Formal Series and Linear Difference Equations

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A doubly infinite complex number sequence

$$(c_k), \quad k \in \mathbb{Z}, \tag{1}$$

will be called a *sequential* solution of an equation of the form

$$a_d(z)y(z+d) + \cdots + a_1(z)y(z+1) + a_0(z)y(z) = 0, \tag{2}$$

$a_1(z), a_2(z), \ldots, a_{d-1}(z) \in \mathbf{C}[z]$, $a_0(z), a_d(z) \in \mathbb{C}[z] \setminus \{0\}$, if

$$a_d(k)c_{k+d} + \cdots + a_1(k)c_{k+1} + a_0(k)c_k = 0$$

for all $k \in \mathbb{Z}$. A sequential solution (1) will be called *subanalytic* sequential (or just subanalytic, for short), if the equation (2) has a solution in the form of a single-valued analytic function $f(z)$ such that $c_k = f(k)$ for all $k \in \mathbb{Z}$.

We discuss a way to compute the values of elements of a subanalytic solution of equation (2) at arbitrary integer points, in particular at the points where the polynomial $a_d(z-d)a_0(z)$ vanishes.

We show that the dimension of the $\mathbb{C}$-linear space of all sequential solutions of (2) is $\geq d$, and for any integer $m \geq d$ there exists an equation of the form (2) such that this dimension is m. The dimension of the space of all subanalytic solutions is positive, but, generally speaking, not all of sequential solutions are subanalytic.

We also show that the application of the discrete Newton-Leibniz formula combined with known computer algebra algorithms for indefinite summation of elements of sequential solutions always gives the correct result in the case when a solution under consideration is subanalytic. However the result can be incorrect for a sequential solution that is not subanalytic. If a meromorphic solution $f(z)$ of equation (2) has some integer poles, then nevertheless it is possible to map $f(z)$ into a sequential solution (the *bottom* of $f(z)$) of (2) such that the summation algorithms mentioned above work correctly for it. The main idea of this mapping can be easily demonstrated for the simplest particular case, when all integer poles are of first order. In this case the bottom of $f(z)$ is the sequence (c_k) such that c_k is equal to zero if k is an ordinary point of $f(z)$, and c_k is equal to the residue of $f(z)$ at k otherwise. It can be proved that the bottom of a meromorphic solution of (2) is a subanalytic sequential solution of (2).

We also consider multivariate hypergeometric sequences. Let $n_1, n_2, \ldots, n_d$ be variables ranging over the integers. d-dimensional *H-systems* are systems of equations for a single unknown sequence

$$(T_{n_1,n_2,\ldots,n_d}), \quad (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d,$$

which have the form

$$f_i(n_1, n_2, \ldots, n_d)T_{n_1,n_2,\ldots,n_i+1,\ldots,n_d} = \tag{3}$$

$$= g_i(n_1, n_2, \ldots, n_d)T_{n_1,n_2,\ldots,n_i,\ldots,n_d},$$

where f_i, g_i are relatively prime non-zero polynomials over $\mathbb{C}$ for $i = 1, 2, \ldots, d$. (The prefix "H" refers to Jakob Horn and to the adjective "hypergeometric" as well.)

A *d-dimensional hypergeometric sequence* is a solution of some d-dimensional H-system when this solution is defined for all $(n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$.

Rational functions $F_1, F_2, \ldots, F_d \in \mathbb{C}(n_1, n_2, \ldots, n_d)$ are *compatible*, if

$$F_i(n_1, n_2, \ldots, n_j + 1, \ldots, n_d)F_j(n_1, n_2, \ldots, n_j, \ldots, n_d) =$$

$$= F_j(n_1, n_2, \ldots, n_i + 1, \ldots, n_d)F_i(n_1, n_2, \ldots, n_i, \ldots, n_d)$$

for all $1 \leq i \leq j \leq d$.

The H-system (3) is *consistent*, if the rational functions

$$F_i = \frac{g_i(n_1, n_2, \ldots, n_d)}{f_i(n_1, n_2, \ldots, n_d)}, \quad i = 1, 2, \ldots, d,$$

are compatible.

The $\mathbb{C}$-linear space of hypergeometric sequences that satisfy a given H-system $\mathcal{H}$ we will denote as $V(\mathcal{H})$.

We prove that

1) dim $V(\mathcal{H}) > 0$ for any consistent H-system $\mathcal{H}$;

2) if m, d are arbitrary natural numbers, then there exists a d-dimensional H-system $\mathcal{H}$ such that $\dim V(\mathcal{H}) = m$;

3) if $d = 1$ for an H-system $\mathcal{H}$, then dim $V(S) < \infty$, but for an arbitrary integer $d > 1$ there exists a d-dimensional H-system $\mathcal{H}$ such that dim $V(\mathcal{H}) = \infty$.

We also give an appropriate corollary of the well-known Ore-Sato theorem on possible forms of d-dimensional hypergeometric sequences. Notice that, contrary to some interpretations found in the literature, the Ore-Sato theorem does *not* imply that every solution of an H-system is of the form

$$R(n_1, n_2, \dots, n_d) \frac{\prod_{i=1}^{p} \Gamma(a_{i,1} n_1 + \cdots + a_{i,d} n_d + \alpha_i)}{\prod_{j=1}^{q} \Gamma(b_{j,1} n_1 + \cdots + b_{j,d} n_d + \beta_j)} u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d},$$

where $R \in \mathbb{C}(n_1, n_2, \dots, n_d)$, $a_{ik}, b_{jk} \in \mathbb{Z}$, and $\alpha_i, \beta_j, u_k \in \mathbb{C}$.

In addition we will discuss some problems connected with searching for power series solutions of ordinary linear differential equations with polynomial coefficients, having in mind the power series whose coefficients can be expressed in closed form as functions of index. A key role in the process of constructing such power series solutions plays an investigation of the family of difference equations, which are satisfied by coefficient sequences of power series solutions and which depend significantly on the expansion point.

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