On Some Decidable and Undecidable Problems Related to Q-Difference Equations with Parameters*

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ABSTRACT

We consider linear q-difference equations with polynomial coefficients depending on parameters. For the case when the ground field is $\mathbb{Q}(q)$ we propose an algorithm recognizing whether or not there exist numerical values of parameters for which a given equation has a non-zero polynomial solution (alternatively, a rational-function solution). We prove that there exists no such algorithm if the parameter values are polynomials or rational functions of q.

Categories and Subject Descriptors

I.1.2 [**Symbolic And Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

General Terms

Algorithms, Theory

Keywords

q-difference equations with parameters, polynomial solutions, rational-function solutions, undecidable problems

1. INTRODUCTION

Suppose that in an equation $L(y) = 0$ the operator L is of the form

$$r_\rho(x, t_1, \dots, t_m)D^\rho + r_{\rho-1}(x, t_1, \dots, t_m)D^{\rho-1} + \dots \\ \dots + r_0(x, t_1, \dots, t_m), \tag{1}$$

where $D = \frac{d}{dx}$, and $r_0, r_1, \dots, r_\rho$ are polynomials over $\mathbb{Q}$ in the specified variables, and $t_1, t_2, \dots, t_m$ are parameters. In the paper [13] of D. Boucher the following result of J.-A. Weil is mentioned: there is no algorithm that, for an arbitrary operator L of form (1) answers whether or not numerical values of parameters $t_1, t_2, \dots, t_m$ exist for which equation $L(y) = 0$ has a solution in the form of a non-zero rational function of x. The proof is based on the David-Matiyasevich-Putnam-Robinson theorem which says that there exists no algorithm which, for an arbitrary polynomial $P(t_1, t_2, \dots, t_m)$ with integral coefficients, determines whether or not the equation $P(t_1, t_2, \dots, t_m) = 0$ has an integral solution [19]. The result by Weil can be easily extended to the problem of existence of polynomial solutions of equation $L(y) = 0$.

Similar results have been obtained for the difference case ([2, 3]). The operator L is of the form

$$r_\rho(x, t_1, \dots, t_m)E^\rho + r_{\rho-1}(x, t_1, \dots, t_m)E^{\rho-1} + \dots \\ \dots + r_0(x, t_1, \dots, t_m), \tag{2}$$

where E is the shift operator: $E(y(x)) = y(x + 1)$, and again $r_0, r_1, \dots, r_\rho$ are polynomials over $\mathbb{Q}$ in the specified variables, $t_1, t_2, \dots, t_m$ are parameters.

In this paper we consider q-difference equations. Differential equations are based on the differentiation operator D, while difference equations are based on the shift operator E. In turn, the q-difference equations are based on the q-shift operator Q:

$$Q(y(x)) = y(qx),$$

where q is a fixed value or an additional variable (q-calculus and the theory and algorithms for q-difference equations are of interest in combinatorics, especially in the theory of partitions [10, Sect. 8.4], [11]). The q-difference analogue of operators (1), (2) is

$$r_\rho(x, t_1, \dots, t_m)Q^\rho + r_{\rho-1}(x, t_1, \dots, t_m)Q^{\rho-1} + \dots \\ \dots + r_0(x, t_1, \dots, t_m), \tag{3}$$

where $r_0, r_1, \dots, r_\rho$ are polynomials in specified variables over a field k of characteristic 0. We assume that $k = k_0(q)$, where k_0 is a subfield of k, and q, x are algebraically independent over k_0.

We show that the situation with the parametric case for q-difference equations in some sense is more interesting than for differential and difference equations. Let, e.g., the ground field k be $\mathbb{Q}(q)$. Then there is an algorithm that recognizes the existence of numerical (real, complex) values of the parameters for which a given linear q-difference equation has a solution in the form of a non-zero polynomial or, alternatively, rational function; it is possible that the right-hand side is a non-zero polynomial in x that contains parameters. (Recall that a rational solution of a linear q-difference equation with polynomial coefficients and polynomial right-hand side without parameters is a rational function of x over k such that substituting it into the equation gives an equality in $k(x)$.)

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At the same time, if the values of parameters are allowed to be arbitrary polynomials or rational functions of q then such algorithm does not exist.

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2. PRELIMINARIES: Q-DIFFERENCE EQUATIONS WITHOUT PARAMETERS AND SYSTEMS OF ALGEBRAIC EQUATIONS

In this section we consider linear q-difference equations with polynomial coefficients and polynomial right-hand sides which do not contain any parameters, i.e., equations of the form

$$L(y) = f(x), \tag{4}$$

where

$$L = r_\rho(x)Q^\rho + \cdots + r_1(x)Q + r_0(x), \tag{5}$$

$r_0(x), r_1(x), \ldots, r_\rho(x), f(x) \in k[x]$. Here k is a field of characteristic 0, $k = k_0(q)$, where k_0 is a subfield of k, and q, x are algebraically independent over k_0. We will assume that $r_\rho(x), r_0(x) \in k[x] \setminus \{0\}$, and ρ will be called the *order* of L (we write $\operatorname{ord} L = \rho$). Below we describe briefly some known algorithms for computing all polynomial and rational-function solutions of equations of this form (see [1], [4]; an implementation of some versions of these algorithms is available in the standard package QDifferenceEquations of Maple computer algebra system [23]). We will need these algorithms later when we consider equations with parameters. An observation given in [12] will be also valuable for us.

In addition we discuss some facts related to systems of algebraic equations.

2.1 An algorithm for finding polynomial solutions

We connect with operator (5) the non-negative integer number ω and the polynomial $I(\lambda) \in k[\lambda]$:

$$\omega = \max_{0 \leqslant j \leqslant \rho} \deg r_j(x), \quad I(\lambda) = \sum_{\substack{0 \leqslant j \leqslant \rho \\ \deg r_j(x) = \omega}} \operatorname{lc}(r_j(x))\lambda^j \tag{6}$$

($\operatorname{lc}(\ldots)$ is the leading coefficient of a polynomial belonging to $k[x] \setminus \{0\}$). The algebraic equation $I(\lambda) = 0$ is called the *indicial equation*, and the integer ω is called the *increment* connected with operator L. Set the degree of zero polynomial to be $-\infty$. The following statement demonstrates the role of the indicial equation in the search for polynomial solutions.

Let $\varphi(x)$ be a polynomial solution of the equation $L(y) = f(x)$, $f(x) \in k[x]$. Then $\deg \varphi(x)$ does not exceed

$$l = \max\{\deg f - \omega,\ \tilde\lambda\}, \tag{7}$$

where $\tilde\lambda = \max(\{h \in \mathbb{N} : I(q^h) = 0\} \cup \{-\infty\})$.

The statement is justified by the fact that if $\varphi(x) \in k[x]$, $\deg \varphi(x) = d$, $I(q^d) \neq 0$ then $\deg L(\varphi(x)) = d + \omega$.

There is an algorithm which allows for any algebraic equation with one unknown λ over the field $k = k_0(q)$ to find all roots of the form q^h, $h \in \mathbb{Z}$: since $q^h \neq 0$ for any $h \in \mathbb{Z}$, we can assume that the algebraic equation has the form

$$a_s(q)\lambda^s + \cdots + a_1(q)\lambda + a_0(q) = 0 \tag{8}$$

$a_1(q), a_2(q), \ldots, a_{s-1}(q) \in k_0[q]$, $a_0(q), a_s(q) \in k_0[q] \setminus \{0\}$. If q^h is a root, then $q^h | a_0(q)$ when $h \geqslant 0$, and $q^{-h} | a_s(q)$ when $h < 0$.

The simplest version of the algorithm for finding the general polynomial solution is to find an upper bound for degrees of all possible polynomial solutions and to use the undeterminate coefficients method. A faster algorithm is described in [6].

2.2 An algorithm for finding rational-function solutions

The general principle of the search for rational solutions that we use is as follows: first of all to find a *universal factor* which is a rational function $V(x)$ over k such that if the original equation has a rational solution, then this solution can be written in the form $z(x)V(x)$, where $z(x)$ is a polynomial. Of course, it is possible that $z(x) \not\perp \operatorname{den} V(x)$ (we write $a(x) \not\perp b(x)$, if polynomials $a(x), b(x) \in k[x]$ have a common factor of positive degree). The substitution

$$y(x) = z(x)V(x) \tag{9}$$

into the original equation reduces the problem of finding rational-function solutions to the problem of finding polynomial solutions.

We describe an algorithm for finding a universal factor.

Any polynomial $\varphi(x) \in k[x] \setminus \{0\}$ can be represented in the form $\varphi(x) = x^v b(x)$, where $v \in \mathbb{N}$ and the polynomial $b(x)$ is not divisible by x. If $\varphi(x)$ is the zero polynomial, then set $v = \infty$. We denote v by $\nu(\varphi(x))$ and (as usual) call it the *valuation* of $\varphi(x)$. If $\nu(\varphi(x)) = \nu(\psi(x)) = 0$ for $\varphi(x), \psi(x) \in k[x]$, then we can consider the *q-dispersion set* (finite) of polynomials $\varphi(x)$ and $\psi(x)$:

$$\operatorname{qds}(\varphi(x), \psi(x)) = \{h \in \mathbb{N} : \varphi(x) \not\perp \psi(q^h x)\} \tag{10}$$

and their *q-dispersion*:

$$\operatorname{qdis}(\varphi(x), \psi(x)) = \max(\operatorname{qds}(\varphi(x), \psi(x)) \cup \{-\infty\}). \tag{11}$$

The set $\operatorname{qds}(\varphi(x), \psi(x))$ can be found, e.g., by computing all the roots having the form $\lambda = q^h$, $h \in \mathbb{N}$, of the equation $R(\lambda) = 0$, where $R(\lambda) = \operatorname{Res}_x(\varphi(x), \psi(\lambda x))$. (In [5] an algorithm is proposed which works also in the case when q is an algebraic number which is not a root of unity.)

A universal factor can be found in the form

$$V(x) = x^{l_0} \cdot \frac{1}{U(x)}, \tag{12}$$

where $l_0 \in \mathbb{Z}$, $U(x) \in k[x]$, $\nu(U(x)) = 0$. The polynomial $U(x)$ can be constructed by the following algorithm ([1], [4]):

Set $A(x) = \tilde r_\rho(q^{-\rho}x)$, $B(x) = \tilde r_0(x)$, where $\tilde r_\rho(x) = \frac{r_\rho(x)}{x^{\nu(r_\rho(x))}}, \tilde r_0(x) = \frac{r_0(x)}{x^{\nu(r_0(x))}}$. Compute $H = \operatorname{qds}(A(x), B(x))$. If $H = \varnothing$, then stop the algorithm with the result $U(x) = 1$ (we assume in the rest of this description of the algorithm that $H = \{h_1, h_2, \ldots, h_s\}$, $h_1 > h_2 > \cdots > h_s$, $s \geqslant 1$). Set $U(x) = 1$ and for all h_i, starting from h_1 in the decreasing *order, execute the following assignments:*

$$\begin{array}{rcl} N(x) & = & \gcd(A(x), B(q^{h_i}x)) \\ A(x) & = & A(x)/N(x) \\ B(x) & = & B(x)/N(q^{-h_i}x) \\ U(x) & = & U(x)\prod_{j=0}^{h_i} N(q^{-j}x). \end{array}$$

The final value of $U(x)$ is a polynomial which can be used to construct a universal factor (12).

For finding l_0 we assign to L of the form (5) and to the equation $L(y) = f(x)$, $f(x) \in k[x]$, the increment

$$\omega_0 = \min_{0 \leqslant j \leqslant \rho} \nu(r_j(x)) \tag{13}$$

and the indicial equation $I_0(\lambda) = 0$, where

$$I_0(\lambda) = \sum_{\substack{0 \leqslant j \leqslant \rho \\ \nu(r_j(x)) = \omega_0}} \operatorname{tc}(r_j(x))\lambda^j \tag{14}$$

($\operatorname{tc}(\dots)$ is the trailing coefficient of a polynomial belonging to $k[x] \setminus \{0\}$). We can set

$$l_0 = \min\{\nu(f(x)) - \omega_0,\ \tilde{\lambda}_0\}, \tag{15}$$

where

$$\tilde{\lambda}_0 = \min(\{h \in \mathbb{Z} \,:\, I_0(q^h) = 0\} \cup \{\infty\}). \tag{16}$$

This can be justified by the fact that if $F(x) \in k(x)$, and $\hat{F}(x)$ is a formal Laurent series in x for $F(x)$, $n = \nu(\hat{F}(x))$ is the valuation of this series (i.e., the minimal exponent of x in non-zero terms; for zero series the valuation is ∞) and $I_0(h^n) \neq 0$, then $\nu(L(\hat{F}(x))) = n + \omega_0$. Notice that in the algorithm from [1], [4] the value l_0 is computed in a different way.

Remark 1. *Let $A(x), B(x)$ be as in the algorithm description given above, and $U(x)$ be the result of applying this algorithm. Let $d \geqslant \operatorname{qdis}(A(x), B(x))$. Using the same reasonings as in [12] for the difference case, one can show that $U(x)$ divides the polynomial $\prod_{i=0}^{d} r_\rho(q^{-\rho-i}x)$. This implies that the latter polynomial can be used in (12) instead of $U(x)$.*

Remark 2. *The existence of the roots having the form $q^h, h \in \mathbb{Z}$, of the equation $I_0(\lambda) = 0$ is a necessary condition for the existence of non-zero rational-function (in particular, polynomial) solutions of $L(y) = 0$.*

Another algorithm for finding a universal factor was described in [18] where difference equations were discussed, but it was noted that the proposed approach can be used in the q-difference case as well. However for the purposes of this paper the algorithm described above (especially in the form mentioned in Remark 1) is more suitable.

2.3 Pairs of systems of algebraic equations

Working with parameters we will face systems of algebraic equations (nonlinear in general). A well-known problem is recognizing whether or not a given system with coefficients in a field k_0 has a solution whose components belong to an extension Λ of k_0. We will consider also a more general problem: given a pair (S_1, S_2) of systems of algebraic equations (possibly empty), decide whether there are values of the unknowns belonging to Λ which satisfy all equations in S_1, but – provided that $S_2 \neq \varnothing$ – not all equations in S_2. (If $S_1 = \varnothing$, then by definition any set of values of the unknowns satisfies S_1.) We will refer to this more general problem as *problem $\mathcal{S}_{k_0,\Lambda}$*.

If the problem $\mathcal{S}_{k_0,\Lambda}$ is decidable for given k_0, Λ then we will denote by $A_{k_0,\Lambda}$ an algorithm which solves this problem. The result of applying $A_{k_0,\Lambda}$ to a pair of systems is one of the words "yes", "no".

If $k_0 = \mathbb{Q}$, then $\mathcal{S}_{k_0,\Lambda}$ is decidable for all Λ from the list

$$\mathbb{C},\ \mathbb{R},\ \overline{\mathbb{Q}},\ \mathbb{R} \cap \overline{\mathbb{Q}}\,. \tag{17}$$

Using the Groebner bases technique an algorithm for $\mathbb{C}$ and $\overline{\mathbb{Q}}$ as Λ can be obtained ([14, Sect. 6], [17, Sect. 21.6], [20, Ch. 4], etc.), and using Tarski's theorem one can obtain an algorithm for $\mathbb{R}$ and $\mathbb{R} \cap \overline{\mathbb{Q}}$ as Λ ([16], [20, Sect. 8.6.3]). It is also known for the case $k_0 = \mathbb{Q}$ that a solution with components in $\Lambda = \mathbb{C}$ exists iff there exists a solution with components in $\Lambda = \overline{\mathbb{Q}}$, while a solution with components in $\Lambda = \mathbb{R}$ exists iff there exists a solution with components in $\Lambda = \mathbb{R} \cap \overline{\mathbb{Q}}$.

If for an arbitrary equation in one variable with coefficients in k_0 we can recognize the existence of a root in Λ, then in the case of one variable an algorithm $A_{k_0,\Lambda}$ can be based on the Euclidean algorithm (see Section 4.2.4).

3. BOUNDING INTEGER EXPONENTS OF ROOTS OF ALGEBRAIC EQUATIONS

Proposition 1. *Let there exist at least one non-zero element among $b_0(q), b_1(q), \dots, b_u(q) \in k_0[q]$. Then the inequality*

$$|h| \leqslant \max_{0 \leqslant j \leqslant u} \deg_q b_j(q) \tag{18}$$

is valid for all $h \in \mathbb{Z}$ such that q^h is a root of the equation

$$b_u(q)\lambda^u + \dots + b_1(q)\lambda + b_0(q) = 0. \tag{19}$$

Proof. See the algorithm for finding the roots of the form $q^h, h \in \mathbb{Z}$, in Section 2.1. $\square$

We will show that the computation of the roots $q^h, h \in \mathbb{Z}$, in algorithms of Sections 2.1, 2.2 for finding polynomial and rational-function solutions can be replaced by finding an upper bound of $|h|$. In Section 4 this will be used for q-difference equations with parameters, but in the current section we still consider equation (4) that does not have any parameters. We can clear denominators in coefficients and $f(x)$ (those denominators are polynomials in q), and assume that $r_0(x), r_1(x), \dots, r_\rho(x), f(x) \in k_0[q][x]$ in (4), (5). It will be convenient for us in some situations to consider coefficients and right-hand sides of q-difference equations as polynomials in q and x over k_0. However we will use as a rule the notation $r_0(x), r_1(x), \dots, r_\rho(x), f(x)$ etc, because the variable x is the main one: we produce the q-shift w.r.t. x. (In some cases we will write just $r_0, r_1, \dots, r_\rho, f$.) When we write, e.g., $\operatorname{lc}(f)$, then we have in mind the leading coefficient of f as a polynomial in x, and this leading coefficient is a polynomial in q over k_0; the same goes for the trailing coefficient $\operatorname{tc}(f)$. However we will use $\deg_x$ resp. $\deg_q$ for degrees of polynomials in x resp. q.

Notice that $\operatorname{lc}(r_j)$ in $I(\lambda)$ (see (6)) and $\operatorname{tc}(r_j)$ in $I_0(\lambda)$ (see (14)) are polynomials in q of degree $\leqslant w_q$.

Proposition 2. *Let the coefficients of operator (5) belong to $k_0[q, x]$, and w_q resp. w_x be maximal degrees in q resp. x of all these coefficients. Let $f \in k_0[q, x]$. Then* *(i) the degree of any polynomial solution of $L(y) = f$ does not exceed*

$$\max\{\deg_x f,\ w_q\}, \tag{20}$$

(ii) any rational-function solution of $L(y) = f$ can be represented as the product of a polynomial and the rational function

$$V(x) = \frac{1}{x^w \prod_{i=0}^{d} r_\rho(q^{-\rho-i}x)}, \tag{21}$$

where $w = \max\{w_x,\ w_q\}$, $d = \rho w_x^2 + 2w_x w_q$.

Proof. (i) The value (20) cannot be less than (7).

(ii) Going back to the algorithm for computing $U(x)$ given in Section 2.2, set $A'(x) = q^{\rho w_x} A(x)$. We have $\operatorname{qdis}(A(x), B(x)) = \operatorname{qdis}(A'(x), B(x))$, and $A'(x), B(x)$ can be considered as polynomials in q and x over k_0. Then

$$\begin{array}{ll} \deg_x A' \leqslant w_x, & \deg_x B \leqslant w_x, \\ \deg_q A' \leqslant w_q + \rho w_x, & \deg_q B \leqslant w_q. \end{array}$$

Taking into account the form of the Sylvester matrix of polynomials $A'(x)$, $B(\lambda x)$ and the algorithm for computing the q-dispersion using a resultant, we get

$$\begin{aligned} \deg_q \operatorname{Res}_x(A'(x), B(\lambda x)) &\leqslant \\ &\leqslant \deg_q A' \deg_x B + \deg_q B \deg_x A' \\ &\leqslant (w_q + \rho w_x) w_x + w_q w_x. \end{aligned}$$

This and Proposition 1 imply $\operatorname{qdis}(A(x), B(x)) \leqslant \rho w_x^2 + 2w_x w_q = d$. So (ii) follows from Remark 1 and from inequalities $-\omega_0 \geqslant -w_x$, $\tilde{\lambda}_0 \geqslant -w_q$ (therefore $w \geqslant -l_0$ for l_0, computed by formula (15)). □

4. Q-DIFFERENCE EQUATIONS WITH PARAMETERS, INDEPENDENT OF Q

We will show that the algorithmic problems mentioned in the Introduction, undecidable in the differential and difference cases, are decidable in the q-difference case when parameters are independent of q.

Computation of roots will be replaced by finding some bounds for the exponents h (see Section 3). Of course, using the bounds instead of exact values of the exponents increases performance time of the algorithms. But, first, concerning q-difference equations with parameters, the problem of finding such exact values appears to be unsolvable. Second, we will be interested only in establishing the existence of algorithms. The effectiveness questions will not be considered (the only exception is Section 4.2.4).

4.1 Basic assumptions

Here we formulate some assumptions which will remain valid throughout Section 4.

1. Λ is an extension of the field k_0 of characteristic 0, and q, x are algebraically independent over Λ.
2. The algorithmic problem $\mathcal{S}_{k_0,\Lambda}$ is decidable, i.e., there exists an algorithm $A_{k_0,\Lambda}$ (see Section 2.3).
3. The operator L has the form

$$r_\rho Q^\rho + r_{\rho-1} Q^{\rho-1} + \cdots + r_0, \tag{22}$$

where $r_0, r_1, \ldots, r_\rho \in k_0[q, x, t_1, t_2, \ldots, t_m]$ and $t_1, t_2, \ldots, t_m$ are parameters. The right-hand side f of the equation $L(y) = f$ also belongs to $k_0[q, x, t_1, t_2, \ldots, t_m]$.

4.2 Recognizing existence of polynomial and rational-function solutions in the homogeneous case

Till Section 4.3 we assume that a given q-difference equation with parameters is homogeneous, i.e., of the form $L(y) = 0$.

First we consider the question of existence of $\tau_1, \tau_2, \ldots, \tau_m \in \Lambda$ such that the equation $L(y) = 0$ after substituting $\tau_1, \tau_2, \ldots, \tau_m$ for $t_1, t_2, \ldots, t_m$ becomes an equation with a non-zero solution in $\Lambda[q, x]$ resp. in $\Lambda(q, x)$ (but notice that the unknown function is denoted by $y(x)$, not by $y(q, x)$). We will refer to the two algorithmic problems related to the existence of parameter values such that the corresponding equation has non-zero polynomial resp. rational-function solutions, as *problem* $\mathcal{P}_{k_0,\Lambda}$ resp. *problem* $\mathcal{R}_{k_0,\Lambda}$. We will show in particular that if $k_0 = \mathbb{Q}$, then both problems are decidable when Λ is any field from the list (17).

Any parameter values belonging to Λ such that a given q-difference equation has a non-zero polynomial resp. rational-function solution will be called *adequate*.

Now we introduce a notion which will be useful in the sequel. Let $\varphi \in k_0[q, x, t_1, t_2, \ldots, t_m]$. The system of algebraic equations in $t_1, t_2, \ldots, t_m$, which is produced by representing φ as a polynomial in q, x with coefficients in $k_0[t_1, t_2, \ldots, t_m]$ and equating each of these coefficients to 0, will be called the 0-*system* corresponding to the polynomial φ.

4.2.1 Decidability of $\mathcal{P}_{k_0,\Lambda}$

We can check whether or not there exist in Λ values of parameters that annihilate all the coefficients of the original equation (with an operator L of the form (22)). To do this we construct the system S' of all equations of 0-systems corresponding to coefficients r_i, $i = 0, 1, \ldots, \rho$, of the operator L, and apply $A_{k_0,\Lambda}$ to $(S', \varnothing)$. If the result of this applying is "yes" then the original q-difference equation with such values of parameters turns into $0 = 0$. Any polynomial is a solution of this equation.

If such values of parameters do not exist, then by Proposition 2(i) the value $l = w_q$ can be used as an upper bound on the degree of any polynomial solution. Of course, for different values of parameters we will get after their substitution into (22) different operators with different values w_q. But none of these w_q's exceeds the value that is found for (22).

The method of undetermined coefficients can be used. Let $y_0, y_1, \ldots, y_l$ be the undetermined coefficients. We get a system S of linear homogeneous algebraic equations for $y_0, y_1, \ldots, y_l$ with coefficients from $k_0[q, t_1, t_2, \ldots, t_m]$, and it is sufficient to recognize whether or not exist in Λ such values of $t_1, t_2, \ldots, t_m$ that the system which is obtained as a result of substituting these values into S, has a non-zero solution with components in $\Lambda(q)$.

We obtain the following algorithm.

Construct S' of all equations of 0-systems corresponding to the coefficients r_i, $i = 0, 1, \ldots, \rho$, of operator L, and apply $A_{k_0,\Lambda}$ to $(S', \varnothing)$; if the result is "yes", then stop the algorithm with the answer "yes" (we will assume in the rest of the description of this algorithm that such values do not exist). Set $l = w_q$. Construct the system of linear algebraic equations for coefficients $y_0, y_1, \ldots, y_l$ of an arbitrary polynomial solution of $L(y) = 0$. Let T be the matrix of this linear system (the elements of T belong to $k_0[q, t_1, t_2, \ldots, t_m]$). Construct the system of algebraic equations, gathering together equations of the 0-systems of all the minors of order $l + 1$ of T, *and apply $A_{k_0,\Lambda}$ to*

$$(S, \varnothing), \tag{23}$$

where S is the constructed system.

So the problem $\mathcal{P}_{k_0,\Lambda}$ is decidable.

REMARK 3. *In contrast to the q-difference case, in the differential and difference cases no independent of the values of parameters upper bound for the degree of polynomial solutions exists in general. For example the differential equation $xy' - ty = 0$ with one parameter t has the polynomial solution x^t of degree t when $t \in \mathbb{N}$. Similarly the difference equation $xy(x+1) - (x+t)y(x) = 0$ has the polynomial solution $x(x+1)\dots(x+t-1)$ of degree t when $t \in \mathbb{N}$.*

4.2.2 Additional constraints

If originally an algebraic system S_1 for $t_1, t_2, \dots, t_m$ is given, then the existence of parameter values which satisfy S_1 and for which the equation $L(y) = 0$ has a non-zero polynomial solution, can be recognized by the above algorithm, provided that we use $S_1 \cup S$ instead of S in (23).

If we investigate the existence of the values of parameters which do not satisfy a non-empty system S_2 and for which the equation $L(y) = 0$ has a non-zero polynomial solution, then we use S_2 instead of $\varnothing$ in (23).

It is also possible to consider two additional systems, the first of which has to be satisfied, while the second one must not be satisfied (if it is not empty).

4.2.3 Decidability of $\mathcal{R}_{k_0,\Lambda}$

Now consider the problem $\mathcal{R}_{k_0,\Lambda}$. For a given equation $L(y) = 0$ with parameters we can use formula (21) to find $V \in \Lambda(q, x, t_1, t_2, \dots, t_m)$ (since our q-difference equation is homogeneous, we can take $w = w_q$). Then we substitute $y = zV$ into $L(y) = 0$, clear denominators and decide whether or not a non-zero polynomial solution of the resulting equation exists.

Note that the corresponding values of parameters should not annihilate the polynomial r_ρ, which is included in the denominator of (21) (but it is easy to show that there is no trouble with the case when r_0 is annihilated). We apply the algorithm from Section 4.2.2, using the system S_2, which is the 0-system corresponding to r_ρ. If such values of parameters do not exist then set $\widetilde{L} = L - r_\rho Q^\rho$, the adequate values have to satisfy the 0-system corresponding to the polynomial r_ρ, and so on.

Now we can give a description of the full algorithm. The algorithm is applicable to an equation $L(y) = 0$ and a system S_1 of algebraic equations, which has to be satisfied by the adequate values of parameters. Even if initially S_1 contains no equations ($S_1 = \varnothing$), and is satisfied by any values of parameters, then non-empty systems S_1 may appear due to recursive calls in this algorithm.

If $L = 0$ then apply $A_{k_0,\Lambda}$ to $(S_1, \varnothing)$ and stop algorithm with the obtained answer (in the rest of the description of this algorithm we will assume that $L \neq 0$). Construct the 0-system S_2 corresponding to the polynomial r_ρ. Find V by formula (21), substitute $y = zV$ into $L(y) = 0$, clearing denominators; this gives an equation $L'(z) = 0$. By the algorithm from Sections 4.2.1, 4.2.2 recognize the existence of parameter values which satisfy S_1 but not S_2 (if the latter system is not empty) and such that the equation $L'(z) = 0$ has a non-zero polynomial solution. Stop the algorithm with the answer "yes" if such values exist, otherwise apply the algorithm recursively to $\widetilde{L}(y) = 0$, $\widetilde{S}_1$, where $\widetilde{L} = L - r_\rho Q^\rho$ and $\widetilde{S}_1 = S_1 \cup S_2$.

So the problem $\mathcal{R}_{k_0,\Lambda}$ is decidable.

4.2.4 The case of a single parameter

Let there be only one parameter, denoted by t. In this case any non-empty algebraic system is equivalent to a single equation $s(t) = 0$, which can be constructed by the Euclidean algorithm. If $s(t)$ is a non-zero polynomial, then we can assume that it is square-free (otherwise we take the quotient of $s(t)$ and $\gcd(s(t), s'(t))$, where $s'(t)$ is the derivative of the polynomial $s(t)$). If both systems in the original pair are non-empty, then we obtain the pair

$$(s_1(t) = 0, \;\; s_2(t) = 0), \tag{24}$$

where each of polynomials $s_1(t)$, $s_2(t)$ is either zero or square-free. In this case

- if $s_2(t)$ is the zero polynomial, then (24) has no solution in Λ,
- if $s_2(t) \in k_0[x] \setminus \{0\}$, but $s_1(t)$ is the zero polynomial, then the set of all solutions of (24) belonging to Λ is the set $\{\lambda \in \Lambda;\; s_2(\lambda) \neq 0\}$,
- if $s_1(t), s_2(t) \in k_0[x] \setminus \{0\}$, then the set of all solutions of (24) belonging to Λ is the set $\{\lambda \in \Lambda;\; s(\lambda) = 0\}$ where $s(t) = s_1(t)/\gcd(s_1(t), s_2(t))$.

Therefore the set of adequate values of the parameter has the form U or $\Lambda \setminus U$, where U is the set of those roots of a concrete polynomial over k_0 which belong to Λ. It easy to see that if M_1, M_2 are sets of this form then the sets $M_1 \cup M_2$, $M_1 \cap M_2$, and $\Lambda \setminus M_1$ are of the same form.

This implies that, e.g., in the case when $m = 1$ and $k_0 = \Lambda = \mathbb{Q}$ we are able to obtain all the desired solutions of the original pair (notice that we did not include $\mathbb{Q}$ in the list (17); we will discuss more about this in Section 4.2.6).

The algorithms in Sections 4.2.1, 4.2.2, 4.2.3 are designed in such a way that if at some point it is detected that adequate parameter values exist, then the algorithms stop. For $m = 1$ these algorithms can be easily modified so that the set of all the adequate values can be presented in simple form.

4.2.5 The main statement for the case of parameters, independent of q

The reasoning given above proves the following theorem.

THEOREM 1. *Let the assumptions 1 – 3, formulated in Section 4.1, be valid. Then*

(i) the question whether or not there exist adequate parameter values can be answered algorithmically, and therefore the problems $\mathcal{P}_{k_0,\Lambda}$, $\mathcal{R}_{k_0,\Lambda}$ are decidable;

(ii) in the case of a single parameter the set of adequate parameter values has the form U or $\Lambda \setminus U$, where U is the set of those roots of a polynomial $h(t) \in k_0[t]$ which belong to Λ. The polynomial $h(t)$ can be constructed algorithmically (it can be the zero polynomial, in this case $\Lambda \setminus U = \varnothing$). This polynomial is independent of Λ.

Recall that algorithms solving the problem $\mathcal{S}_{k_0,\Lambda}$ for the fields Λ from the list (17) are known for $k_0 = \mathbb{Q}$.

4.2.6 The case $k_0 = \Lambda = \mathbb{Q}$

Let $k_0 = \Lambda = \mathbb{Q}$. It is not clear whether the problem of existence of $\tau_1, \tau_2, \dots, \tau_m \in \Lambda$ such that after substituting the values $\tau_1, \tau_2, \dots, \tau_m$ for $t_1, t_2, \dots, t_m$ in $L(y) = 0$ the resulting equation has a non-zero polynomial (rational-function) solution, is decidable. Let us show that if it is decidable then the problem of the existence of a solution with components belonging to $\mathbb{Q}$ of a given algebraic equation with integer coefficients is decidable too (the question of the decidability of the later problem is still open, the common opinion of experts is that this problem is undecidable, see, e.g., [22]). Indeed, let $P(t_1, t_2, \dots, t_m)$ be an arbitrary polynomial with integral coefficients. Then for any values $\tau_1, \tau_2, \dots, \tau_m \in \mathbb{Q}$, the indicial equation $I_0(\lambda) = 0$ (see Remark 2) of the q-difference equation

$$y(qx) - (1 + P(\tau_1, \tau_2, \dots, \tau_m))y(x) = 0 \qquad (25)$$

is $\lambda - 1 - P(\tau_1, \tau_2, \dots, \tau_m) = 0$. This indicial equation has a root of the form q^h, $h \in \mathbb{Z}$, only if $P(\tau_1, \tau_2, \dots, \tau_m) = 0$. Then $h = 0$, and the q-difference equation (25) is satisfied, e.g., by the polynomial $y(x) = 1$.

4.2.7 On possible values of q

If q is an additional variable besides x, then q is transcendental over any of the fields (17). When $k_0 = \mathbb{Q}$ the previous results are valid also if q is a transcendental number (i.e., $q \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ or, in the real case, $q \in \mathbb{R} \setminus \overline{\mathbb{Q}}$), and Λ is one of $\overline{\mathbb{Q}}$, $\mathbb{R} \cap \overline{\mathbb{Q}}$.

4.3 Inhomogeneous equations

4.3.1 Polynomial right-hand sides

In the Introduction we listed some concrete undecidable problems, connected with differential and difference linear homogeneous equations with numerical parameters. We described above algorithms for solving those problems in the case of q-difference equations. Similar algorithms can be applied in the case of linear *inhomogeneous* q-difference equations, when the right hand side f is a polynomial in x with coefficients in $k_0[q, t_1, t_2, \dots, t_m]$. It follows from (7) that we can use $\max\{\deg_x f,\ w_q\}$ as an upper bound for degrees of polynomial solutions. For constructing rational-function solutions we can use the algorithm from Section 4.2.3 using the same bounding rule for polynomial solutions.

Checking the existence of polynomial solutions, we obtain an inhomogeneous system of linear algebraic equations whose matrix T and right-hand sides consist of elements of $k_0[q, t_1, t_2, \dots, t_m]$. By means of algorithms considered above we can recognize whether or not there exist parameter values annihilating the right-hand side of this system such that the corresponding homogeneous system has a non-zero solution. The condition (on parameters values) that the right-hand side of the system is not annihilated we call the *inhomogeneity condition*. Suppose that the inhomogeneity condition is satisfied. Using, e.g., step-by-step consideration of minors and Kronecker-Capelli's theorem, we can recognize whether there exist parameter values for which the system is compatible (there exists a non-zero minor of some order n of the matrix T while any minor of order $n + 1$ of the augmented matrix $\bar{T}$ are equal to zero). This analysis can be done by the algorithm $A_{k_0,\Lambda}$. In the case of a single parameter the set of adequate parameter values can be presented as in Section 4.2.4.

4.3.2 Parametric summation

If k_0, Λ are such that the problem $\mathcal{R}_{k_0,\Lambda}$ is decidable in the inhomogeneous case then, e.g., the parametric problem of q-hypergeometric summation is decidable also, and in the q-difference case it is possible to consider a parametric version of Gosper's algorithm, since in this algorithm one can use the universal factors instead of the special Gosper form of rational functions representation. (Parametric versions of algorithms that are based on Gosper's algorithm [21] probably exist, too; see, e.g., [9, Sect. 3].) It is also possible to propose q-difference version of the accurate integration (summation) algorithm [7, 8]. In the one-parametric case we not only can *recognize the existence* of adequate values of parameters, but can also *find* them. However one should not forget that the algorithms discussed above have high complexity. As mentioned, the aim of this paper is only to establish decidability of some algorithmic problems "in principle".

5. WHEN PARAMETERS DEPEND ON Q

Let the assumptions 1 and 3, formulated in Section 4.1, be valid. We will consider algorithmic problems similar to $\mathcal{P}_{k_0,\Lambda}$ and $\mathcal{R}_{k_0,\Lambda}$ (the homogeneous case) investigated above, allowing parameter values belong to $\Lambda[q]$ or $\Lambda(q)$. From this point on we will consider the problems

$$\mathcal{P}_{k_0,\Lambda[q]}, \quad \mathcal{R}_{k_0,\Lambda[q]} \qquad (26)$$

and

$$\mathcal{P}_{k_0,\Lambda(q)}, \quad \mathcal{R}_{k_0,\Lambda(q)}. \qquad (27)$$

In (26) parameter values belong to the ring $\Lambda[q]$, in (27) they belong to the field $\Lambda(q)$.

5.1 Two theorems of J. Denef

In our investigation of problems (26), (27) the key role will be played by two theorems of Denef [15]. Before formulating them we introduce two notions following [15]: Let R be a commutative ring with unity and let R' be a subring of R. We say that the *diophantine problem for R with coefficients in R'* is undecidable (decidable) if there exists no (an) algorithm to decide whether or not a polynomial equation (in several variables) with coefficients in R' has a solution in R.

The following results are proved in [15]:

THEOREM A. *Let R be an integral domain of characteristic zero; then the diophantine problem for $R[T]$ with coefficients in $\mathbb{Z}[T]$ is undecidable. ($R[T]$ denotes the ring of polynomials over R, in one variable T.)*

THEOREM B. *Let K be a formally real field, i.e., -1 is not the sum of squares in K. Then the diophantine problem for $K(T)$ with coefficients in $\mathbb{Z}[T]$ is undecidable. ($K(T)$ denotes the field of rational functions over K, in one variable T.)*

As a consequence of Theorems A, B we obtain the following:

The diophantine problem for $\Lambda[q]$ with coefficients in $\mathbb{Z}[q]$ is undecidable. If the field Λ is formally real, then the diophantine problem for $\Lambda(q)$ with coefficients in $\mathbb{Z}[q]$ is also undecidable.

5.2 Undecidability in the case of parameters depending on q

Now we engage in problems (26), (27) in earnest.

LEMMA 1. *Let $P(t_1, t_2, \ldots, t_m)$ be an arbitrary polynomial with coefficients in $\Lambda[q]$ (in particular, in $\mathbb{Z}[q]$). Then the equation*

$$y(qx) - (1 + P^2(t_1, t_2, \ldots, t_m))y(x) = 0 \qquad (28)$$

with some rational functions (in particular, polynomials) $t_1 = \tau_1(q),\ t_2 = \tau_2(q),\ \ldots,\ t_m = \tau_m(q)$ over Λ has a non-zero solution y in $\Lambda(q)(x)$ iff $P(\tau_1(q), \tau_2(q), \ldots, \tau_m(q)) = 0$.

Proof. Since q is transcendental over $\Lambda(x)$, q can be considered as a variable. If $P(\tau_1(q), \tau_2(q), \ldots, \tau_m(q)) \in \Lambda(q) \setminus \{0\}$ has the form of a fraction $\frac{f(q)}{g(q)}$ with relatively prime polynomials $f(q), g(q)$ over Λ, then

$$I_0(\lambda) = \lambda - 1 - \frac{f^2(q)}{g^2(q)}$$

in the corresponding indicial equation. But the equation $I_0(\lambda) = 0$ has no roots of the form q^h, $h \in \mathbb{Z}$ (see Remark 2). Indeed, $h \neq 0$, because otherwise $\frac{f(q)}{g(q)}$ is the zero rational function. If $h > 0$, then we would have the equality $\left(q^h - 1\right) g^2(q) = f^2(q)$ in $\Lambda[q]$. However the irreducible factor $q - 1$ appears in the left-hand side with an odd exponent, while in the right-hand side it appears with an even exponent – a contradiction. If $h < 0$ then for $h_0 = -h$ we have $-\left(q^{h_0} - 1\right) g^2(q) = f^2(q)q^{h_0}$. This is impossible for the same reasons.

If $P(\tau_1(q), \tau_2(q), \ldots, \tau_m(q)) = 0$, then the equation (28) has, e.g., the solution that is identically equal to 1. □

THEOREM 2. *The problems (26) are undecidable. In addition, if Λ is a formally real field then the problems (27) are undecidable as well.*

Proof. By the consequence of Theorems A, B formulated in Section 5.1, and by Lemma 1. □

Let $k_0 = \mathbb{Q}$. If q is a variable, $\Lambda \in \{\mathbb{C},\ \mathbb{R},\ \mathbb{Q},\ \overline{\mathbb{Q}},\ \mathbb{R} \cap \overline{\mathbb{Q}}\}$ then the problems (26) are undecidable. The same is true if q is a transcendental number and $\Lambda \in \{\mathbb{Q},\ \overline{\mathbb{Q}},\ \mathbb{R} \cap \overline{\mathbb{Q}}\}$. In turn, the problems (27) are undecidable if, e.g., q is a variable and $\Lambda \in \{\mathbb{R},\ \mathbb{Q},\ \mathbb{R} \cap \overline{\mathbb{Q}}\}$, or if q is a transcendental number and $\Lambda \in \{\mathbb{Q},\ \mathbb{R} \cap \overline{\mathbb{Q}}\}$.

5.3 The case $\Lambda = \mathbb{C}$

It is not clear whether or not the problems (27) are decidable when, e.g., $k_0 = \mathbb{Q}$, $\Lambda = \mathbb{C}$ (q is a variable). However it follows from Lemma 1 that if at least one of them is decidable then the diophantine problem for $\mathbb{C}(q)$ with coefficients from $\mathbb{Z}[q]$ is decidable as well. Notice that the latter problem is still open, but the common opinion of experts is such that it is undecidable — we again refer to the survey [22].

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