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Improved Universal Denominators

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Abstract—The paper presents an algorithm for improvement (degree reduction) of universal denominators, which are used, for example, for constructing rational solutions of linear differential and difference systems with polynomial coefficients. A variant of Zeilberger's algorithm is described; it uses construction of universal denominator instead of application of Gosper's algorithm.

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1. INTRODUCTION

For a system of first-order linear differential equations

$$b_i(x)y_i'(x) + a_{i1}(x)y_1(x) + \dots + a_{in}(x)y_n(x) = c_i(x), \quad i = 1, \dots, n, \tag{1}$$

where a_{ij}, b_i, and c_i are polynomials over some field $\mathbf{K}$ of zero characteristic, $y_i' = \frac{dy_i}{dx}$, $i = 1, \dots, n$, we will extend the notion of *universal denominator* [1—3] in the following way: vector $(u_1, \dots, u_n) \in (\mathbf{K}[x])^n$ will be referred to as universal denominator for the left-hand side of system (1) if any component y_i in an arbitrary vector $(y_1, \dots, y_n) \in (\mathbf{K}(x))^n$ that turns the left-hand sides of the system into polynomials can be represented in the form $\frac{z_i}{u_i}$, $z_i \in \mathbf{K}[x]$.

Similarly, we can define universal denominator for the left-hand side of a system of first-order difference equations

$$b_iy_i(x+1) + a_{i1}y_1(x) + \dots + a_{in}y_n(x) = c_i, \quad i = 1, \dots, n, \tag{2}$$

where $a_{ij}, b_i, c_i \in \mathbf{K}[x]$, $i, j = 1, \dots, n$.

Universal denominator for the left-hand side of (1) is determined ambiguously: indeed, together with $(u_1, \dots, u_n)$, any vector $(U_1, \dots, U_n) \in (\mathbf{K}[x])^n$ with components satisfying $u_i | U_i$, $i = 1, \dots, n$, has the above property. A similar assertion is valid for (2).

After the universal denominator $(u_1, \dots, u_n)$ for the left-hand side of one of the systems (1) or (2) has been calculated, the task of finding rational solution $(y_1, \dots, y_n)$ of the system is reduced to finding polynomial solution $(z_1, \dots, z_n)$ to a new system obtained by the substitution

$$y_i = \frac{z_i}{u_i}, \quad i = 1, \dots, n. \tag{3}$$

Methods for calculating universal denominators are well known for both differential and difference cases [1–5].

The paper discusses methods to decrease the degrees of universal denominator components, which allows us to decrease degrees of coefficients in the system obtained by substitution (3) and, in some cases, to reduce time costs for its solution. In addition, in the case of differential equations, the proposed method can allow finding relations of the form $w_i | z_i'$, where w_i is a known polynomial. These relations do not immediately decrease degrees of universal denominator components, but they can be used to reduce computational time costs for solving the system obtained by substitution (3).

All results obtained for the difference case can easily be transferred to the q-difference case (finding of rational solutions of q-difference systems is discussed in [5, 6]).

For difference and q-difference scalar equations of an arbitrary order, such an approach to improving universal denominator is proposed in [7]. The approach described in [8] is applicable not only to difference scalar equations but also to systems of equations; however, it is significantly more complicated than that proposed in this paper and, moreover, cannot be directly extended to the differential case.

The balancing method proposed in Section 2.2 can be used not only for improving universal denominators but also for finding any solution of a differential or difference system in the neighborhood of the point that is an isolated pole for its components.

In the last section, we show that construction of universal denominator can be used in Zeilberger's algorithm [9] instead of Gosper's algorithm [10] application. This simplifies the structure of Zeilberger's algorithm, facilitates the adjustment of the algorithm for the difference and q-difference cases, and allows us to avoid difficulties in the situation where the recurrent relation obtained by Zeilberger's algorithm is homogeneous (see [11]).

2. IMPROVEMENT OF UNIVERSAL DENOMINATOR

2.1. Preliminary System Simplification

Let $g_1, \ldots, g_n$ be polynomials such that the substitution

$$y_i = \frac{\tilde{y}_i}{g_i}, \quad i = 1, \ldots, n \tag{4}$$

converts system (1) to a system with polynomial coefficients in the left-hand side (i.e., $g_i|a_{ji}$ for $j \neq i$, and there exist $\tilde{b}_i$, $\tilde{a}_{ii} \in \mathbf{K}[x]$ such that $b_i y_i' + a_{ii}y_i = \tilde{b}_i(g_iy_i)' + \tilde{a}_{ii}g_iy_i$). If some of the polynomials $g_1, \ldots, g_n$ are not constant, substitution (4) removes common factors of the corresponding coefficients.

Proposition 1. *Among the polynomials satisfying conditions of substitution* (4), *polynomials*

$$g_i^* = \gcd(a_{1i}, \ldots, a_{i-1,i}, a_{ii} - b_i', a_{i+1,i}, \ldots, a_{ni}, b_i), \tag{5}$$

$i = 1, \ldots, n$, *have the greatest degree.*

Proof. From $b_i y_i' + a_{ii}y_i = \tilde{b}_i(g_iy_i)' + \tilde{a}_{ii}g_iy_i$, it follows that $b_i = \tilde{b}_i g_i$, $a_{ii} = \tilde{b}_i g_i' + \tilde{a}_{ii}g_i$, $a_{ii} - b_i' = (\tilde{b}_i' + \tilde{a}_{ii})g_i$; hence, $g_i|b_i$, $g_i|(a_{ii} - b_i')$. It is easy to see that, together with $g_i|a_{ji}, j \neq i$, the last two relations present a sufficient condition for substitution $y_i = \frac{\tilde{y}_i}{g_i}$ to transform (1) to a system with polynomial coefficients. ■

Example 1. For the system

$$\begin{cases} x^3y_1' + 4x^2y_1 - (x-1)y_2 = 0, \\ x(x-1)^2y_2' + x^3y_1 + (x-1)(x-2)y_2 = 0, \end{cases}$$

$g_1^* = x^2$, $g_2^* = x - 1$. By performing substitution (5), we obtain the system

$$\begin{cases} x\tilde{y}_1' + 2\tilde{y}_1 - \tilde{y}_2 = 0, \\ x(x-1)\tilde{y}_2' + x\tilde{y}_1 - 2\tilde{y}_2 = 0. \end{cases}$$

The universal denominator for the left-hand side of this system (obtained by applying the algorithm from [4]) is equal to (x^2, x^2); the universal denominator for the left-hand side of the original system is $(x^4(x - 1), x^4(x - 1))$.

In the above example, substitution (4) simplifies significantly solution of the system. Moreover, this substitution can also simplify the process of finding universal denominator. Computational costs for calculating g_1^*, $\ldots$, g_n^* are insignificant compared to the costs for finding universal denominator.

In order to avoid introducing new notation, we will assume that such substitutions have been already performed in constructing (1), i.e., $g_i^* = 1$, $i = 1, \ldots, n$.

A similar simplifying substitution for systems of difference equations is proposed in [2], where

$$g_i^* = \gcd(a_{1i}(x), \ldots, a_{ni}(x), b_i(x-1)),$$
$$i = 1, \ldots, n.$$

2.2. Balancing A Priori Estimates of Upper Bounds for the Orders of Poles of the System Solution Components

Let system (1) satisfy the following condition: the components $y_1, \ldots, y_n$ of some solution to this system have poles at point x_0 of orders that do not exceed α_1, $\ldots$, α_n, respectively. By performing substitution

$$y_i = \frac{\tilde{y}_i}{(x - x_0)^{\alpha_i}}, \quad i = 1, \ldots, n, \tag{6}$$

we obtain a system whose corresponding solution $(\tilde{y}_1, \ldots, \tilde{y}_n)$ has no poles at this point. Let coefficients of $\tilde{y}_i'$, $\tilde{y}_1, \ldots, \tilde{y}_n$ in the ith equation of the new system have poles at point x_0 of orders $\beta_0, \beta_1, \ldots, \beta_n$, respectively.

Proposition 2. *Let* $\max_{j=0,\ldots,n} \beta_j$ *be attained at a single* $j = j_0$, *and let* $\delta = \beta_{j_0} - \max_{j \neq j_0} \beta_j$. *Then, the corresponding solution component* $\tilde{y}_{j_0}$ *for* $j_0 > 0$, *or the derivative* $\tilde{y}_i'$ for $j_0 = 0$, *has a zero at point* x_0 *of order* δ.

Proof. Let us assume this is not so and multiply both sides of the ith equation by $(x - x_0)^{\beta_{j_0} - \delta}$. Only one term on the left-hand side of the obtained equation has a pole at point x_0; therefore, the sum cannot be equal to the polynomial on the right-hand side. ■

Proposition 2 allows us to improve the orders of the poles of solution components of system (1) if the system obtained from (1) by substitution (6) contains an equation satisfying the conditions of Proposition 2 with $j_0 > 0$. In this case, the order of pole y_{j_0} at point x_0 does not exceed $\alpha_{j_0} - \delta$, and, after performing substitution (6) with new $\alpha_1, \ldots, \alpha_n$, we can obtain again an improved estimate for the order of pole for one of the solution components by means of this proposition. It makes sense to repeat these operations until the estimates cease to change. The described procedure will be referred to as balancing of a priori estimates of upper bounds for the orders of poles at point x_0.

Example 2. Preliminary estimation demonstrates that the orders of poles of the solution components of the system

$$\begin{cases} x^3 y_1'(x) + 60 y_3(x) = 0, \\ y_2'(x) - y_1(x) = 0, \\ y_3'(x) - y_2(x) = 0 \end{cases}$$

at point $x_0 = 0$ do not exceed 5. By setting $\alpha_1 = \alpha_2 = \alpha_3 = 5$ and performing substitution (6) we obtain in the first equation $\beta_0 = 3$, $\beta_3 = 5$; consequently, y_3 has a pole of order not greater than 3. Let us set $\alpha_1 = \alpha_2 = 5$, $\alpha_3 = 3$ and apply substitution (6) again. In the third equation of the obtained system, $\beta_0 = 4$, $\beta_2 = 5$; therefore, the order of the pole y_2 does not exceed 4. Substitution (6) with $\alpha_1 = 5$, $\alpha_2 = 4$, and $\alpha_3 = 3$ does not result in any improvements.

A similar procedure can be proposed for the difference case. Let (2) be a system such that the components $y_1, \ldots, y_n$ of some solution of this system have poles at the points $x_0 + k$ ($k \in M$, M is a finite subset of $\mathbf{Z}$) of orders not greater than $\alpha_{k1}, \ldots, \alpha_{kn}$, respectively, and do not have poles at points $x_0 + \tilde{k}$, $\tilde{k} \in \mathbf{Z} \backslash M$. By applying the substitution

$$y_i = \frac{\tilde{y}_i}{\prod_{k \in M} (x - x_0 - k)^{\alpha_{ki}}}, \quad i = 1, \ldots, n, \tag{7}$$

we obtain a system whose corresponding solution ($\tilde{y}_1$, $\ldots$, $\tilde{y}_n$) does not have poles at the points $x_0 + k$, $k \in \mathbf{Z}$. Let coefficients of $\tilde{y}_i'$, $\tilde{y}_1$, $\ldots$, $\tilde{y}_n$ in the ith equation of the obtained system have poles at the point x_0 of orders $\beta_0, \beta_1, \ldots, \beta_n$, respectively.

Proposition 3. *Let* $\max\limits_{j = 0, \ldots, n} \beta_j$ *be attained for a single* $j = j_0$, *and let* $\delta = \beta_{j_0} - \max\limits_{j \neq j_0} \beta_j$. *Then, the corresponding solution component* $\tilde{y}_{j_0}(x)$ *for* $j_0 > 0$, *or* $\tilde{y}_i(x+1)$ *for* $j_0 = 0$, *has a zero at point* x_0 *of order* δ.

Proof is similar to that of Proposition 2. ■

Example 3. Let solution components of the system

$$\begin{cases} x(x+3)(2x+3)y_1(x+1) \\ -(x-1)(x+2)(2x-1)y_2(x) = 0, \\ y_2(x+1) - y_1(x) = 0 \end{cases}$$

have poles of order not greater than 1 at points –2, –1, 0, 1, $-\frac{1}{2}$, and $\frac{1}{2}$, and have no other poles. Let us choose $x_0 = 0$ and perform the substitution

$$y_i = \frac{\tilde{y}_i}{(x-1)x(x+1)(x+2)}, \quad i = 1, 2.$$

From the first equation, we obtain $x | \tilde{y}_2(x)$, $(x + 2) | \tilde{y}_1(x+1)$; from the second equation, we have $(x - 1) | \tilde{y}_1(x)$, $(x+3) | \tilde{y}_2(x+1)$. Then, we perform another substitution

$$y_i = \frac{\tilde{y}_i}{\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right)}, \quad i = 1, 2,$$

and obtain from the second equation $\left(x + \frac{3}{2}\right) \Big| \tilde{y}_2(x+1)$, $\left(x - \frac{1}{2}\right) \Big| \tilde{y}_1(x)$. Therefore, y_1 has poles of order not greater than 1 at points –2, 0, and $-\frac{1}{2}$, and y_2 has poles at points –1, 1, and $\frac{1}{2}$.

If we additionally consider the right-hand sides of the system, the balancing of upper bounds can be performed for zero and negative bounds for the order of poles of the system solution components, which allows establishing existence of zeros for some components.

Example 4. Preliminary estimation shows that the orders of poles of solution components of the system

$$\begin{cases} x^4 y_1' - x^3 y_1 + y_2 = -x^2, \\ y_2' - 2x^2 y_1 = 0, \\ x y_3' - y_1 + y_2 - y_3 = x^2 \end{cases}$$

at the point $x_0 = 0$ do not exceed 1. After substitution (6), the first equation of the system takes the form

$$x^3 \tilde{y}_1' - 2x^2 \tilde{y}_1 + \frac{1}{x} \tilde{y}_2 = -x^2.$$

The right-hand side and coefficients of $\tilde{y}_1$, $\tilde{y}_1'$ have zeros of order 2 at x_0; therefore, the component $\tilde{y}_2$ should have zero of order not less than 3 at point x_0.

2.3. Universal Denominators

The balancing procedure proposed in Section 2.2 can be used not only for the pole order bounds, but also for any bounds of order of irreducible factors included in the universal denominator under consideration.

Example 5. The universal denominator for the left-hand side of the system

$$\begin{cases} (x^2+1)^2 y_1' + 3x(x^2+1)y_1 - y_2 = 0, \\ y_2' - y_1 = 0 \end{cases}$$

calculated by the algorithm from [4] is equal to $((x^2+1)^2, (x^2+1)^2)$. Let us perform substitution (3). The first equation of the obtained system is

$$z_1' - \frac{x}{x^2+1} z_1 - \frac{1}{(x^2+1)^2} z_2 = 0;$$

therefore, $(x^2+1)|z_2$. Substitution (3) for the obtained universal denominator $((x^2+1)^2, x^2+1)$ does not lead to further improvement.

Let us assume that, after we have obtained improved upper bounds for the orders of occurrence of v in the universal denominator component, some additional relations of the form $(x-x_0)^\delta | z_i'$ have been derived. Such relations can be used directly in the course of system solving by representing z_i as a Hermite interpolation polynomial and equating coefficients of $x - x_0, \ldots, (x-x_0)^\delta$ to zero. This makes it possible to decrease the number of unknown coefficients z_i.

Example 6. The system

$$\begin{cases} (x+1)y_1' + 2y_1 = c, \\ (x+1)y_2' - 2y_1 = -c \end{cases}$$

by substitution $z_1 = \frac{y_1}{(x+1)^2}$, $z_2 = \frac{y_2}{(x+1)^2}$ takes the form

$$\begin{cases} \dfrac{z_1'}{x+1} = c, \\ \dfrac{z_2'}{x+1} - \dfrac{2z_1}{(x+1)^2} - \dfrac{2z_2}{(x+1)^2} = -c, \end{cases}$$

which allows us to use the relation $(x+1)|z_1'$ for solving it. After we have obtained an estimate $\deg_x z_1 \le 2$, we can write $z_1(x) = h_2(x+1)^2 + h_0$, which yields $h_2 = c/2$.

If we have several additional relations $v_j^{\delta_j} | z_i'$, $j = 1, \ldots, k$, and some of irreducible polynomials v_j have a degree greater than 1, it is convenient to use the following method: to find the upper bound d of the order z_i, to set $z_i = h_d x^d + \ldots + h_0$ and to divide $dh_d x^{d-1} + \ldots + h_1$ with remainder by

$$w_i = \prod_{j=1}^{k} v_j^{\delta_j}.$$

The remainder r must be equal to zero, which also reduces the number of unknown coefficients h_j.

Example 7. Let the derivative of polynomial z_i satisfy relations $(x-1)|z_i'$, $(x+1)|z_i'$, $(x^2+1)|z_i'$, $(x^4+1)|z_i'$, and let $\deg z_i \le 12$. Dividing $12h_{12}x^{11} + \ldots + h_1$ by $x^8 - 1$ and equating the remainder to zero, we obtain $h_8 = h_7 = h_6 = h_5 = 0$, $h_4 = -3h_{12}$, $h_3 = -\frac{11}{3}h_{11}$, $h_2 = -5h_{10}$, $h_1 = -9h_9$.

By using the equation $r = 0$ we can reduce considerably the time for solving the system if its right-hand side contains parameters, since $r = 0$ is easily solved and does not contain parameters. However, the addition of such equations to the system may also increase the computation time.

In the difference case, all relations obtained in the course of balancing the bounds of order of irreducible factors can be used for improvement of the universal denominator.

Example 8. The universal denominator for the left-hand side of the system

$$\begin{cases} (x+2)(x+5)y_1(x+1) - 4xy_1(x) \\ -(x^2-2x+2)(x^2+1)y_2(x) = 0, \\ (x^2+2x+2)y_2(x+1) \\ -(x^2-2x+2)y_2(x) = 0 \end{cases}$$

obtained by the algorithm from [2] is equal to $(x(x+1)(x+2)(x+3)(x+4)(x^2+1)(x^2-2x+2), x(x+1)(x+2)(x+3)(x+4)(x^2+1)(x^2-2x+2))$. By performing substitution (3), from the first equation, we obtain $(x^2+2x+2)|z_1(x+1)$, $(x^2-2x+2)|z_1(x)$, $x|z_2(x)$; from the second equation, we obtain $(x+5)|z_2(x+1)$. Further improvements give the universal denominator $(x(x+1), (x^2+1)(x^2-2x+2))$.

3. ZEILBERGER'S ALGORITHM BASED ON UNIVERSAL DENOMINATOR CONSTRUCTION

3.1. Difference Case

Let the hypergeometric term $T(n, k)$ be given by its certificates that are rational functions

$$\mathcal{C}_n(T) = \frac{E_n T(n,k)}{T(n,k)}, \quad \mathcal{C}_k(T) = \frac{E_k T(n,k)}{T(n,k)},$$

$E_n f(n) = f(n+1)$, $E_k f(k) = f(k+1)$.

Let $d \ge 0$ be a fixed integer, and we try to find, for $T(n, k)$, a k-free operator $L \in \mathbf{K}[n, E_n]$ of order d such that $LT(n,k) = E_k G(n,k) - G(n,k)$ for some hypergeometric term $G(n, k)$. Application of any operator from $\mathbf{K}(n,k)[E_n, E_k]$ to $T(n, k)$ gives a hypergeometric term of form $R(n,k)T(n,k)$ (it is not excluded that $R(n,k) = 0$). In particular, application of a k-free operator $L = a_d(n)E_n^d + \ldots + a_0(n)$ with undefined coefficients $a_0(n), \ldots, a_d(n)$ gives the rational function

$$R(n, k, a_0, \dots, a_d) = \frac{p(n, k, a_0, \dots, a_d)}{q(n, k)}, \tag{8}$$

where

$$p(n, k, a_0, \dots, a_d) \\ = p_0(n, k)a_0 + \dots + p_d(n, k)a_d, \tag{9}$$

and polynomials $q(n, k)$, $p_0(n, k)$, …, $p_d(n, k) \in \mathbf{K}[n, k]$ are calculated using $\mathscr{C}_n(T)$ and d. Our goal is to find a_0, …, $a_d \in \mathbf{K}(n)$, $a_d \neq 0$, such that the following equality is valid:

$$LT(n, k) = (E_k - 1)G(n, k), \tag{10}$$

where $G(n, k) = F(n, k)T(n, k)$, $F(n, k) \in \mathbf{K}(n, k)$. From (10), we obtain the equation for the rational function $F(n, k)$:

$$\mathscr{C}_k(T)F(n, k+1) - F(n, k) \\ = \frac{p(n, k, a_0, \dots, a_d)}{q(n, k)}. \tag{11}$$

Setting

$$\mathscr{C}_k(T) = S(n, k) = \frac{s(n, k)}{t(n, k)}, \quad s(n, k) \perp t(n, k),$$

and introducing a new unknown function

$$\tilde{F}(n, k) = \frac{F(n, k)}{t(n, k-1)},$$

we obtain

$$q(n, k)s(n, k)\tilde{F}(n, k+1) \\ - q(n, k)t(n, k-1)\tilde{F}(n, k) \\ = p(n, k, a_0, \dots, a_d). \tag{12}$$

Considering $q(n, k)s(n, k)$ and $q(n, k)t(n, k-1)$ as polynomials in k with coefficients lying in $\mathbf{K}[n]$, we find the universal denominator $u(n, k) \in \mathbf{K}[n, k]$ for the left-hand side of (12). Making a substitution $\tilde{F}(n, k) = \frac{f(n, k)}{u(n, k)}$ in (12) and excluding denominators from the left-hand side, we obtain the equality

$$g_1(n, k)f(n, k+1) - g_0(n, k)f(n, k) \\ = h(n, k, a_0, \dots, a_d), \tag{13}$$

where $h(n, k, a_0, \dots, a_d) = h_0(n, k)a_0 + \dots + h_d(n, k)a_d$, and $h_0(n, k)$, …, $h_d(n, k)$ are the known polynomials from $\mathbf{K}[n, k]$. The problem of finding a_0, …, $a_d \in \mathbf{K}(n)$, $a_d \neq 0$, for which equation (13) has solution $f(n, k) \in \mathbf{K}(n)[k]$ is resolved in the same way as in the initial version of Zeilberger's algorithm [9].

Example 9. Let

$$T(n, k) = \frac{(2n+k)(k+2)!(k+5)!}{k!}.$$

For $d = 1$,

$$s(n, k)q(n, k) = (k+3)(k+6)(2n+k)(2n+k+1),$$

$$t(n, k-1)q(n, k) = k(2n+k-1)(2n+k).$$

After the preliminary substitution $\tilde{F}(n, k) = \frac{f_0(n, k)}{(2n+k-1)(2n+k)}$, the left-hand side of equation (12) takes the form

$$(k+3)(k+6)f_0(n, k+1) - kf_0(n, k),$$

and the universal denominator is equal to $k(k+1)(k+2)(k+3)(k+4)(k+5)$. After finding the improved universal denominator $u = k(k+1)(k+2)$ as described in Section 2.3 and performing the substitution $f_0(n, k) = \frac{f(n, k)}{u}$, we obtain the equation

$$(k+6)f(n, k+1) - f(n, k) \\ = (k+1)(k+2)((2n+k)a_0 + (2n+k+2)a_1);$$

hence, $f(n, k) = -22k^2 - 6k - 4$, $a_1 = 22n - 47$, $a_0 = -22n + 25$.

If we succeed in finding the corresponding values a_0, …, a_d and the expression (9) is equal to 0 for these values, this does not obstruct obtaining equality (10); we will merely have $G(n, k) = 0$ in this equality (some implementations of the initial version of the algorithm work inappropriately in this case, see [11] for details).

The described procedure presents one step of the algorithm for a given d. If we fail to find a_0, …, a_d satisfying (13), the algorithm proceeds to the next step, increasing d by one.

3.2. Differential Case

Let $d \geq 0$ be a fixed integer, and we need to find a y-free operator $L = a_d(x)D_x^d + \dots + a_0(x)$, $D_x = \frac{\partial}{\partial x}$, for a *hyperexponential* function $F(x, y)$ ($F(x, y)$ is called hyperexonential if $\frac{\partial F(x, y)}{\partial x}/F(x, y)$, $\frac{\partial F(x, y)}{\partial y}/F(x, y) \in \mathbf{K}(x, y)$) such that

$$LF(x, y) = \frac{\partial}{\partial y}G(x, y), \tag{14}$$

where $G(x, y) = S(x, y)F(x, y)$, $S(x, y) \in \mathbf{K}(x, y)$. An analogue of equation (11) in the differential case is the equation

$$S'_y(x, y) + \frac{s(x, y)}{t(x, y)}S(x, y) \\ = \frac{p(x, y, a_0, \dots, a_d)}{q(x, y)}, \tag{15}$$

where $\frac{s(x, y)}{t(x, y)} = \frac{F'_y(x, y)}{F(x, y)}$, $s(x, y) \perp t(x, y)$. Multiplying both sides of (15) by $q(x, y)$ and making the substitution

$$S(x, y) = \tilde{S}(x, y)t(x, y),$$

we obtain the equation with polynomial coefficients

$$\begin{aligned} &q(x, y)t(x, y)\tilde{S}'_y(x, y) + q(x, y)(s(x, y) \\ &+ t'_y(x, y))\tilde{S}(x, y) = p(x, y, a_0, \ldots, a_d). \end{aligned} \tag{16}$$

We will now calculate the universal denominator $u(x, y)$ for the left-hand side of (16), make the substitution $\tilde{S}(x, y) = \frac{z(x, y)}{u(x, y)}$, and get rid of the denominators. The obtained equation

$$\begin{aligned} &g_1(x, y)z'_y(x, y) - g_0(x, y)z(x, y) \\ &= h(x, y, a_0, \ldots, a_d) \end{aligned} \tag{17}$$

is solved for unknowns $z \in \mathbf{K}[x, y]$, $a_0, \ldots, a_d \in \mathbf{K}[x]$ (one can also use additional relations $v_j^{\delta_j} \big| z'_y(x, y)$ if they have been derived).

3.3. Use of Previous Steps of the Algorithm

Besides applying the proposed methods of improving the universal denominator, the efficiency of Zeilberger's algorithm (both the proposed version based on universal denominators and the original version) can be increased by applying at each step the intermediate results of the previous steps.

These results can be used, for example, in converting $R_d(n, k) = R(n, k, a_0, \ldots, a_d)$ from (8) into an irreducible form. Redundant factors in $q(n, k)$ can increase significantly the time for finding universal denominator for the left-hand side of (12) and the time required for solving (13). Elimination of these factors by finding the greatest common divisor also requires much time due to the presence of parameters $a_0, \ldots, a_d$ in the numerator.

For $d = 1$, one can use

$$\begin{aligned} p(n, k, a_0, a_1) &= a_0\hat{q}(n, k) + a_1\hat{p}(n, k), \\ q(n, k) &= \hat{q}(n, k), \end{aligned}$$

where

$$\frac{\hat{p}(n, k)}{\hat{q}(n, k)} = \mathscr{C}_n(T), \quad \hat{p}(n, k) \perp \hat{q}(n, k).$$

Let the denominator $q_d(n, k)$ be known for some $d \geq 1$. By virtue of the relations

$$\begin{aligned} R_{d+1}(n, k) &= R_d(n, k) + a_{d+1}\prod_{i=0}^{d} E_n^i \mathscr{C}_n(T) \\ &= \frac{a_0 p_0(n, k) + \ldots + a_d p_d(n, k)}{q_d(n, k)} \\ &+ a_{d+1}\frac{p_d(n, k)}{q_d(n, k)}\frac{\hat{p}(n+d, k)}{\hat{q}(n+d, k)} \end{aligned}$$

and $q_d(n, k) | q_{d+1}(n, k)$, we obtain $q_{d+1}(n, k) = \tilde{q}_d\, q_d(n, k)$, where

$$\tilde{q}_d = \frac{\hat{q}(n+d, k)}{\gcd(\hat{q}(n+d, k), p_d(n, k))}.$$

In a similar way, $R_d(x, y)$ is converted to an irreducible form in the differential case.

In the difference case, in calculating the universal denominator, one can use intermediate results of the previous steps (with smaller values of d). To demonstrate this, we need the notion of *dispersion set* of polynomials $f(k)$, $g(k)$ (which is denoted as $\mathrm{ds}(f(k), g(k))$), i.e., the set of all nonnegative integers h such that $f(k + h)$ and $g(k)$ are not coprime. The universal denominator for the left-hand side of equation (12) is calculated using

$$M_d = \mathrm{ds}(q(n, k)t(n, k-1), q(n, k-1)s(n, k-1)). \tag{18}$$

Direct calculation of M_d can be replaced by calculation of $M'_d = \mathrm{ds}(Q(n, k+1)t(n, k), Q(n, k)s(n, k))$, where

$$Q(n, k) = \prod_{i=0}^{d-1}\hat{q}(n+i, k),$$

with the subsequent checking whether the elements of M'_d belong to M_d. Under such an approach, it is sufficient to find the decomposition of $s(n, k)$, $t(n, k)$, and $\hat{q}(n, k)$ into irreducible factors only once for the entire time of Zeilberger's algorithm operation. The dispersion set calculated for d can be used at the next step of the algorithm (i.e., for $d + 1$), since

$$\begin{aligned} M'_{d+1} = M'_d &\cup \mathrm{ds}(\hat{q}(n+d, k+1), \hat{q}(n, k)s(n, k)) \\ &\cup \mathrm{ds}(t(n, k)\hat{q}(n, k+1), \hat{q}(n+d, k)). \end{aligned}$$

The proposed method for calculating the dispersion set for $d \geq 2$ may occur considerably more efficient than the direct calculation; for $d = 0$, $d = 1$, the difference is insignificant.

Example 10. For the hypergeometric term

$$T(n, k) = \frac{\binom{n}{k+1}}{2n-3k},$$

we have $M'_3 = \{0, 1, 2\}$. Using the decompositions into irreducible factors obtained earlier for the polynomials

$s(n, k) = (n-k-1)(2n-3k)$, $t(n, k) = (k+2)(2n-3k-3)$, $\hat{q}(n, k) = (n-k)(2n-3k+2)$, it is easy to find $\mathrm{ds}(\hat{q}(n+3, k+1), \hat{q}(n, k)) = \{1, 2\}$, $\mathrm{ds}(\hat{q}(n+3, k+1), s(n, k)) = \{3\}$, $\mathrm{ds}(t(n, k)\hat{q}(n, k+1), \hat{q}(n+3, k)) = \emptyset$, from which we obtain $M'_4 = \{0, 1, 2, 3\}$. After the preliminary substitution $\tilde{F}(n, k) = \dfrac{f_0(n, k)}{g^*}$,

$$\begin{aligned} g^* &= \gcd(s(n, k-1)q_4(n, k-1), t(n, k-1)q_4(n, k)) \\ &= (n-k)(n-k+1)(n-k+2)(n-k+3), \end{aligned}$$

equation (12) takes the form

$$\begin{aligned} &h_1 f_0(n, k+1) - h_0 f_0(n, k) \\ &= p(n, k, a_0, \ldots, a_4), \end{aligned} \tag{19}$$

$h_1 = (2n-3k)(2n-3k+2)(2n-3k+4)(2n-3k+6)(2n-3k+8)(n-k+3)$, $h_0 = (2n-3k)(2n-3k+2)(2n-3k+4)(2n-3k+6)(2n-3k+8)(k+1)$. Let us find $r_i = \gcd(h_0, E_k^{-i-1}h_1)$ for all $i \in M'_4 \setminus \{0\}$ (common factors for $i = 0$ have been eliminated by the preliminary substitution): $r_3 = 1$, $r_2 = 1$, $r_1 = (2n-3k+6)(2n-3k+8)$. Therefore, $\mathrm{ds}(h_0, E_k^{-1}h_1) = \{1\}$, and the universal denominator for the left-hand side of (19) is equal to $(2n-3k+3)(2n-3k+5)(2n-3k+6)(2n-3k+8)$.

In the differential case calculation of the universal denominator is simplified by the fact that, at any step, all irreducible polynomials included in it divide $s(x, y)\hat{q}(x, y)$.

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