2(n)$ such that $T_2(n) = V(n)\prod_{k=n_0}^{n-1} F(k)$ where the degrees of the numerator and denominator of F equal those of D. We show that for any other pair of terms $T_1(n)$, $T_2(n)$ it is impossible to decrease the degree of the denominator of V without increasing the degrees of the numerator and denominator of F. Preliminary publications on this topic have appeared as Abramov and Petkovšek (2001a,b).

Throughout the paper, K is a field of characteristic zero, and $\mathbb{N}$ denotes the set of non-negative integers. A sequence $T(n)$ of elements of K defined for all integers $n \geq n_0$ is a *hypergeometric term* if there are polynomials $p, q \in K[x] \setminus \{0\}$ such that $q(n)T(n+1) = p(n)T(n)$ for all $n \geq n_0$. Note that for every hypergeometric term $T(n)$ there is an integer $n_1 \geq n_0$ such that either $T(n) \neq 0$ for all $n \geq n_1$, or $T(n) = 0$ for all $n \geq n_1$. If $T(n)$ is eventually non-zero then the rational function p/q is unique and is called the *certificate* of T. A hypergeometric term $T(n)$ is *rational* if there is a rational function $R \in K(x)$ such that $T(n) = R(n)$ for all large enough n. Hypergeometric terms T_1 and T_2 are *similar* if there is a rational function $R \in K(x)$ such that $T_1(n) = R(n)T_2(n)$ for all large enough n. We write $p \perp q$ to indicate that polynomials $p, q \in K[x]$ are relatively prime.

As usual, if $R = p \oslash q$ where $p, q \in K[x]$, $p \perp q$ and q is monic, we call p the *numerator* of R, q the denominator of R, and write $p = \text{num}R$, $q = \text{den}R$. The *leading coefficient* of a rational function is the quotient of the leading coefficients of its numerator and denominator. A rational function is *monic* if its leading coefficient is 1. We denote the shift operator by E, and let it act on both sequences by $ET(n) = T(n+1)$, and on rational functions by $ER(x) = R(x+1)$. We write $\Delta = E - 1$. A rational function $R \in K(x)$ is *shift-reduced* if there are $a, b \in K[x]$ such that $R = a \oslash b$ and $a \perp E^k b$ for all $k \in \mathbb{Z}$. A polynomial $p \in$ *shift-free* if $p \perp E^k p$ for all $k \in \mathbb{Z} \oslash \{0\}$.

2. Rational Normal Forms

Following Paule (1995) we introduce the notion of shift-equivalence among polynomials.

Definition 1. Irreducible polynomials $p, q \in K[x]$ are *shift-equivalent* if $p \mid E^k q$ for some $k \in \mathbb{Z}$. In this case we write $p \overset{\text{sh}}{\sim} q$. A rational function $R \in K(x)$ is *shift-homogeneous* if all non-constant irreducible factors of $\operatorname{num} R$ and $\operatorname{den} R$ belong to the same shift-equivalence class, which we call the *type* of R.

It is clear that by grouping together shift-equivalent irreducible monic factors of its numerator and denominator every rational function can be written in the form

$$R(n) = z\, R_1(n) R_2(n) \cdots R_k(n) \tag{4}$$

where $z \in K$, $k \geq 0$, each R_i is a monic shift-homogeneous rational function, and $R_i R_j$ is not shift-homogeneous whenever $i \neq j$. We call (4) a *shift-homogeneous factorization* of R.

LEMMA 1. *Let $R(n) = z\, R_1(n) R_2(n) \cdots R_k(n) = w\, S_1(n) S_2(n) \cdots S_k(n)$ be two shift-homogeneous factorizations of R such that R_i and S_i have the same type. Then $z = w$ and $R_i = S_i$ for all i.*

PROOF. Clearly $z = w$ because they are both equal to the leading coefficient of R. Therefore

$$\frac{R_i(n)}{S_i(n)} = \prod_{\substack{j=1 \\ j \neq i}}^{k} \frac{S_j(n)}{R_j(n)} \quad (i = 1, 2, \ldots, k).$$

As every non-constant irreducible factor of the left-hand fraction is shift-inequivalent to every such factor of the right-hand fraction, $R_i(n)/S_i(n) = 1$, and $R_i = S_i$, for all i. □

The following well-known form is used in algorithms for hypergeometric summation (Gosper, 1978), finding hypergeometric solutions of difference equations (Petkovšek, 1992), and rational summation (Pirastu and Strehl, 1995).

DEFINITION 2. Let $R \in K(x)$ be a rational function. If $z \in K$ and monic polynomials $a, b, c \in K[x]$ satisfy

(i) $R = z \cdot \frac{a}{b} \cdot \frac{Ec}{c}$,
(ii) $a \perp E^k b$ for all $k \in \mathbb{N}$,

then (z, a, b, c) is a *polynomial normal form* (*PNF*) of R. If in addition,

(iii) $a \perp c$ and $b \perp Ec$,

then (z, a, b, c) is a *strict PNF* of R.

Every non-zero rational function has a unique strict PNF. For a proof of this, and for an algorithm to compute it, see Petkovšek (1992) or Petkovšek *et al.* (1996).

LEMMA 2. *If (a, b, c) is a strict PNF of p/q where $p, q \in K[x]$, then $a \mid p$ and $b \mid q$.*

PROOF. We have $pbc = aqEc$, hence $a \mid pbc$ and $b \mid aqEc$. By (ii) and (iii), $a \perp bc$ and $b \perp aEc$, so $a \mid p$ and $b \mid q$. □

Instead of (ii) we will need the stronger property that a/b is shift-reduced. Therefore we allow c to be a rational function.

Definition 3. Let $R \in K(x)$ be a rational function. If $z \in K$ and monic polynomials $r, s, u, v \in K[x]$ satisfy

(i) $R = z \cdot \frac{r}{s} \cdot \frac{E(u/v)}{u/v}$ where $u \perp v$,
(ii) $r \perp E^k s$ for all $k \in \mathbb{Z}$,

then (z, r, s, u, v) is a *rational normal form* (*RNF*) of R. If in addition,

(iii) $r \perp u \cdot Ev$ and $s \perp Eu \cdot v$,

then (z, r, s, u, v) is a *strict RNF* of R.

Sometimes we write the RNF $\varphi = (z, r, s, u, v)$ of R more succinctly as (F, V) where $F = zr/s$ and $V = u/v$. Then $F, V \in K(x)$, and

(i) $R = F \cdot \frac{EV}{V}$,
(ii) F is shift-reduced.

We call F the *kernel* of φ.

The following example shows that a rational function can have several RNF's, even strict ones.

Example 1. Let $R(x) = x(x+2)/((x-1)(x+1)^2(x+3))$. Then we can write $R = F(EV)/V$ where

$$\begin{array}{lll}
R = \frac{1}{(x-1)(x+3)} \cdot \frac{x(x+2)}{(x+1)^2}, & F = \frac{1}{(x-1)(x+3)}, & V = \frac{x+1}{x}, \\
R = \frac{1}{(x+1)^2} \cdot \frac{x(x+2)}{(x-1)(x+3)}, & F = \frac{1}{(x+1)^2}, & V = \frac{x-1}{x+2}, \\
R = \frac{1}{(x+1)(x+3)} \cdot \frac{x(x+2)}{(x-1)(x+1)}, & F = \frac{1}{(x+1)(x+3)}, & V = (x-1)(x+1), \\
R = \frac{1}{(x-1)(x+1)} \cdot \frac{x(x+2)}{(x+1)(x+3)}, & F = \frac{1}{(x-1)(x+1)}, & V = \frac{1}{x(x+2)},
\end{array}$$

so R has four different strict RNF's.

Proposition 1. *Let $\varphi = (z, r, s, u, v)$ be an RNF of $R = p/q$ where $p, q \in K[x]$.*

(i) *If φ is strict then $r \mid p$ and $s \mid q$.*
(ii) *$\varphi^{-1} = (1/z, s, r, v, u)$ is an RNF of $1/R$. If φ is strict then so is φ^{-1}.*
(iii) *If (z, r, s, u', v') is another RNF of R then $u' = u$ and $v' = v$.*
(iv) *The set of strict RNF's of R is finite.*

Proof. (i) As $p\,s\,u\,Ev = z\,q\,r\,v\,Eu$, and $r \perp s\,u\,Ev$, it follows that $r \mid p$. Similarly $s \mid q$.
(ii) Clearly,

$$\frac{1}{z} \cdot \frac{s}{r} \cdot \frac{E(v/u)}{v/u} = \left(z \cdot \frac{r}{s} \cdot \frac{E(u/v)}{u/v}\right)^{-1} = \frac{1}{R}.$$

Properties (ii) and (iii) of RNF are invariant on exchanging r with s and u with v, so they remain satisfied for the new form.

(iii) Write $V' = u'/v'$. Then $u\,Ev\,EV' - v\,Eu\,V' = 0$ which, given u and v, is a first-order homogeneous linear recurrence with polynomial coefficients for the unknown function V', with general solution $V' = Cu/v$ where C is an arbitrary constant. As u, v, u', v' are monic, $u \perp v$, and $u' \perp v'$, this implies that $u = u'$ and $v = v'$.

(iv) By (i), there are only finitely many candidates for r and s. By (iii), each choice of r and s leads to at most one RNF of R. □

Theorem 1. *Every rational function $R \in K(x)$ has a strict RNF.*

Proof. If $R = 0$ take $z = 0$ and $r = s = u = v = 1$. Otherwise let (z, a, b, c) be a strict PNF of R, $(1, s, r, d)$ a strict PNF of b/a, and $c/d = u/v$ where $u, v \in K[x]$ are monic and $u \perp v$. We claim that (z, r, s, u, v) is a strict RNF of R. Indeed,

$$z \cdot \frac{r}{s} \cdot \frac{E(u/v)}{u/v} = z \cdot \frac{r}{s} \cdot \frac{d}{Ed} \cdot \frac{Ec}{c} = z \cdot \frac{a}{b} \cdot \frac{Ec}{c} = R,$$

proving (i). Because $s \perp E^k r$ for $k \geq 0$, we have $r \perp E^k s$ for $k \leq 0$. By Lemma 2, $s \mid b$ and $r \mid a$. As $a \perp E^k b$ for $k \geq 0$, it follows that $r \perp E^k s$ for $k \geq 0$ as well, proving (ii). To prove (iii), note that $u \mid c$ and $v \mid d$. Because $(1, s, r, d)$ is a strict PNF we have $s \perp v$ and $r \perp Ev$. Because (z, a, b, c) is a strict PNF we have $r \perp u$ and $s \perp Eu$. □

The proof of Theorem 1 provides the following algorithm for computing a strict RNF of R.

Algorithm *RNF*

`input:` $R \in K[x]$, $R \neq 0$;
`output:` a strict RNF of R.

$(z, a, b, c) := \textit{strict\_PNF}(R)$;
$(1, s, r, d) := \textit{strict\_PNF}(b/a)$;
$g := \gcd(c, d)$; *(take g monic)*
$u := c/g$; $v := d/g$;
`return` (z, r, s, u, v).

Example 2. Take $R(x) = (x^2 - 1)/(x^2 + 2x)$. As

$$R(x) = \frac{x-1}{x+2} \cdot \frac{x+1}{x},$$

we have $z = 1$, $a = x - 1$, $b = x + 2$, $c = x$. Next,

$$\frac{x+2}{x-1} = \frac{x(x+1)(x+2)}{(x-1)x(x+1)},$$

so $s = r = 1$, $d = x(x^2 - 1)$, $u = 1$, and $v = x^2 - 1$. Thus $(1, 1/(x^2 - 1))$ is a strict RNF of R. Incidentally, we have discovered that $R = EV/V$ where $V \in K(x)$ (cf. Petkovšek, 1992, Lemma 5.1).

Even though RNF is not unique, the RNF's representing the same rational function are closely related. To describe their relationship, we use localization to shift-equivalence classes.

Lemma 3. *If (z, r, s, u, v) is an RNF of $z \in K \setminus \{0\}$ then $r = s = u = v = 1$.*

Proof. We have
$$r \cdot Eu \cdot v = s \cdot u \cdot Ev. \tag{5}$$
Let $t \in K[x] \setminus K$ be an irreducible factor of r. It follows from (5) that $t \,|\, u \cdot Ev$. We distinguish two cases.

(a) If $t \,|\, u$ then $Et \,|\, Eu$, so (5) implies that $Et \,|\, u \cdot Ev$. As $u \perp v$, it follows that $Et \,|\, u$. By induction, $E^n t \,|\, u$ for all $n \in \mathbb{N}$, hence $t \in K$.
(b) If $t \,|\, Ev$ then $E^{-1}t \,|\, v$, so (5) implies that $E^{-1}t \,|\, u \cdot Ev$. As $u \perp v$, it follows that $E^{-1}t \,|\, Ev$. By induction, $E^{-n}t \,|\, Ev$ for all $n \in \mathbb{N}$, hence $t \in K$.

Thus we conclude that $r = 1$. In the same way we find that $s = 1$. Now (5) implies that $E(u/v) = u/v$, hence $u/v \in K$ as well. But u, v are monic and $u \perp v$, so $u = v = 1$. □

Lemma 4. *Let $R \in K(x)$ be shift-homogeneous. If (z, r, s, u, v) is an RNF of R then r, s, u, v are shift-homogeneous of the same type as R.*

Proof. Let $r = r_1 \cdots r_k$, $s = s_1 \cdots s_k$, $u = u_1 \cdots u_k$, $v = v_1 \cdots v_k$ be shift-homogeneous factorizations where polynomials with the same subscript are of the same type, and r_1, s_1, u_1, v_1 are of the same type as R. Write $r' = r/r_1$, $s' = s/s_1$, $u' = u/u_1$, $v' = v/v_1$. Then Lemma 1 implies that $(1, r', s', u', v')$ is an RNF of 1. Hence by Lemma 3, $r' = s' = u' = v' = 1$. So $r = r_1$, $s = s_1$, $u = u_1$, $v = v_1$, proving the assertion. □

Lemma 5. *Let $R \in K(x)$ be shift-homogeneous. If (z, r, s, u, v) and (z, r_1, s_1, u_1, v_1) are two RNF's of R then $r = r_1 = 1$ and $\deg s = \deg s_1$, or $s = s_1 = 1$ and $\deg r = \deg r_1$.*

Proof. From
$$z \cdot \frac{r}{s} \cdot \frac{E(u/v)}{u/v} = z \cdot \frac{r_1}{s_1} \cdot \frac{E(u_1/v_1)}{u_1/v_1}$$
we obtain $r\, s_1\, Eu\, u_1\, v\, Ev_1 = r_1\, s\, u\, Eu_1\, Ev\, v_1$, so $\deg r - \deg r_1 = \deg s - \deg s_1$. Lemma 4 implies that r and s are shift-homogeneous of the same type. As r/s is shift-reduced, it follows that $r = 1$ or $s = 1$. In the same way, $r_1 = 1$ or $s_1 = 1$. We distinguish four cases: if $r = r_1 = 1$ then $\deg s = \deg s_1$. If $s = s_1 = 1$ then $\deg r = \deg r_1$. If $r = s_1 = 1$ then $\deg s + \deg r_1 = 0$, so $s = r_1 = 1$. If $r_1 = s = 1$ then $\deg r + \deg s_1 = 0$, so $r = s_1 = 1$. In all four cases, the assertion is true. □

Theorem 2. *Let (z, r, s, u, v) and (z', r', s', u', v') be two RNF's of $R \in K(x)$. Then*

(i) $z = z'$,
(ii) $\deg r = \deg r'$ *and* $\deg s = \deg s'$,
(iii) *there is a one-to-one correspondence f between the multisets of non-constant irreducible monic factors of r and r' such that $p \overset{\text{sh}}{\sim} f(p)$ for all $p \,|\, r$,*

(iv) *there is a one-to-one correspondence g between the multisets of non-constant irreducible monic factors of s and s' such that $q \overset{\text{sh}}{\sim} g(q)$ for all $q \mid s$.*

Proof. Obviously $z = z'$ because they both equal the leading coefficient of R. Let $r = r_1 \cdots r_k$, $s = s_1 \cdots s_k$, $u = u_1 \cdots u_k$, $v = v_1 \cdots v_k$, and likewise for r', s', u', v', be shift-homogeneous factorizations where polynomials with the same subscript are of the same type. For $i = 1, 2, \ldots, k$ write

$$R_i = \frac{r_i}{s_i} \cdot \frac{E(u_i/v_i)}{u_i/v_i}, \qquad R'_i = \frac{r'_i}{s'_i} \cdot \frac{E(u'_i/v'_i)}{u'_i/v'_i}.$$

Then, clearly, (r_i, s_i, u_i, v_i) is an RNF of R_i, and (r'_i, s'_i, u'_i, v'_i) is an RNF of R'_i. As $R = R_1 R_2 \cdots R_k = R'_1 R'_2 \cdots R'_k$, Lemma 1 implies that $R_i = R'_i$, for all i. By Lemma 5, $\deg r_i = \deg r'_i$ and $\deg s_i = \deg s'_i$. It follows that $\deg r = \deg r'$ and $\deg s = \deg s'$. To obtain the desired correspondences f resp. g, let the non-constant irreducible monic factors of r_i (resp. s_i) correspond to the non-constant irreducible monic factors of r'_i (resp. s'_i). □

3. The Minimal Multiplicative Representation Problem

If $T(n)$ is a hypergeometric term then there is a rational function $R \in K(x)$ and an integer $n_0 \in \mathbb{Z}$ such that

$$T(n) = T(n_0) \prod_{k=n_0}^{n-1} R(k)$$

for all $n \geq n_0$. This motivates the following definition.

Definition 4. Let $T(n)$ be a hypergeometric term. A *multiplicative representation* of T is a triple (F, V, n_0) where $F, V \in K(x)$, $n_0 \in \mathbb{Z}$, and

(i) $T(n) = V(n) \prod_{k=n_0}^{n-1} F(k)$, for all integers $n \geq n_0$,
(ii) if $V \neq 0$ then F, V have neither a pole nor a zero at any integer $n \geq n_0$.

This representation is *minimal* if for any other multiplicative representation (G, W, n_1) of T we have $\deg \operatorname{num} F \leq \deg \operatorname{num} G$ and $\deg \operatorname{den} F \leq \deg \operatorname{den} G$.

If $V = 0$ we simply write 0 instead of $(F, 0, n_0)$.

Proposition 2. *Let $R \in K(x)$ have neither a pole nor a zero at integers $n \geq n_0$, and let (z, r, s, u, v) be a strict RNF of R. Then the polynomials r, s, u, v have no zero at integers $n \geq n_0$.*

Proof. For r and s this follows from Proposition 1 (i). Write $p = \operatorname{num} R$ and $q = \operatorname{den} R$. Then

$$p \cdot s \cdot Ev \cdot u = z \cdot q \cdot r \cdot Eu \cdot v. \tag{6}$$

Assume that $n_1 \geq n_0$ is a zero of u. Then (6) implies that n_1 is a zero of Eu, hence $n_1 + 1$ is a zero of u. By induction, each $n \geq n_1$ is a zero of u, which is impossible. This shows that u has no zero at integers $n \geq n_0$. For v the proof is analogous. □

Using the concept of RNF, we can compute minimal multiplicative representations of hypergeometric terms. Unlike the decomposition problems of integration and summation where the degree of the numerator of the remaining integrand resp. summand is not important, the degree of the numerator of F in (i) is important. Luckily it is possible to minimize the degrees of the numerator and denominator of F simultaneously.

THEOREM 3. *Let (z, r, s, u, v) be an RNF of $R \in K(x)$. If*

$$R = \frac{p}{q} \cdot \frac{EV}{V}$$

where $p, q \in K[x]$ and $V \in K(x)$, then $\deg r \le \deg p$ *and* $\deg s \le \deg q$.

PROOF. Let (z', r', s', u', v') be a strict RNF of p/q. Then $(z'r'/s', V\,u'/v')$ is an RNF of R, and Theorem 2 implies that $\deg r = \deg r'$ and $\deg s = \deg s'$. By Proposition 1 (i), $r' \mid p$ and $s' \mid q$, hence $\deg r \le \deg p$ and $\deg s \le \deg q$. $\Box$

THEOREM 4. *Let $T(n)$ be a hypergeometric term with multiplicative representation $(R, T(n_0), n_0)$. If (F, V) is an RNF of R, then (F, W, n_0) where $W(n) = V(n)T(n_0)/V(n_0)$ is a minimal multiplicative representation of T.*

PROOF. Proposition 2 guarantees that F and V have neither zeros nor poles at integers $n \ge n_0$. A short computation

$$T(n) = T(n_0) \prod_{k=n_0}^{n-1} R(k) = T(n_0) \prod_{k=n_0}^{n-1} F(k) \frac{V(k+1)}{V(k)} = \frac{T(n_0)}{V(n_0)} V(n) \prod_{k=n_0}^{n-1} F(k)$$

shows that (F, W, n_0) is indeed a multiplicative representation of T. If (G, U, n_1) is another then $T(n) = U(n) \prod_{k=n_1}^{n-1} G(k)$, therefore

$$R(n) = \frac{T(n+1)}{T(n)} = G(n) \frac{U(n+1)}{U(n)}.$$

By Theorem 3, $\deg \operatorname{num} F \le \deg \operatorname{num} G$ and $\deg \operatorname{den} F \le \deg \operatorname{den} G$, so (F, W, n_0) is minimal. $\Box$

EXAMPLE 3. Consider the hypergeometric term $T(n)$ defined by

$$\begin{aligned} T(0) &= 2, \\ \frac{T(n+1)}{T(n)} &= \frac{(n+3)(2n+5)(3n+1)(4n+1)}{(n+1)(n+4)(2n+1)(3n+4)} \qquad (n \ge 0). \end{aligned}$$

We can express this hypergeometric term explicitly as

$$T(n) = 2 \prod_{k=0}^{n-1} \frac{(k+3)(2k+5)(3k+1)(4k+1)}{(k+1)(k+4)(2k+1)(3k+4)}.$$

As an RNF of $T(n+1)/T(n)$ is

$$(4, n + \tfrac{1}{4}, n + 4, (n+1)(n+2)(n + \tfrac{1}{2})(n + \tfrac{3}{2}), n + \tfrac{1}{3}),$$

we can also write

$$T(n) = \frac{(n+1)(n+2)(2n+1)(2n+3)}{3(3n+1)} \prod_{k=0}^{n-1} \frac{4k+1}{k+4}$$

where the factors in the product have numerators and denominators of minimal possible degrees.

4. The Minimal Additive Decomposition Problem

4.1. introduction

Definition 5. A hypergeometric term T is *summable* if there is a hypergeometric term T_1 such that $T = \Delta T_1$. A rational term T is *rational-summable* if there is a rational term T_1 such that $T = \Delta T_1$.

By means of RNF, we can now state the problem of minimal additive decomposition of hypergeometric terms:
Given a hypergeometric term T, find hypergeometric terms T_1, T_2 such that

(*1*) $T = \Delta T_1 + T_2$,
(*2*) *if T is summable then $T_2 = 0$,*
(*3*) *if T is not summable then $(ET_2)/T_2$ has an RNF (F, V) where V's denominator is of minimal possible degree.*

We call any pair of terms T_1, T_2 such that $T = \Delta T_1 + T_2$ an *additive decomposition* of T with *summable component* T_1 and *non-summable component* T_2.

This formulation agrees with the minimal additive decomposition problem for rational functions (Abramov, 1975, 1995; Pirastu and Strehl, 1995) because if $T_2 \in K(x)$, then $r = s = 1$ and v is the denominator of T_2.

In the rest of this section we prepare some tools that we need in the sequel. In particular, we define dispersion of two polynomials, and describe relations among multiplicative decompositions of hypergeometric terms T, T_1 and T_2 which satisfy $T = \Delta T_1 + T_2$. In Section 4.2 we describe algorithm *dterm* which, given a hypergeometric term T, constructs an additive decomposition of T. In Section 4.3 we prove that this decomposition is minimal, and hence that our algorithm solves the additive decomposition problem. Finally, in Section 4.4 we extend it to algorithm *hg_add_dec* which also recognizes when T is summable.

Definition 6. Let $a, b \in K[x] \setminus \{0\}$. The *dispersion* $\mathrm{dis}(a, b)$ is the largest $n \in \mathbb{N}$ such that $a(x)$ and $b(x+n)$ have a non-constant common divisor. If no such n exists then $\mathrm{dis}(a, b) = -1$.

Note that $\mathrm{dis}(a, b)$ can be computed as the largest non-negative integer root of the polynomial $R(n) = \mathrm{Res}_x(a(x), b(x+n))$. An alternative way of computing $\mathrm{dis}(a, b)$ consists in factoring a and b into irreducible factors over K, then finding all pairs u, v of factors of a resp. b such that $u(x) = v(x+n)$ for some $n \in \mathbb{N}$, and selecting the largest such n.

LEMMA 6. *Let (D, U, n_0) be a multiplicative representation of a term T, $n_1 \geq n_0$, and*

$$V(n) = U(n)\frac{T(n_1)}{U(n_1)} = U(n)\prod_{k=n_0}^{n_1-1} D(k).$$

Then (D, V, n_1) is a multiplicative representation of T.

PROOF. A direct check. □

We will need an algorithm which, given multiplicative representations of two similar terms, computes a multiplicative representation of their sum.

Algorithm *sum_of_terms*

`input:` multiplicative representations (D_1, U_1, n_1), (D_2, U_2, n_2)
of similar terms T_1, T_2;
`output:` multiplicative representation of $T_1 + T_2$.

let $(F,\ S)$ be an RNF of D_1/D_2;
find $n_3 \geq n_1, n_2$ s.t. $S(n)$ has neither a pole nor a zero for $n \geq n_3$;
$\alpha = \prod_{k=n_1}^{n_3-1} D_1(k)/S(n_3)$;
$\beta = \prod_{k=n_2}^{n_3-1} D_2(k)$;
$G := \alpha S U_1 + \beta U_2$;
`if` $G = 0$ `then return` 0
`fi;`
find $n_4 \geq n_3$ s.t. $G(n)$ has neither a pole nor a zero for $n \geq n_4$;
$\gamma = \prod_{k=n_3}^{n_4-1} D_2(k)$;
`return` $(D_2,\ \gamma G,\ n_4)$.

THEOREM 5. *Given multiplicative representations (D_1, U_1, n_0) resp. (D_2, U_2, n_1) of similar terms T_1 resp. T_2, algorithm **sum_of_terms** constructs a multiplicative representation of $T_1 + T_2$.*

PROOF. Since T_1 and T_2 are similar, the ratio of their certificates is of the form ER/R where $R \in K(x)$, $T_1 = RT_2$, and

$$\frac{ER}{R} = \frac{D_1}{D_2} \cdot \frac{E(U_1/U_2)}{(U_1/U_2)}.$$

This implies that

$$\frac{D_1}{D_2} = \frac{E(RU_2/U_1)}{RU_2/U_1},$$

hence $F = 1$ and $D_1/D_2 = (ES)/S$. Therefore

$$\begin{aligned}\gamma G(n) \prod_{k=n_4}^{n-1} D_2(k) &= G(n) \prod_{k=n_3}^{n-1} D_2(k) \\ &= (\alpha S(n) U_1(n) + \beta U_2(n)) \prod_{k=n_3}^{n-1} D_2(k)\end{aligned}$$

$$= \alpha S(n) U_1(n) \prod_{k=n_3}^{n-1} D_1(k) \frac{S(k)}{S(k+1)} + \beta U_2(n) \prod_{k=n_3}^{n-1} D_2(k)$$

$$= \alpha S(n_3) U_1(n) \prod_{k=n_3}^{n-1} D_1(k) + U_2(n) \prod_{k=n_2}^{n-1} D_2(k)$$

$$= U_1(n) \prod_{k=n_1}^{n-1} D_1(k) + U_2(n) \prod_{k=n_2}^{n-1} D_2(k)$$

$$= T_1(n) + T_2(n). \ \square$$

Lemma 7. *Let the triples (D, U, n_0) and (D, U_1, n_0) be multiplicative representations of (similar) terms T and T_1. Then the certificate of $T_2 = T - \Delta T_1$ is*

$$D \frac{EU_2}{U_2} \tag{7}$$

where

$$U_2 = U - D(EU_1) + U_1. \tag{8}$$

Proof. For all integer $n \geq n_0$ we have

$$T_2(n) = U(n) \prod_{k=n_0}^{n-1} D(k) - \Delta \left(U_1(n) \prod_{k=n_0}^{n-1} D(k) \right)$$

$$= U(n) \prod_{k=n_0}^{n-1} D(k) - U_1(n+1) \prod_{k=n_0}^{n} D(k) + U_1(n) \prod_{k=n_0}^{n-1} D(k)$$

$$= (U(n) - D(n) U_1(n+1) + U_1(n)) \prod_{k=n_0}^{n-1} D(k).$$

It follows that ET_2/T_2 agrees with (7) for all integers $n \geq n_0$ which proves the claim. $\square$

Lemma 8. *Let (D, U, n_0) be a multiplicative representation of a term T, and let $U_1, U_2 \in K(x)$ satisfy $U_2 = U - D(EU_1) + U_1$. Then there are terms T_1, T_2 such that*

(*1*) $T = \Delta T_1 + T_2$,

(*2*) *if $U_i \neq 0$ then T_i has a multiplicative representation of the form $(D, \beta U_i, n_1)$ where $n_1 \geq n_0$ and $\beta \in K$ $(i = 1, 2)$.*

Proof. Choose $n_1 \geq n_0$ such that if $U_i \neq 0$, then U_i has neither a pole nor a zero for $n \geq n_1$, $i = 1, 2$. Let

$$T_1(n) = \beta U_1(n) \prod_{k=n_1}^{n-1} D(k), \tag{9}$$

$$T_2(n) = \beta U_2(n) \prod_{k=n_1}^{n-1} D(k), \tag{10}$$

where $\beta = \prod_{k=n_0}^{n_1-1} D(k)$. Then

$$\Delta T_1(n) + T_2(n) = \beta\,(U_1(n+1)D(n) - U_1(n) + U_2(n)) \prod_{k=n_1}^{n-1} D(k)$$
$$= U(n) \prod_{k=n_1}^{n-1} D(k) = T(n). \;\Box$$

4.2. ALGORITHM *dterm*

The following lemma and its proof contain the main idea of our algorithm.

LEMMA 9. *Let (z, d_1, d_2, u_1, u_2) be a strict RNF of some $R \in K(x)$. Write $D = zd_1/d_2$, $U = u_1/u_2$. Then there are $U_1 \in K(x)$, $v_1, v_2 \in K[x]$ and $i, j \in \{0,1\}$ such that*

(i) $U - D(EU_1) + U_1 = \frac{v_1}{(E^{-1}d_1)^i d_2^j v_2}$ *where* $v_1 \perp (E^{-1}d_1)^i d_2^j v_2$,
(ii) $v_2 \perp E^{-h}d_1$, $v_2 \perp E^h d_2$ *for all* $h \geq 0$,
(iii) v_2 *is shift-free.*

PROOF. Let q be an irreducible factor of u_2. Write $u_2 = u_2' q^k$ where $q \perp u_2'$. Then, by the partial fraction decomposition, there are $a, b \in K[x]$ such that

$$U = \frac{a}{u_2'} + \frac{b}{q^k}. \tag{11}$$

We distinguish two cases.

(a) There is an integer $h \geq 0$ such that $E^h q \,|\, d_1$. Let ${U_1}' = -b/q^k$. Then $U - D(E{U_1}') + {U_1}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{d_2} + \frac{c_2}{(Eq)^l}$$

where $l \leq k$ and $c_0, c_1, c_2 \in K[x]$.

(b) There is an integer $h \leq 0$ such that $E^h q \,|\, d_2$. Let ${U_1}' = E^{-1}\big(b/(Dq^k)\big)$. Then $D(E{U_1}') = b/q^k$, so $U - D(E{U_1}') + {U_1}'$ can be written as

$$\frac{c_0}{u_2'} + \frac{c_1}{E^{-1}d_1} + \frac{c_2}{(E^{-1}q)^l}$$

where $l \leq k$ and $c_0, c_1, c_2 \in K[x]$.

Since D is shift-reduced, at most one of the cases (a), (b) can occur. Repeating these steps if necessary (using ${U_1}'', {U_1}''', \ldots$) we obtain a rational function $U - DE({U_1}' + {U_1}'' + \cdots) + ({U_1}' + {U_1}'' + \cdots)$ whose denominator is of the form $(E^{-1}d_1)^i d_2^j v_2'$ where v_2' has no irreducible factor q such that $E^h q \,|\, d_1$ or $E^{-h} q \,|\, d_2$ with $h \geq 0$.

We proceed similarly with the remaining irreducible factors of u_2 (those that are not shift-equivalent to q), and finally obtain U_1, v_1, v_2 which satisfy (i) and (ii). If v_2 is not shift-free then there is an integer $h > 0$ and an irreducible $q \in K[x]$ such that q and $E^h q$ both divide v_2. In this case we further transform U_1 in the same way as U was transformed in (a) above. $\Box$

THEOREM 6. *Let T be a hypergeometric term. Then there exists a term T_1 similar to T such that the certificate of the term $T_2 = T - \Delta T_1$ has an RNF of the form (z, f_1, f_2, v_1, v_2) which satisfies the following two properties:*

(A) v_2 *is shift-free,*
(B) $v_2 \perp E^{-h} f_1$, $v_2 \perp E^h f_2$ *for all* $h \geq 0$.

PROOF. Combining Lemmas 8 and 9 we obtain hypergeometric terms T_1 and $T_2 = T - \Delta T_1$ with certificates $ET_1/T_1 = D(EU_1)/U_1$ and $ET_2/T_2 = D(EU_2)/U_2$ where

$$U_2 = \frac{v_1}{(E^{-1}d_1)^i d_2^j v_2}$$

with v_1, v_2, d_1, d_2, i, j as in Lemma 9. To remove the factors $(E^{-1}d_1)^i$ and d_2^j from the denominator of U_2 we set

$$F \;=\; D\,\frac{\left(E^{-1}d_1/d_1\right)^i}{\left(Ed_2/d_2\right)^j}, \qquad V = \frac{v_1}{v_2}.$$

Then $D(EU_2)/U_2 = F(EV)/V$ and F is still shift-reduced, proving the theorem. □

The proofs of Theorem 6 and Lemma 9 contain an algorithm to compute the terms T_1, T_2 (mentioned in Theorem 6) that we now state explicitly. In case (a) of the proof of Lemma 9 we considered the irreducible q's and integers $h \geq 0$ such that $q \mid u_2$ and $E^h q \mid d_1$. All the q's (say $q_1, \ldots, q_\kappa$) that relate to the maximal possible h can be considered together. Using the concept of dispersion, we find the maximal value of h along with $q' = q_1^{\nu_1} \ldots q_\kappa^{\nu_\kappa}$, $q' \mid u_2$, $\nu_1, \ldots, \nu_\kappa > 0$, then compute $\tilde{q} = q_1^{\mu_1} \ldots q_\kappa^{\mu_\kappa}$, where $\mu_1, \ldots, \mu_\kappa$ are the maximal possible such that $q_1^{\mu_1} \ldots q_\kappa^{\mu_\kappa} \mid u_2$. For this, we use the following simple algorithm:

Algorithm *pump*

`input:` $f, g \in K[x]$ such that $f \mid g$;
`output:` $\tilde{f}, \tilde{g} \in K[x]$ such that $f \mid \tilde{f}$, $q \mid \tilde{f} \wedge q$ irreducible $\Rightarrow q \mid f$, $\tilde{f}\tilde{g} = g$, $\tilde{f} \perp \tilde{g}$.

$\tilde{f} := f$; $\tilde{g} := g/f$;
`repeat` $d = \gcd(\tilde{f}, \tilde{g})$;
  $\tilde{f} := \tilde{f}d$; $\tilde{g} := \tilde{g}/d$;
`until` $\deg d = 0$;
`return` $(\tilde{f}, \tilde{g})$.

With $(\tilde{q}, \tilde{u}_2) = pump(q, u_2)$, we compute a partial fraction decomposition

$$U = \frac{\tilde{a}}{\tilde{u}_2} + \frac{\tilde{b}}{\tilde{q}} \tag{12}$$

which serves in place of (11).

In case (b) of the proof of Lemma 9, we proceed similarly. Thus we have the following algorithm:

Algorithm *dcert*

`input:` $D, U \in K(x)$ where (D, U) is a strict RNF of some $R \in K(x)$;
`output:` U_1, F, $V \in K(x)$ such that

1. if $F = 0$ then $U = D(EU_1) - U_1$,
2. if $F \neq 0$ then
   (a) $F(EV)/V = D(EU_2)/U_2$ where $U_2 = U - D(EU_1) + U_1$,
   (b) $f_1 = \operatorname{num} F$, $f_2 = \operatorname{den} F$, $v_1 = \operatorname{num} V$, $v_2 = \operatorname{den} V$ have properties (A), (B) of Theorem 6.

```
U_1 := 0; U_2 := U;
u_2 := den U;
d_1 := num D;  d_2 := den D;
N_1 := dis(d_1, u_2);
M := dis(u_2, u_2);
if M = 0 then M := −1;
N_1 = max{N_1, M};
for h := N_1 downto 0 do
    q := gcd(u_2, E^{−h} d_1);
    if h > 0  then
        t := u_2/q;
        q := q gcd(t, E^{−h} t)
    fi;
    (q̃, ũ_2) := pump(q, u_2);
    write U_2 = ã/ũ_2 + b̃/q̃  where  ã, b̃ ∈ K[x];
    U_1' := −b̃/q̃;
    U_2 := U_2 − D(EU_1') + U_1';   U_1 := U_1 + U_1';
    u_2 := den U_2
od;
N_2 := −dis(d_2(−n), u_2(−n));
for h := N_2 to 0 do
    q := gcd(u_2, E^{−h} d_2);
    (q̃, ũ_2) := pump(q, u_2);
    write U_2 = ã/ũ_2 + b̃/q̃  where  ã, b̃ ∈ K[x];
    U_1' := E^{−1}(b̃/(Dq̃));
    U_2 := U_2 − D(EU_1') + U_1';   U_1 := U_1 + U_1';
    u_2 := den U_2
od;
v_1 := num U_2;  v_2 := u_2;
if E^{−1} d_1 | v_2
    then v_2 := v_2/(E^{−1} d_1);  f_1 := E^{−1} d_1
    else f_1 := d_1
fi;
if d_2 | v_2
    then v_2 := v_2/d_2;  f_2 := Ed_2
    else f_2 := d_2
```

`fi;`
$F := f_1/f_2;\ V := v_1/v_2;$
`return` $(U_1,\ F,\ V)$.

Using Lemma 8 it is now easy to write down the algorithm *dterm*.

Algorithm *dterm*

`input:` multiplicative representation $t = (D, U, n_0)$ of a term T
where D is shift-reduced;
`output:` multiplicative representations t_1, t_2 of terms T_1, T_2 such that

1. $T = \Delta T_1 + T_2$,
2. if $T_2 \neq 0$ then $(ET_2)/T_2 = F(EV)/V$ where $f_1 = \operatorname{num} F$, $f_2 = \operatorname{den} F$, $v_1 = \operatorname{num} V$, $v_2 = \operatorname{den} V$ have properties (A), (B) of Theorem 6.

$(U_1,\ F,\ V) := dcert(D, U)$;
`if` $U_1 = 0$ `then`
  `return` $(0,\ t)$
`fi;`
find $n_1 \geq n_0$ s.t. $U_1(n)$, and also $F(n)$, $V(n)$ if $V \neq 0$,
  have neither a pole nor a zero for $n \geq n_1$;
$\beta = \prod_{k=n_0}^{n_1-1} D(k)$;
$t_1 := (D, \beta U_1, n_1)$;
`if` $V = 0$ `then`
  `return` $(t_1,\ 0)$
`fi;`
$U_2(n_1) := U(n_1) - D(n_1)U_1(n_1+1) + U_1(n_1)$;
$t_2 := (F,\ \beta U_2(n_1)/V(n_1)V,\ n_1)$;
`return` $(t_1,\ t_2)$.

Example 4. Applying *dterm* to $D(n) = 1/(n+2)$, $U(n) = 1/(n+1) - 1/n$, $n_0 = 1$ which is a multiplicative representation of the term

$$T(n) = \left(\frac{1}{n+1} - \frac{1}{n}\right)\frac{2}{(n+1)!}$$

results in the additive decomposition $T(n) = \Delta T_1(n) + T_2(n)$ where

$$T_1(n) = \frac{2}{n\,n!}, \quad T_2(n) = \frac{2}{(n+1)!}.$$

We show in Section 4.3 that algorithm *dterm* constructs a decomposition where the denominator v_2 of V from the certificate of T_2 has minimal possible degree. In Abramov and Petkovšek (2001b), it is shown that in addition, we can also reduce the degree of the *numerator* v_1 of V so that it is less than

$$\lambda = \begin{cases} \deg v_2 + \deg f_2 & \text{if } \deg(f_2 - f_1) > \deg f_1, \\ \deg v_2 + \deg f_1 & \text{if } \deg(f_2 - f_1) = \deg f_1 \\ & \quad \text{or } \deg(f_2 - f_1) < \deg f_1 - 1, \\ \deg v_2 + \deg f_1 + \tau & \text{if } \deg(f_2 - f_1) = \deg f_1 - 1, \end{cases}$$

where in the last case τ is equal to $\mathrm{lc}\,(f_2 - f_1)/\mathrm{lc}\, f_1$ if this is a non-negative integer, and -1 otherwise.

Example 5. Consider the rational term

$$T(n) = \frac{1}{8} \frac{(n+3)(n+2)(n+4)(43\,n+35)}{(2\,n+1)(2\,n+3)(2\,n+5)(2\,n+7)}.$$

An application of *dterm* yields

$$T_1 = -\frac{15}{256} \frac{168\,n^2 + 460\,n + 251}{(2\,n+1)(2\,n+3)(2\,n+5)}, \qquad T_2 = \frac{86\,n + 457}{256\,n + 896}.$$

Using techniques from Abramov and Petkovšek (2001b) this can be rewritten as

$$T_2 = \Delta\left(\frac{43}{128}\,n\right) + \frac{156}{256\,n + 896},$$

hence

$$T = \Delta\left(\frac{1}{256} \frac{688\,n^4 + 3096\,n^3 + 1436\,n^2 - 5610\,n - 3765}{(2\,n+1)(2\,n+3)(2\,n+5)}\right) + \frac{156}{256\,n + 896}.$$

4.3. Proof of minimality of decomposition constructed by *dterm*

Definition 7. A rational function $F \in K(x)$ is *adequate* for a hypergeometric term $T(n)$ if the certificate ET/T has an RNF with F as its kernel.

Let T, T_1, T_2 satisfy $T = \Delta T_1 + T_2$. Note that these terms are similar (cf. Petkovšek *et al.*, 1996, Proposition 5.6.2), hence any rational function adequate for one of them is also adequate for the other two.

First we prove that the additive decomposition produced by *dterm* is minimal if we consider only RNF's having the same kernel F as the one constructed by *dcert*.

Theorem 7. *Let the terms T, T_1, T_1' be such that $T_2 = T - \Delta T_1$, $T_2' = T - \Delta T_1'$, and $F = f_1/f_2$ is a shift-reduced rational function adequate for these terms. Let $ET_2/T_2 = F(EV)/V$ where $F, V \in K(x)$ have properties (A), (B) of Theorem 6, and $ET_2'/T_2' = F(EV')/V'$. If $V = v_1/v_2$ and $V' = v_1'/v_2'$ where $v_1, v_2, v_1', v_2' \in K[x]$ and $v_1 \perp v_2$, then $\deg v_2 \leq \deg v_2'$.*

Proof. We have

$$T_2' = T_2 - \Delta(T_1' - T_1).$$

Suppose that the certificate of $T_1' - T_1$ is equal to $F\frac{EW}{W}$ where $W = w_1/w_2$ and $w_1 \perp w_2$. Then, by Lemma 7,

$$\frac{v_1'}{v_2'} = \frac{v_1}{v_2} - \frac{f_1}{f_2}\frac{Ew_1}{Ew_2} + \frac{w_1}{w_2}. \tag{13}$$

Consider an arbitrary irreducible $p \in K[x]$ such that $p \mid v_2$. We set

$$k = \max\{\alpha;\ p^\alpha \mid v_2\}$$

and claim that $E^l p^k \mid v_2'$ for some $l \in \mathbb{Z}$. Since the pair F, V has property (A), this claim implies the statement of the theorem. Suppose that p^k does not divide v_2'. Equation (13) implies that v_2 and hence p^k divides the lcm of v_2', $f_2 E w_2$, and w_2. By property (B) we have $p \perp f_2$, therefore $p^k \mid E w_2$ or $p^k \mid w_2$.

Let $p^k \mid E w_2$. Then

$$E^{-1} p^k \mid w_2. \tag{14}$$

Set $l = \min\{m : E^m p^k \mid w_2\}$. Evidently $E^l p^k$ does not divide $E w_2$. It follows from (14) that $l \le -1$; together with property (B) this gives $E^l p \perp f_2$. As v_2 is shift-free and $p \mid v_2$, it follows that $E^l p^k$ does not divide v_2. Therefore (13) implies

$$E^l p^k \mid v_2'. \tag{15}$$

Let $p^k \mid w_2$. Then

$$E p^k \mid E w_2. \tag{16}$$

Set $l = \max\{m : E^m p^k \mid E w_2\}$. Evidently $E^l p^k$ does not divide w_2. It follows from (16) that $l \ge 1$; together with property (B) this gives $E^l p \perp f_1$. Therefore (13) implies (15) in this case as well. □

Corollary 1. *Let $F, U, S_1, S_2 \in K(x)$ where F is shift-reduced. Let the rational functions*

$$V_1 = U - FES_1 + S_1,\ V_2 = U - FES_2 + S_2$$

be such that the pairs F, V_1 and F, V_2 have properties (A), (B) of Theorem 6. Then the degrees of the denominators of V_1 and V_2 are the same.

In the rest of this section we prove that algorithm *dterm* gives a complete solution to the additive decomposition problem. If rational functions $F_1, F_2 \in K(x)$ are both adequate for a term T then there exists $G \in K(x)$ such that

$$\frac{F_1}{F_2} = \frac{EG}{G}. \tag{17}$$

Indeed, for some $U_1, U_2 \in K(x)$ we have

$$F_1 \frac{EU_1}{U_1} = F_2 \frac{EU_2}{U_2},$$

and therefore $G = U_1^{-1} U_2$. The case where $G \in K[x]$ is of special interest.

Theorem 8. *Let F_1, F_2 be rational functions adequate for a term T, and such that* (17) *holds with $G \in K[x]$. If the pair F_1, V has properties (A), (B) of Theorem* 6 *then* $\operatorname{den} V \perp G$, *and the pair F_2, GV also has properties (A), (B) of Theorem* 6.

Proof. First we prove that $\operatorname{den} V \perp G$. If they have a common irreducible factor p then the set $\{\nu;\ E^\nu p \mid G\}$ is non-empty and finite. Suppose that m resp. M are the minimal resp. the maximal elements of this set. Write

$$W = \frac{G}{EG} = \frac{F_2}{F_1} = \frac{w_1}{w_2}, \qquad w_1 \perp w_2.$$

Then $E^{M+1}p \,|\, w_2$ and $E^m p \,|\, w_1$. We have $F_2 = WF_1$. As p divides the denominator of V and the pair F_1, V has properties (A) and (B), the numerator of F_1 is not divisible by $E^{M+1}p$ since $M+1>0$. Similarly the denominator of F_1 is not divisible by $E^m p$ since $m \le 0$. Therefore the numerator of F_2 is divisible by $E^m p$ while the denominator of F_2 is divisible by $E^{M+1}p$. But F_2 is shift-reduced by Definition 3(ii), a contradiction.

Now we prove that the pair F_2, GV has properties (A) and (B). We have

$$F_2 = \frac{G}{EG} F_1$$

and the pair F_2, GV has property (A) because the denominator of GV divides the denominator of V. Now we shall be concerned with (B). Let p be an irreducible from $K[x]$ that divides the denominator of GV and thereby divides the denominator of V. Let $E^h p$, $h \le 0$, divide the denominator of F_2. Then $E^h p$ does not divide the denominator of F_1 since the pair F_1, V has properties (A) and (B). The equality $(EG)F_2 = GF_1$ implies that $E^h p \,|\, EG$. Set $h_0 = \min\{\nu : E^\nu p \,|\, EG\}$. Then $h_0 \le h \le 0$ and $E^{h_0-1}p \,|\, G$, but $E^{h_0-1}p$ does not divide EG. The denominator of F_1 is not divisible by $E^{h_0-1}p$ since the pair F_1, V has properties (A) and (B). Therefore $E^{h_0-1}p$ divides the numerator of F_2. But as $E^h p$ divides the denominator of F_2, this contradicts the fact that F_2 is shift-reduced.

Similarly it can be shown that $E^h p$, $h \ge 0$, cannot divide the numerator of F_2. □

Lemma 10. *Let $F, F_1, U, U_1 \in K(x)$, $G \in K[x]$ be such that $F/F_1 = EG/G$, $G \in K[x]$ and $F\frac{EU}{U} = F_1\frac{EU_1}{U_1}$. Then there exists $\overline{G} \in K[x]$ such that $\overline{G}U = U_1$ and for any $S \in K(x)$ we have*

$$\overline{G}(U - FES + S) = U_1 - F_1E(\overline{G}S) + \overline{G}S.$$

Proof. We have

$$\frac{E(U^{-1}U_1)}{U^{-1}U_1} = \frac{EG}{G}.$$

It follows from this that there exists $\alpha \in K$ such that $U^{-1}U_1 = \alpha G$. Set $\overline{G} = \alpha G$. We get

$$\frac{E\overline{G}}{\overline{G}} F_1 = F, \qquad U_1 = \overline{G}U.$$

Substituting U_1 for $\overline{G}U$ and $(E\overline{G}/\overline{G})F_1$ for F in $\overline{G}U - (\overline{G}F)ES + \overline{G}S$ gives $U_1 - F_1E(\overline{G}S) + \overline{G}S$. □

Theorem 9. *Let F_1, F_2 be rational functions that are adequate for a term T. Let $U_1, U_2, R \in K(x)$ be such that*

$$F_1\frac{EU_1}{U_1} = F_2\frac{EU_2}{U_2} = R. \tag{18}$$

For $S_1, S_2 \in K(x)$, let

$$V_1 = U_1 - F_1ES_1 + S_1,\ V_2 = U_2 - F_2ES_2 + S_2 \tag{19}$$

be such that the pairs F_1, V_1 and F_2, V_2 have properties (A) and (B) of Theorem 1. Then the denominators of V_1 and V_2 are of the same degree.

Proof. First of all we show that there exists a shift-reduced rational function a/b, such that for the rational functions

$$F_0 = \frac{a}{b}, \qquad F_{-1} = \frac{E^{-1}a}{b}, \qquad F_{-2} = \frac{a}{Eb}, \qquad F_{-3} = \frac{E^{-1}a}{Eb} \tag{20}$$

the equalities

$$\frac{F_i}{F_1} = \frac{EG_i'}{G_i'}, \qquad \frac{F_i}{F_2} = \frac{EG_i''}{G_i''}, \qquad G_i', G_i'' \in K[x], \tag{21}$$

hold for $i = -1, -2, -3$. It is sufficient to prove the theorem for shift-homogeneous F_1, F_2 which belong to the same shift-homogeneous class. Then, by Lemma 5, either both F_1 and F_2 are polynomials, or both F_1 and F_2 are reciprocals of polynomials. By Theorem 2(ii) we have

$$F_1 = \prod_{i=1}^{\tau} E^{h_i} p, \qquad F_2 = \prod_{i=1}^{\tau} E^{l_i} p, \tag{22}$$

in the former case, and

$$F_1 = \frac{1}{\prod_{i=1}^{\tau} E^{h_i} p}, \qquad F_2 = \frac{1}{\prod_{i=1}^{\tau} E^{l_i} p} \tag{23}$$

in the latter, where $p \in K[x]$ is irreducible. In the case of (22), set

$$a = \prod_{i=1}^{\tau} E^{\max\{h_i, l_i\}+1} p, \qquad b = 1,$$

and in the case of (23), set

$$a = 1, \qquad b = \prod_{i=1}^{\tau} E^{\min\{h_i, l_i\}-1} p.$$

It is easy to see that if $F_0, F_{-1}, F_{-2}, F_{-3}$ are defined as in (20) then the equalities (21) hold for some polynomials G_i', G_i''.

Considering the RNF of R with the kernel a/b and using algorithm *dcert* we can get $i,\ -3 \leq i \leq 0$, and $F, U, V, S \in K[x]$ such that

- $F = F_i$;
- $R = F\frac{EU}{U}$, $U = \frac{u_1}{u_2}$, $u_1 \perp u_2$;
- $V = U - FES + S$;
- the pair F, V has properties (A) and (B).

Set

$$G' = G_i',\ G'' = G_i''$$

for the computed i. By Lemma 10 there exists a polynomial $\overline{G}'$ such that

$$\overline{G}'V = \overline{G}'(U - FES + S) = U_1 - F_1 E(\overline{G}'S) + \overline{G}'S.$$

By Theorem 8 the pair F_1, $U_1 - F_1 E(\overline{G}'S) + \overline{G}'S$ has properties (A) and (B) and the degree of denominator of $\overline{G}'V$ is equal to the degree of the denominator of V. By Corollary 1 the denominator of V is of the same degree as the denominator of V_1, and similarly for the degrees of the denominators of V and V_2. The claim follows. □

The following is the main result of this section.

THEOREM 10. *Let T, T_1, T_1' be similar terms. Let the certificates of the terms $T_2 = T - \Delta T_1, T_2' = T - \Delta T_1'$ be written in the form*

$$F\frac{EV}{V}, \qquad F'\frac{EV'}{V'}$$

with shift-reduced F, F'. Let the pair F, V have properties (A) and (B) of Theorem 1. *Then* $\deg \operatorname{den} V \leq \deg \operatorname{den} V'$.

PROOF. Since $ET_2'/T_2' = F'(EV'/V')$, where F' is shift-reduced, there exists $U \in K(x)$ such that $ET/T = F'(EU/U)$. Now applying *dcert* to the input F', U yields $\tilde{U}, \tilde{V} \in K(x)$ such that the term T has the decomposition $T = \Delta\tilde{T}_1 + \tilde{T}_2$ where $\tilde{T}_1$, $\tilde{T}_2$ have the certificates $F'(E\tilde{U}/\tilde{U})$ and $F'(E\tilde{V}/\tilde{V})$, resp. with $F', \tilde{V}$ satisfying properties (A) and (B). From Theorem 9, one concludes that the denominators of V and $\tilde{V}$ are of the same degree. The claim now follows, since it is clear from Theorem 7 that $\deg \operatorname{den} \tilde{V} \leq \deg \operatorname{den} V'$. □

4.4. THE ISSUE OF SUMMABILITY

Any algorithm to solve the decomposition problem for rational functions guarantees that if the input rational function T is rational-summable, then it will return a rational function T_1 such that

$$T = \Delta T_1.$$

It would be natural to expect that an algorithm to solve the same problem for hypergeometric terms would exhibit analogous behaviour. It is clear, however, that by simply applying *dterm* one will not achieve this goal. One solution is to apply an indefinite hypergeometric summation algorithm (such as Gosper's algorithm (Gosper, 1978)) first, and only in the case of failure proceed with the additive decomposition. But we can also detect summability from the minimal additive decomposition as follows.

THEOREM 11. *Let T and T_1 be hypergeometric terms such that $\Delta T_1 = T$. If $ET/T = F(EV)/V$ and $ET_1/T_1 = F(ER)/R$ where $F, V, R \in K(x)$, F is shift-reduced, and the pair F, V has properties (A), (B) of* Theorem 6, *then V, R are polynomials.*

PROOF. Set $V = v_1/v_2$, $F = f_1/f_2$, $v_1, v_2, f_1, f_2 \in K[x]$. If $\Delta T_1 = T$ then there exists $\mu \in K$ such that $F\,E(\mu R) - \mu R = V$. Set $S = \mu R$ then

$$FES - S = V, \tag{24}$$

or equivalently,

$$v_2\, f_1\, ES - v_2\, f_2\, S = f_2\, v_1. \tag{25}$$

Since the pair F, V has properties (A), (B) of Theorem 6, the dispersion of $f_2 v_2$, $f_1 v_2$ cannot be a positive integer. Therefore there is no non-polynomial rational function S that satisfies (25) (see Abramov (1989)). Consequently, $S, R \in K[x]$. It follows from (24) that $V \in K[x]$. □

Consider the following algorithm for the case where $V \in K[x]$:

Algorithm *dpol*

```
input:  multiplicative representation (F, V, n_0) of a term T where
        F, V have properties (A), (B) of Thm 6, F is shift-reduced,
        and V ∈ K[x];
output: multiplicative representation of a term T_1 such that
        T = ΔT_1 if it exists, and 0 otherwise.

    if the equation FEy − y = V has a polynomial solution
       then set S to an arbitrary polynomial solution
       else return 0
    fi;
    find integer n_1 ≥ n_0 s.t. S(n) has neither a pole nor a zero
       at n ≥ n_1;
    β = ∏_{k=n_0}^{n_1−1} F(k);
    return (F, βS, n_1).
```

Finally, we present algorithm *hg_add_dec* that solves the additive decomposition problem, and also recognizes summability of its input.

Algorithm *hg_add_dec*

input: multiplicative representation $t = (D, U, n_0)$ of a term T where D is shift-reduced;

output: multiplicative representations t_1, t_2 of terms T_1, T_2 such that

1. $T = \Delta T_1 + T_2$,
2. if T is summable then $T_2 = 0$,
3. if T is not summable then $(ET_2)/T_2$ has an RNF (F, V) where V's denominator is of minimal possible degree.

```
(U_1, F, V) := dcert(D, U);
(t_1, t_2) := dterm(D, U, n_0);
if t_2 = 0 or V ∉ K[x]
   then return (t_1, t_2)
fi;
t_3 := dpol(t_2);
if t_3 = 0 then return (t_1, t_2)
        else return (sum_of_terms (t_1, t_3), 0)
fi.
```

Example 6. For the hypergeometric term

$$T(n) = \frac{1}{n\,(n+1)} \prod_{k=1}^{n-1} \frac{k^2}{k^2+k+1},$$

applying *dterm* results in

$$T_1(n) = -\frac{1}{3}\,\frac{n^2-n+1}{n\,(n-1)^2}\prod_{k=2}^{n-1}\frac{k^2}{k^2+k+1}, \qquad T_2(n) = -\frac{1}{3}\prod_{k=2}^{n-1}\frac{(k-1)^2}{k^2+k+1}$$

where T_2 has a multiplicative decomposition (F, V, n_0) with

$$F = \frac{f_1}{f_2}, \qquad f_1 = (n-1)^2, \qquad f_2 = n^2+n+1, \qquad V = -\frac{1}{3}.$$

Since $V \in K[n]$, we apply *dpol*. The equation

$$f_1\,Ey - f_2\,y = f_2\,V$$

has a polynomial solution

$$y = -\frac{1}{3}\,(n-2)\,(n^2-n+1).$$

Therefore

$$T_2(n) = \Delta\left(-\frac{1}{21}\,(n-2)\,(n^2-n+1)\prod_{k=3}^{n-1}\frac{(k-1)^2}{k^2+k+1}\right),$$

hence T is summable as well, and *hg_add_dec* returns

$$T(n) = \Delta\left(-\frac{1}{21}\,\frac{n^4-3\,n^3+4\,n^2-3\,n+1}{n}\prod_{k=3}^{n-1}\frac{(k-1)^2}{k^2+k+1}\right).$$

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