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TRUNCATED SERIES IN THE ROLE OF MATRIX ELEMENTS

S. A. Abramov and A. A. Ryabenko UDC 519.7

Abstract. Square matrices over a ring of formal power series are considered. The matrix elements (series) are given "approximately": for each of them only a finite number of initial terms are known, and these numbers may not coincide for different matrix elements. Issues related to the possibility of guaranteeing the nonsingularity of a matrix for which only this kind of "approximate" representation is known are discussed.

To the memory of E. V. Pankratiev

1. Introduction

In [1], an algorithm is proposed that is applicable to an arbitrary nonsingular square matrix $P = (p_{ij})$, in which all elements p_{ij} are polynomials in x over a field K of characteristic 0. The algorithm allows one to check whether the matrix $P + Q$ is nonsingular for any matrix Q with elements from the ring $K[[x]]$ of formal power series that has, first, the same size as P and, second, has the property

$$\operatorname{val} Q \geq 1 + \deg P. \tag{1}$$

Recall that the valuation $\operatorname{val} f(x)$ of a power series or polynomial is the lowest power of x that has a nonzero coefficient in $f(x)$ (for example, for $f(x) = -3x^2 + 5x^3 + \ldots$ we have $\operatorname{val} f(x) = 2$; by definition we set $\operatorname{val} 0 = \infty$). The valuation of a matrix composed of power series or polynomials is the smallest of the valuations of all elements of this matrix. The degree of a matrix composed of polynomials is considered to be the largest of all the degrees of the elements, wherein $\deg 0 = -\infty$.

A matrix P for which the matrix $P + Q$ is nonsingular whenever for a matrix Q of the same size as P, the inequality (1) is satisfied, is called in [1] *strongly nonsingular*. A strongly nonsingular matrix remains nonsingular when we add "tails" to its elements, turning the polynomials into series with coefficients in K. When adding "tails", it is necessary that the lowest power (valuation) of each such "tail" exceed $\deg P$.

Let a matrix P be nonsingular and be obtained by truncating some matrix

$$M = (m_{ij}), \tag{2}$$

whose elements are formal power series, and the degree of truncation for all elements is equal to a fixed nonnegative integer d: all terms of higher degrees than d are removed. Then, if $d = \deg P$, using the algorithm proposed in [1], we can check whether the matrix M (we do not know this matrix) cannot be singular. For this purpose, the strong nonsingularity of the matrix P is checked.

Remark 1. If d is greater than a power of some p_{ij}, then it is implied that the coefficients at the powers $\deg p_{ij} + 1, \ldots, d$ in m_{ij} are equal to zero.

We further consider the problem of checking strong nonsingularity in a more general version, in comparison with [1], by expanding the very concept of strong nonsingularity, omitting the assumption that the degrees of truncations of all elements of the original matrix are the same and, in particular, assuming that the degree of any (and even each) element of the matrix P can be less than d (see Remark 1).

Below, P is a nonsingular $(n \times n)$-matrix whose elements are polynomials in x over an field K. (The degrees of these polynomials may differ from each other.)

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In [2, 3] the question was considered of how, for a given nonsingular numerical real matrix, in whose elements after the decimal point there is only a finite number of digits, to check whether this matrix will remain nonsingular after arbitrarily adding digits to some (explicitly specified in advance) of its elements? It was found that this problem is algorithmically decidable. A computer implementation of the proposed algorithmic solution was discussed. It was assumed that different elements of the original matrix P can have different numbers of known digits after the decimal point. As already mentioned, below a similar problem is considered for power series, which have their own specifics.

2. Preliminaries

In [1, Proposition 1], the criterion was proved, i.e., necessary and sufficient condition for the strong nonsingularity of the matrix P:

$$\deg P + \operatorname{val} P^* \geq \operatorname{val} \det P, \tag{3}$$

where, in accordance with the definition of the val and deg,

- $\deg P = \max\limits_{i,j} \deg p_{ij}$,
- $\operatorname{val} P^* = \min\limits_{i,j} \operatorname{val} f_{ij}$; here f_{ij} is the minor of the matrix P obtained by crossing out the ith row and jth column. In general, the matrix P^* is considered as the transpose of the cofactor matrix of P.

Criterion (3) can be generalized to cover the case where the truncation degree d exceeds the degree of matrix P. It can be noted that according to (3) the matrix

$$P = \begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}$$

is not strongly nonsingular. But at the same time, there is no (2×2)-matrix Q, $\operatorname{val} Q > 2$, such that matrix $P + Q$ is singular.

We change the formulation of [1, Proposition 1].

Proposition 1. *Let $P = (p_{ij})$, $p_{ij} \in K[x]$, be a nonsingular polynomial $(n \times n)$-matrix and let d be an integer such that $d \geq \deg p_{ij}$, $i, j = 1, \ldots, n$. Then the matrix $P+Q$ is nonsingular for any $(n \times n)$-matrix Q with elements from the ring $K[[x]]$, possessing property* (1) *if and only if*

$$d + \operatorname{val} P^* \geq \operatorname{val} \det P. \tag{4}$$

The proof in the new formulation remains the same as the proof for Proposition 1 in [1].

This version of the criterion will be used substantially further in the algorithm for solving the problem under consideration.

For a field K, we assume that there exists an algorithm that checks the consistency of an arbitrary given polynomial system of equations over K, provided that the coefficients of the system belong to K and the system has a solution with components in K. In the case of an algebraically closed field, such an algorithm is based on Gröbner bases [5]. In the case of a real closed field, the cylindrical decomposition method works with quantifier elimination [4, 6–8], which is a generalization of the Tarski algorithm [11]. If a polynomial system has rational coefficients, then Tarski's algorithm itself allows us to answer the question of whether this system has a solution with real components. In relation to the original problem, this means that if the elements of a given matrix P are polynomials with rational coefficients, then we can find out whether there are continuations of these polynomials in the form of series with real coefficients for which the matrix becomes singular. Note that if the answer to this last question is positive, we can assume that the coefficients of the polynomial system and the components of its solution belong to the field K of real algebraic numbers (which is a real closed field in the sense of [7]).

Remark 2. As for solutions with rational components, it is currently unknown whether it is possible to algorithmically resolve the question of their existence. Thanks to the results of Yu. V. Matiyasevich [10] it is known that this problem is not decidable if we are talking about integer components, but it does not follow directly from this result that the problem is also not decidable for rational numbers.

To check for the presence of solutions in the field K of real algebraic numbers of a polynomial system of equations with rational number coefficients, one can first apply an algorithm based on Gröbner bases, which is significantly less time-consuming than the cylindrical decomposition algorithm. If the answer is negative, this will mean that the system is inconsistent over the field of algebraic numbers and, consequently, over the field K. But a positive answer does not mean that the system has a solution with components in K. Additionally, the cylindrical decomposition algorithm can be used to obtain the final answer.

Together with the matrix P, the integers $d_{ij} \geq -1$, $i, j = 1, \ldots, n$, are considered given. Let

$$P = (p_{ij}), \quad p_{ij} \in K[x], \quad \deg p_{ij} \leq d_{ij}, \quad d = \max_{i,j} d_{ij}. \tag{5}$$

With respect to the element m_{ij} of the matrix (2), it is assumed that $m_{ij} = p_{ij} + q_{ij}$, where $q_{ij} \in K[[x]]$ is some series, such that $\operatorname{val} q_{ij} \geq d_{ij} + 1$. The equality $d_{ij} = -1$ means that we have no information about the element m_{ij} of the matrix M, except for its membership in $K[[x]]$.

This problem of checking the nonsingularity of a matrix M is more difficult than that considered in [1], where none of the added terms can have a degree less than $d + 1$.

Thus, the added term does not affect the initial terms of the determinant. The solution of this new problem is reduced below to a series of consistency checks (in other words, checks for the presence of solutions, solvability) of systems of polynomial equations.

We will assume that among the integers d_{ij} there are nonnegative ones. Thus, $d \geq 0$ in (5).

In Sec. 6, we additionally consider the question of the number of initial terms of the series that serve as elements of the matrix P^{-1} and coincide with the initial terms of the elements of the matrix M^{-1}. The answer is given under the assumption of strong nonsingularity (in the new, extended sense) of the matrix P.

If all d_{ij} are equal to each other, then the matrix P will be called *flat*.

3. Extension of the Concept of Strong nonsingularity of a Matrix

Definition 1. A matrix P of the form under consideration (5) will be called *strongly nonsingular* if for any $(n \times n)$-matrix $Q = (q_{ij})$ whose elements belong to $K[[x]]$, $\operatorname{val} q_{ij} \geq d_{ij} + 1$, the matrix $P + Q$ is nonsingular. (If $d_{ij} = -1$, then q_{ij} can be any element of $K[[x]]$, and for any q_{ij} the matrix $P + Q$ is nonsingular.)

It will be shown that for a given matrix P, under certain assumptions about the field K, it is possible to find out algorithmically whether this matrix is strongly nonsingular in the sense of Definition 1.

Definition 2. Let (5) hold. For a matrix P, its K-*completion* is the matrix obtained by adding to each p_{ij} for $d_{ij} < d$ some monomials of degrees $d_{ij} + 1, \ldots, d$ with coefficients from K. *Formal completion* is a completion in which all added monomials have different indefinite (symbolic) coefficients.

It is easy to see that the following proposition is valid.

Proposition 2. *A matrix P is strongly nonsingular if and only if any of its K-completions* (*each of which is a flat matrix*) *is strongly nonsingular.*

In what follows, considering statements that are valid for all solutions of some polynomial system of equations, we assume that if the system is empty (does not contain equations), then any set of values of the unknowns is its solution.

Theorem 1. *Let $p(x; t_1, \ldots, t_m)$ be a polynomial in x whose coefficients are polynomials over K in unknowns $t_1, \ldots, t_m$. Let S be a polynomial over K consistent system of equations* (*possibly empty*) *with respect to unknowns $t_1, \ldots, t_m$. There exists an algorithm that allows one to find all values of* $\operatorname{val} p$ *that* *are realized (arise) for some values $t_1, \dots, t_m \in K$ that satisfy the system S. Together with each found value ν of the valuation, this algorithm finds a consistent system S_ν of polynomial over K equations, the solutions of which are all those solutions of the system S for which* $\operatorname{val} p = \nu$. *The result of the algorithm is a list of* conditional valuations *of the form (ν, S_ν).*

Proof. Let us indicate the corresponding algorithm (let us call it $\mathcal{A}$). Let for some n

$$p = p_0(t_1, \dots, t_m) + \dots + p_n(t_1, \dots, t_m)x^n, \tag{6}$$

where $p_n(t_1, \dots, t_m)$ is a nonzero polynomial. Let $\tau_1, \dots, \tau_m \in K$ be some specific values of the unknowns $t_1, \dots, t_m$. Let ν be some integer, $0 \le \nu \le n$. The equality $\nu = \operatorname{val} p$ for given values of unknowns, satisfying the system S, is satisfied if and only if for $\tau_1, \dots, \tau_m$ the relations

$$S,\ p_0 = \dots = p_{\nu-1} = 0,\ p_\nu \ne 0 \tag{7}$$

are satisfied (the equations of the system S are included in (7)). The system (7) can be written as a system of polynomial equations, including the equations of the system S:

$$S,\ p_0 = 0, \dots, p_{\nu-1} = 0,\ p_\nu - r = 0,\ rs - 1 = 0, \tag{8}$$

where r, s are additional unknowns involved in the system. If $\nu = 0$, then the equations $p_0 = 0, \dots, p_{\nu-1} = 0$ are absent from (8). The last two equations of the system (8) ensure that the inequality $p_\nu(t_1, \dots, t_m) \ne 0$ is satisfied.

The algorithm sequentially considers all a priori values $\nu = 0, \dots, n$, for each of them it constructs a system S_ν of polynomial equations of the form (8) and checks its consistency. If the system S_ν is consistent, then the conditional valuation (ν, S_ν) is included in the resulting list. □

4. Algorithm for Checking Strong nonsingularity of a Matrix

Algorithm $\mathcal{A}$ and criterion (4) lead to an algorithm for solving the main problem. The idea is as follows: using $\mathcal{A}$, find all pairs

$$\operatorname{val} \det P,\ \operatorname{val} P^*, \tag{9}$$

that are realized for certain specific K-completions of the original system; for each of these pairs, check the satisfying of inequality (4).

Thus, let the determinant $\det \tilde{P}$ found for the formal completion $\tilde{P}$ of the matrix P have the form (6), where the unknowns are the symbolic notations used in the formal completion.

For the polynomial p written in the form (6), the algorithm first checks the consistency of the system $p_0 = 0, \dots, p_n = 0$. If it is consistent, the calculations are terminated and it is declared that P is not a strongly nonsingular matrix.

Applying $\mathcal{A}$ to $\det \tilde{P}$ and the empty system, we obtain a finite list of conditional valuations. Next, we consider these conditional valuations. Let $(\nu,\ S_\nu)$ be the next one. For the matrix $(\tilde{P})^*$, we find, using $\mathcal{A}$, its conditional valuations ν_0 taking into account the system S_ν. A list of conditional valuations of the form (ν_0, ν, S_{ν_0}) is obtained, in which all systems S_{ν_0} are consistent. Consistency guarantees that each of the obtained pairs (9) is realized for some values of $t_1, \dots, t_m$. We examine each of the pairs (9) using the criterion (4). This may indicate the existence of a K-completion of the matrix P for which (4) does not hold, then P is not strongly nonsingular. If this does not happen, then any K-completion yields a strongly nonsingular matrix, and by Proposition 2 the matrix P is strongly nonsingular.

5. Implementation in Maple

The above algorithm for checking the strong nonsingularity of a matrix with elements in the form of truncated series has been implemented by us in the environment of the computer algebra Maple 2024 (see [9]) as the `StronglyNonSingular` procedure. The Maple library containing this procedure, as well as the Maple session with examples of using the procedure, are available at the link `http://www.ccas.ru/ca/_media/StronglyNonSingularSeriesMatrix.zip`.

To determine the consistency of polynomial over K systems that arise during the operation of the procedure, where K is the field of real algebraic numbers, the `Solve` procedure of the `Groebner` package built into Maple is first used. This procedure determines whether the given system is consistent over the field $\bar{\mathbb{Q}}$ of algebraic numbers. If the system is consistent over $\bar{\mathbb{Q}}$, then the `QuantifierEliminate` procedure from the Maple package `QuantifierElimination` (see [12]), which implements the cylindrical decomposition method, is used to check the consistency of the system over K.

The first argument of our `StronglyNonSingular` procedure is a square matrix P, the second is the name of the variable (in our examples, x). The truncated elements of the matrix are given in the form $p_{ij}(x) + O(x^{d_{ij}+1})$, where $d_{ij} \geq -1$ and $p_{ij}(x)$ is a polynomial of degree at most d_{ij} over the field of rational numbers. (Here and below, the notation $O(x^k)$ is used for some (unspecified) formal series, whose valuation is greater than or equal to k.) The procedure returns `true` if P is strongly nonsingular, otherwise `false`. It also prints some intermediate results.

Example 1. Let it be checked whether the matrix

$$\begin{pmatrix} x & 0 \\ 1 & x \end{pmatrix}$$

is strongly nonsingular for truncation degrees $d_{11} = 1$, $d_{12} = 2$, $d_{21} = 2$, and $d_{22} = 1$. We perform the assignment

$$> P := \begin{bmatrix} x + O(x^2) & O(x^3) \\ 1 + O(x^3) & x + O(x^2) \end{bmatrix} :$$

We apply our procedure:

> `StronglyNonSingular`(P, x);

```
completion:
```

$$\begin{bmatrix} x + x^2\, t_1 & 0 \\ 1 & x + x^2 t_2 \end{bmatrix}$$

```
d = 2
detP = t[1]*t[2]*x^4+(t[1]+t[2])*x^3+x^2
    val_detP = 2
    S2 = {}
         d >= val_detP --> true
                                            true
```

The procedure prints intermediate results: the formal completion of the given matrix, maximum truncation degree $d = 2$, etc.

The answer `true` indicates that the given matrix is strongly nonsingular.

Example 2. Now the first argument of the procedure is not strongly nonsingular matrix.

$$> \texttt{StronglyNonSingular}\left(\begin{bmatrix} x + O(x^3) & O(x^2) \\ 1 + O(x^3) & x + x^2 + O(x^3) \end{bmatrix}, x\right);$$

```
completion:
```

$$\begin{bmatrix} x & x^2\, t_1 \\ 1 & x + x^2 \end{bmatrix}$$

```
d = 2
detP = x^3+(-t[1]+1)*x^2
    val_detP = 2
    S2 = {-t[1]+1-r[2] = 0
          r[2]*s[2]-1 = 0}
```

```
adjoint:
```

$$\begin{bmatrix} x+x^2 & -t_1\,x^2 \\ -1 & x \end{bmatrix}$$

```
        val_Padj = 0
        S0 = {-t[1]+1-r[2] = 0
              r[2]*s[2]-1 = 0}
        d + val_Padj >= val_detP --> true
    val_detP = 3
    S3 = {-t[1]+1 = 0}
        val_Padj = 0
        S0 = {-t[1]+1 = 0}
        d + val_Padj >= val_detP --> false
```

false

The answer `false` indicates that the given matrix is not strongly nonsingular.

6. Inverse Matrix

For a given flat strongly nonsingular matrix P, $d = \deg P$, we can calculate the initial terms of the series that make up the inverse matrix, in the amount of $h + 1$, where

$$h = d + \operatorname{val} P^{-1} = d + \operatorname{val} P^* - \operatorname{val}\det P \tag{10}$$

[1, Proposition 3]. That is, the elements of the matrices P^{-1} and M^{-1} coincide, at least to the degree $\operatorname{val} P^{-1} + h$. In particular, $\operatorname{val} P^{-1} = \operatorname{val} M^{-1}$.

Let the truncation P of the matrix M be strongly nonsingular, but not necessarily flat. For the formal completion $\tilde{P}$ of the matrix P, we can, using the algorithm proposed, find all admissible pairs (9). From all the obtained pairs, we choose the smallest $\nu = \operatorname{val}(\tilde{P})^{-1}$ and set $h = d + \nu$.

Each (i, j)th element of the matrix $(\tilde{P})^{-1}$ is calculated to the degree $\nu + h$:

$$a_\nu x^\nu + \cdots + a_{\nu+h} x^{\nu+h} + \ldots .$$

Some of the initial coefficients, may depend on the unknowns $t_1, t_2, \ldots$ introduced during the formal completion. Let $h_{i,j}$ be such that $a_\nu, \ldots, a_{\nu+h_{i,j}}$ do not depend on the unknowns, while $a_{\nu+h_{i,j}+1}$ depends. Then (i, j)th element of the matrix M^{-1} to the degree $\nu + h_{i,j}$ are known.

Example 3. Consider a truncation with the formal completion

$$\tilde{P} = \begin{pmatrix} 1+x & 0 \\ 1+t_1\,x & 1-x \end{pmatrix}.$$

Here $d = 1$, $\det \tilde{P} = -x^2 + 1$. Since $\operatorname{val}\det \tilde{P} = 0$ independently of the value t_1 and $d > \operatorname{val}\det \tilde{P}$, that (4) is satisfied. We compute

$$(\tilde{P})^{-1} = \begin{pmatrix} \frac{1}{x+1} & 0 \\ \frac{x\,t_1+1}{-x^2+1} & \frac{1}{x-1} \end{pmatrix}.$$

We obtain $\operatorname{val}(\tilde{P})^{-1} = 0$ and terms of the power series expansion till the degree 1:

$$(\tilde{P})^{-1} = \begin{pmatrix} 1-x+O(x^2) & O(x^2) \\ -1-t_1\,x+O(x^2) & 1+x+O(x^2) \end{pmatrix}.$$

Discarding the term depending on t_1, we finally obtain

$$P^{-1} = \begin{pmatrix} 1-x+O(x^2) & O(x^2) \\ -1+O(x) & 1+x+O(x^2) \end{pmatrix}.$$

Our `StronglyNonSingular` procedure provides the ability to construct the initial terms of the elements of the matrix M^{-1} in cases where we know a strongly nonsingular matrix P, which is a truncation for M. For this, the procedure is additionally passed a name which will be assigned to the truncation of the inverse matrix.

$$> \texttt{StronglyNonSingular}\left(\begin{bmatrix} 1+x+O(x^2) & O(x^2) \\ 1+O(x) & 1-x+O(x^2) \end{bmatrix},\ x,\ \texttt{'inverse'} = \texttt{'}Pinv\texttt{'}\right);$$

```
completion:
```

$$\begin{bmatrix} 1+x & 0 \\ 1+t_1\,x & 1-x \end{bmatrix}$$

```
d = 1
detP = -x^2+1
    val_detP = 0
    S0 = {}
         d >= val_detP --> true
```

$$\texttt{true}$$

$> Pinv$;

$$\begin{bmatrix} 1-x+O(x^2) & O(x^2) \\ -1+O(x) & 1+x+O(x^2) \end{bmatrix}$$

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S. A. Abramov
Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences,
Russia, 119333 Moscow, Vavilov str., 40
E-mail: `sergeyabramov@mail.ru`

A. A. Ryabenko
Federal Research Center "Computer Science and Control" of the Russian Academy of Sciences,
Russia, 119333 Moscow, Vavilov str., 40
E-mail: `anna.ryabenko@gmail.com`