On Incomplete Rank Matrices

S.A. Abramov[1], M. Petkovšek[2,*], A.A. Ryabenko[1]

[1] *Federal Research Center "Computer Science and Control" of RAS, Russia*

[2] *University of Ljubljana, Faculty of Mathematics and Physics, Slovenia*

e-mail: sergeyabramov@mail.ru, anna.ryabenko@gmail.com

Abstract

Consider a matrix A of size $m \times n$ over a field K with $r = \operatorname{rank} A$ and $d = \min\{m, n\} - r > 0$, which implies that the rank of A is not full. We demonstrate that in such cases, it is possible to choose d elements from A such that, upon replacement of their values with other values from K, yield a matrix $\tilde{A}$ of full rank (when $m = n$, $\tilde{A}$ is nonsingular). We discuss as well the implications of this result for matrices with truncated formal series as their elements.

Keywords: matrices over fields, full and incomplete rank matrices, formal power series, formal Laurent series, truncated series

1. Introduction

Matrices are used in all areas of mathematics. The rank serves as an essential characteristic of a matrix. If K is a field and $m \times n$-matrix A over K (i.e. $A \in K^{m\times n}$), $r = \operatorname{rank} A$ then the situation of incomplete rank is possible, i.e. the situation in which $d = \min\{m, n\} - r > 0$. This is an obstacle to carrying out some transformations of the matrix A and performing calculations related to A. The case of matrices with elements in the form of truncated series is considered especially. The series themselves and matrices, whose elements are series, can be given in a truncated form, when instead of each infinite series one of its initial segments is specified. This can be viewed as an approximation of the data, or more generally, as incomplete information about the original data.

2. Rank regulation

To prove Lemma 1 below, the notion of a basic minor of a matrix A will be important. In [1] this notion is defined as follows: "The determinant of a submatrix C of order k is a basic minor if and only if it is nonzero and all submatrices of order $k + 1$ which contain C have zero determinant. The system of rows (columns) of a basic minor form a maximal linearly independent subsystem of the system of all rows (columns) of the matrix." (See also [2].)

Lemma 1. *Let K be a field, let m, n be positive integers, and let $A \in K^{m\times n}$ be a matrix with rank $A < \min\{m, n\}$. Then*

(i) replacing any one element of the matrix A by some other element belonging to K cannot increase $\operatorname{rank} A$ *by more than* 1;

(ii) the matrix A contains at least one element such that its replacement by any element belonging to the field K that is not equal to it increases $\operatorname{rank} A$ *by 1.*

Proof. (i) Assertion (i) is almost trivial. Nevertheless, we give for completeness its proof:

Let $A = [a_{ij}]$ and assume that replacement of some a_{ij} by $\tilde{a}_{ij} \in K$ increases the rank of A by $\rho > 1$. Denote by $\tilde{A}$ the matrix resulting from this substitution, and choose some

*Our friend and co-author Marko Petkovšek passed away on March 24, 2023 (*S.Abramov, A.Ryabenko).*

basic minor of $\tilde{A}$ includes the element $\tilde{a}_{ij}$. The Laplace expansion of this minor along the row containing $\tilde{a}_{ij}$, is a sum of products, one of the factors in each of which is equal to zero (up to the sign, that factor is equal to the determinant of a submatrix of A of order $\operatorname{rank} A + \rho - 1 > \operatorname{rank} A$). Hence for $\rho > 1$, the minor under consideration cannot be a basic one; thus $\rho \leq 1$.

(ii) Let B be a basic minor of the matrix A. Since $\operatorname{rank} A < n$, the matrix A contains a row and a column that are not related to the minor B; let them be the i-th row and the j-th column of A. In A, replace the element a_{ij} with some $\tilde{a}_{ij} \neq a_{ij}$, and append the i-th row and the j-th column of the modified matrix $\tilde{A}$ to the set of rows and columns related to the minor B. The modified minor $\tilde{B}$ is non-zero: up to a sign, its determinant equals

$$(\tilde{a}_{ij} - a_{ij}) \det B,$$

which follows from the Laplace expansion of $\det \tilde{B}$ along the row containing $\tilde{a}_{ij}$. Thus this minor, whose order equals $\operatorname{rank} A + 1$, is nonzero. By virtue of (i), this minor is basic for $\tilde{A}$. □

A matrix element, whose replacement increases the rank of the matrix, will be called a *rank-regulating element.*

The following example shows that not every element is rank-regulating.

Example 1.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

It can be seen that, for example, the element a_{33} does not affect the rank, unlike, say, a_{13}.

Proposition 1. *Let K be a field, $n > 1$ and $A \in K^{n \times n}$. Let $\operatorname{rank} A = r < n$. Then in the matrix A one can choose $n - r$ such elements that their replacement by any elements of the field K unequal to them will give a full rank matrix.*

Proof. Consider some basic minor B of the matrix A and write out the numbers $i_1, \ldots, i_{n-r}$ of the rows and the numbers $j_1, \ldots, j_{n-r}$ of the columns not related to the minor B. By using Lemma 1(ii) repeatedly, we see that the elements in the list

$$(a_{i_1, j_1}, \ \ldots \ , a_{i_{n-r}, j_{n-r}}) \tag{1}$$

have the desired property. □

Of course, for $n - r > 1$ such a list will not be unique. For example, for any mapping φ of the set $\{1, \ldots, n-r\}$ onto itself, the elements of $(a_{i_1, j_{\varphi(1)}}, \ \ldots \ , a_{i_{n-r}, j_{\varphi(n-r)}})$ have the desired property, too.

Example 2. Consider the following 5×5-matrix over the field of rational numbers:

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1 & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5 & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}. \tag{2}$$

Its rank is 3, and a basic minor can be obtained, for example, by selecting rows and columns with numbers 1, 3, 5. It is not hard to see that a_{22} and a_{44} are rank-regulating elements of

A. Replacing them by zeros, we obtain a matrix $\tilde{A}$ with $\det \tilde{A} = 55$; if instead we add 1 to each of the initial a_{22}, a_{44}, then for the resulting matrix $\tilde{\tilde{A}}$ we have $\det \tilde{\tilde{A}} = -11$ (obviously, $\operatorname{rank} \tilde{A} = \operatorname{rank} \tilde{\tilde{A}} = 5$).

Example 3. Consider a non-square matrix. Let A be the following 6×4-matrix over the field of rational numbers:

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 4 & 5 & 6 & 3 \\ 1 & -2 & 0 & 0 \\ 1 & 1 & 4 & 1 \end{bmatrix}. \tag{3}$$

Its rank is 3, and a basic minor can be obtained, for example, by selecting rows and columns with numbers 1, 2, 3.

Thus any of $a_{4,4}, a_{5,4}, a_{6,4}$ is a rank-regulating element of A. For example, replacing $a_{5,4}$ by 1 we obtain $\tilde{A}$ of full rank: $\operatorname{rank} \tilde{A} = 4 = \min\{6, 4\}$. A basic minor of $\tilde{A}$ can be obtained, for example, by selecting rows 1, 2, 3, 5 and all the matrix columns.

3. Matrices over truncated formal series

In this section, we consider the field K as the formal Laurent series field $F((x))$ over a field F. The field $F((x))$ is the quotient field of the formal power series ring $F[[x]]$. Let the elements of the matrix A be polynomials, which are considered as truncated power series. If $\det A = 0$ then A has obviously a prolongation which is a singular matrix belonging to $F[[x]]^{n\times n}$: such a prolongation can be obtained by adding to each element of A an infinite sequence of zero terms. On the other hand, using the recipe from Lemma 1 and Proposition 1, we can construct a prolongation which gives a nonsingular matrix $\tilde{A}$. To do this, we can, for example, add to each of the rank-regulating elements some terms that have degrees higher (say, by 1) than the degrees of the elements of the matrix A.

Thus, the following proposition is valid:

Proposition 2. *For an incomplete rank polynomial matrix $A = [a_{ij}] \in F[x]^{m\times n}$, there exists and can be constructed a polynomial full rank matrix $\tilde{A} = [\tilde{a}_{ij}] \in F[x]^{m\times n}$ which is a prolongation of A; wherein, if $a_{ij} = 0$ then $\tilde{a}_{ij} = 0$ or $\deg \tilde{a}_{ij} = 0$, otherwise $\deg \tilde{a}_{ij} \le \deg a_{ij} + 1$, $i = 1, \dots, m$, $j = 1, \dots, n$.*

Example 4. A simple example is given by the following polynomial matrix over the field of rational numbers:

$$A = \begin{bmatrix} 1 & x \\ x & x^2 \end{bmatrix}.$$

Its rank is 1, and all its first-order minors are basic. Thus, the prolongation of any one of its elements by a non-zero term of degree 1 for a_{11}, of degree 2 for a_{12} or a_{21}, and of degree 3 for a_{22} results in a matrix of rank 2. Take, for example, the element a_{12} and add $-x^2$ to it. This gives

$$\tilde{A} = \begin{bmatrix} 1 & x - x^2 \\ x & x^2 \end{bmatrix}$$

with $\det \tilde{A} = x^3$.

Example 5. Consider the matrix (2) as truncated. Adding x to a_{22} and $-2x$ to a_{44} we get

$$\tilde{A} = \begin{bmatrix} 1 & -1 & 2 & 3 & 4 \\ 2 & 1+x & -1 & 2 & 0 \\ -1 & 2 & 1 & 1 & 3 \\ 1 & 5 & -8 & -5-2x & -12 \\ 3 & -7 & 8 & 9 & 13 \end{bmatrix}$$

with $\det \tilde{A} = 22x^2$, $\operatorname{rank} \tilde{A} = 5$.

Acknowledgements

The authors are grateful to A. Aparicio Monforte for useful comments.

References

[1] Encyclopedia of Mathematics. Retrieved 10.04.2023 from ULR: https://encyclopediaofmath.org/wiki/Minor

[2] *Shilov G.E.* Linear Algebra, revised English edition. Translated from the Russian and edited by Richard A. Silverman. Dover Publications, Inc., New York. (1977).