On Ramification Indices of Formal Solutions of Constructive Linear Ordinary Differential Systems

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Abstract

We consider full rank linear ordinary differential higher-order systems whose coefficients are computable power series. It is shown that the algorithmic problems connected with the ramification indices of irregular formal solutions of a given system are mostly undecidable even if we fix a conjectural value r of the ramification index. This enables us to obtain a strengthening of the theorem which has been proven earlier and states that we are not able to compute algorithmically the dimension of the space of all formal solutions although we can construct a basis for the subspace of regular solutions. In fact, it is impossible to compute algorithmically this dimension even if, in addition to the system, we know the list of all values of the ramification indices. However, there is nearby an algorithmically decidable problem: if a system S and integers r, d are such that for S the existence of d linearly independent formal solutions of ramification index r is guaranteed then one can compute such d solutions of S.

Keywords: system of linear differential equations, formal solution, ramification index, undecidable algorithmic problem

1. Introduction

Some of the problems that can be considered as related to computer algebra are algorithmically undecidable. In Abramov & Barkatou (2014), it was in particular shown that testing the existence of an irregular formal solution for a given higher-order full rank linear ordinary differential system S having computable power series coefficients is algorithmically undecidable. This follows from the fact that it is possible to construct algorithmically the subspace of all regular solutions of a given system Abramov & Khmelnov (2014), and as a consequence, to compute the dimension of that subspace, but it is impossible to compute algorithmically the dimension of the space of all solutions Abramov & Barkatou (2014), Prop. 4. In the present paper, the attention is concentrated

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on irregular solutions. An irregular formal solution involves Puiseux series generated by a factor of the form e^Q where Q is a polynomial in $1/x^r$, $r \in \mathbb{Z}_{>0}$. The ramification of such series is the ramification index of the irregular formal solution. In accordance to Abramov & Barkatou (2014), we are not able to construct algorithmically the list of all values of the ramification indices related to a given system S and even are not able to test whether that list is empty. We show in Section 2 that even when a conjectural value r is fixed, we are not able to test algorithmically the existence of an irregular solution of ramification index r for a given system S. This implies evidently that we are not able to compute the maximal number of linearly independent solutions of ramification index r when r is a given number. In Section 3 we prove that we cannot perform such computation even if we know in advance that such solutions exist. We also cannot construct the list of all polynomials Q which correspond to r in irregular solutions.

This enables us to obtain a strengthening of the statement that we are not able to compute algorithmically the dimension of the space of all formal solutions. In fact, it is impossible to compute algorithmically this dimension even when, in addition to the system, we know the list of all values of the ramification indices. This result indicates that from the algorithmic point of view, the dimension of the space of all formal solutions is a "deeply hidden" characteristic of a constructive system, i.e., of a full rank system having computable power series in the role of coefficients.

Similarly to Abramov & Barkatou (2014), the results of this paper supplement known results on the zero testing problem and some algorithmically undecidable problems related to differential equations (see, e.g., Denef & Lipshitz (1984), van der Hoeven & Shackell (2006)).

The present paper contains a "positive result" as well (Section 4): there exists an algorithm which, given a system S and integers r, d such that the existence for S of d linearly independent solutions of ramification index r is guaranteed, computes such d formal solutions; e.g., if we know in advance that there exists an irregular solution of ramification index r then we can construct such a solution.

2. Existence of Formal Solutions of Given Ramification Index

Let K be an algebraically closed field of characteristic 0. It is known that a system

$$y' = A(x)y, \tag{1}$$

$A(x) \in \mathrm{Mat}_m(K((x)))$, $y = (y_1, \ldots, y_m)^T$, has m linearly independent formal solutions.

Definition 1. *A* proper formal solution *of a system is a solution of the form*

$$e^{Q(\frac{1}{t})} t^{\lambda} \Phi(t), \quad x = t^r \tag{2}$$

where $\lambda \in K$; $Q(\frac{1}{t})$ is a polynomial in $\frac{1}{t}$ over K and the constant term of this polynomial is equal to zero; r is a positive integer; $\Phi(t)$ is a column vector with *components in the form* $\sum_{i=0}^{k} g_i(t)\log^i(t)$ *and all* $g_i(t)$ *are power series over* K. *The product* $t^\lambda\Phi(t)$ *is the* regular part *of (2), and* $e^{Q\left(\frac{1}{t}\right)}$ *is its* exponential part.

If r *has the minimal possible value in the representation (2) of a proper formal solution then* r *is the* ramification index *of that solution. If* $r = 1$ *and* $Q = 0$ *then solution (2) is* regular. *Otherwise it is* irregular.

A formal solution *is a finite linear combination with coefficients from* K *of terms as (2).*

We will deal with objects that are constructive.

Definition 2. *A ring (field) is said to be* constructive *if there exist algorithms for performing the ring (field) operations and an algorithm for zero testing in the ring (field).*

Let K be a constructive field. We will consider infinite power series belonging to $K[[x]]$. As it was noted in Abramov & Barkatou (2014), the problem of infinite series representation is important for computer algebra. A general formula that expresses the coefficients of a series is not always available and may even not exist. A natural way to represent the series is the algorithmic one, i.e., providing an algorithm which computes its coefficients.

We denote by $K[[x]]|_C$ the ring of *computable series* (i.e., the series whose sequences of coefficients can be represented algorithmically; arbitrary deterministic algorithms which are applicable to non-negative integer numbers and return elements of K are allowed).

Definition 3. *Let* n, m *be integers,* $n \geqslant 1$, $m \geqslant 2$. *A full-rank linear differential system*

$$A_n(x)y^{(n)} + \cdots + A_1(x)y' + A_0(x)y = 0 \tag{3}$$

with $A_i(x) \in \mathrm{Mat}_m(K[[x]]\,|_C)$, $i = 0, 1, \ldots, n$, $y = (y_1, \ldots, y_m)^T$, *having a non-zero leading matrix* $A_n(x)$ *(i.e. the system is of order* n*), is a* constructive system.

An algorithmic representation of a concrete computable series is not unique. This non-uniqueness is one of the reasons for undecidability of the zero testing problem for such computable series. This implies undecidability of some of problems concerned with formal solutions of systems, although for scalar equations the same problems are decidable.

In Abramov & Barkatou (2014), it has been proven that for the space of formal solutions of a constructive system there exists a basis whose elements are of the form (2), and all power series involved into them are computable. However, one cannot find such a basis algorithmically.

Remark 1. *If* K *is not algebraically closed then there exists a simple algebraic extension* K_1 *of* K *(specific for each system) such that system (1) has* m *linearly independent solutions of the form (2) with* $\lambda \in K_1$, $Q(t) \in K_1[t^{-1}]$, $\Phi(t) \in K_1^m[[t]][\ln t]$ *(Barkatou (1997)).*

Let $r \geqslant 1$ be an integer, S a constructive system. We denote by $N_r(S)$ the maximal number of linearly independent proper formal irregular solutions of the system S having ramification index r.

Proposition 1. *There exists no algorithm which, given a constructive system S and $r \in \mathbb{Z}_{>0}$, tests the existence of a proper formal solution of ramification index r for the system S, i.e., tests the inequality $N_r(S) > 0$.*

Proof. If S is as in (3) then the ramification index of a solution for S does not exceed mn. This follows, first, from the fact that in the case $m = 1$ (i.e., in the case of a scalar equation of order n) the ramification index of any proper formal solution does not exceed n (see (Barkatou, 1988, §2)), and, second, the fact that there exists a so-called l-embracing system (Abramov & Khmelnov (2011, 2012)) of the same form (3) whose leading matrix is invertible and whose space of solutions contains all the solutions of (3). Finally, recall that such a system with invertible leading matrix is equivalent to a scalar equation of order mn.

If we had an algorithm for testing the existence of a proper formal solution of given ramification index then we would select from the set $\{1, \ldots, mn\}$ all the numbers r for which there exists a proper formal solution of ramification index r: trying all r such that $1 \leqslant r \leqslant mn$ we would know whether a given system has an irregular formal solution. This means that we would test the existence of an irregular solution for S. But such algorithmic testing is impossible by Abramov & Barkatou (2014); Abramov & Khmelnov (2014). □

3. When Solutions of Given Ramification r Exist

Next, we prove that even if we know in advance that $N_r(S) > 0$, i.e., that a constructive system S has a proper formal solution of ramification index $r \geqslant 1$, we are generally not able to compute $N_r(S)$.

Lemma 1. *Given $r \in \mathbb{Z}_{>0}$ and $s(x) = \sum_{i=0}^{\infty} s_i x^i \in K[[x]]|_C$, $s_0 = 0$, one can find $g_0(x), \ldots, g_{2r}(x) \in K[[x]]|_C$ such that the scalar equation*

$$x^{2r} g_{2r}(x) z^{(2r)} + \cdots + x g_1(x) z' + g_0(x) z = 0 \tag{4}$$

(a) is of order larger than or equal to r, i.e., at least one of the series

$$g_{2r}(x), g_{2r-1}(x), \ldots, g_r(x)$$

is nonzero,

(b) has at least r linearly independent proper formal solutions of ramification index r,

(c) has more than r linearly independent proper formal solutions of ramification index r if and only if $s(x) = 0$.

Proof. Consider the computable series $u(x) = \sum_{i=0}^{\infty} u_i x^i$:

$$u_i = \begin{cases} s_{\frac{i}{r}} & \text{if } r \mid i, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

$i = 0, 1, \ldots$ The series $u(x)$ is non-zero if and only if the series $s(x)$ is non-zero and then $r \mid \operatorname{val} u(x)$. The latter relation implies that r and $\operatorname{val} u(x) + 1$ are relatively prime.

The scalar equation

$$x^{r+1} z^{(r)} + z = 0 \tag{6}$$

has r proper formal solutions of ramification index r: the Newton polygon of this equation contains two vertices: $(0, 0)$ and $(r, 1)$. The slop of the corresponding edge is $1/r$, and the order of the equation (6) is r.

If $u(x) = 0$ then the scalar equation

$$x^{r+1} u(x) z^{(r)} + z = 0 \tag{7}$$

possesses only zero solution. In the case $u(x) \neq 0$, the Newton polygon of (7) contains two vertices: $(0, 0)$ and $(r, \operatorname{val} u(x) + 1)$. The slop of the corresponding edge is $(\operatorname{val} u(x) + 1)/r$. This fraction is reduced due to (5), and the order of equation (7) is r. This implies that in the case $s(x) \neq 0$ the equation (7) has r proper formal solutions of ramification index r, and it has only zero solution if $s(x) = 0$.

Let $L, \tilde{L}$ be the operators corresponding to equations (6), (7):

$$L = x^{r+1} \frac{d^r}{dx^r} + 1, \;\; \tilde{L} = x^{r+1} u(x) \frac{d^r}{dx^r} + 1.$$

Using the standard Euclidean algorithm for scalar differential operators (see, e.g., Bronstein & Petkovšek (1996)) we can detect that the operators have no non-trivial common right divisor (i.e., $\operatorname{gcrd}(L, \tilde{L}) = 1$), and find operators F, G of minimal orders such that $FL + G\tilde{L} = 0$:

$$F = 1 + L \frac{u(x)}{1 - u(x)}, \;\; G = L \frac{1}{1 - u(x)};$$

note that $\operatorname{val}(1 - u(x)) = 0$ since $\operatorname{val} u(x) > 0$. Thus, we get the least common left multiple $\operatorname{lclm}(L, \tilde{L})$ as the product $FL = \left(1 + L \frac{u(x)}{1-u(x)}\right) L$. After multiplication by some factor of the form x^k, $k \in \mathbb{Z}_{\geqslant 0}$, the latter operator can be represented as

$$M = x^{2r} g_{2r}(x) \frac{d^{2r}}{dx^{2r}} + \cdots + x g_1(x) \frac{d}{dx} + g_0(x) \in K[[x]]|_{\mathrm{C}} \left[\frac{d}{dx}\right] \tag{8}$$

(since $u(x)$ is a computable power series, the series $g_i(x)$, $i = 0, 1, \ldots, 2r$, are also computable). The equation $M(z) = 0$ can be used as equation (4). Indeed, $\operatorname{ord} \operatorname{lclm}(L, \tilde{L}) =\geqslant \operatorname{ord} L = r$, this proves (a). Each solutions of the equation

$L(z) = 0$ is a solution of $M(z) = 0$ as well, this proves (b). Finally, if $u(x) = 0$ then $M = L$, and if $u(x) \neq 0$ then $\operatorname{ord} M = 2r$ and M has all the formal solutions of the equations $\tilde{L}(z) = 0$, $L(z) = 0$. Since $u(x) = 0$ if and only if $s(x) = 0$, we get that the number of linearly independent proper formal solutions of ramification index r for the equation $M(z) = 0$ is r if $u(x) = 0$ and $2r$ otherwise. This proves (c). □

Now we can prove the key statement of this section.

Proposition 2. *There exists no algorithm which, given $r \in \mathbb{Z}_{>0}$ and a constructive system S having a proper formal solution of ramification index r, computes $N_r(S)$, i.e., the maximal number of linearly independent proper formal solutions of S having the ramification index r.*

Proof. Consider the systems S:

$$\begin{pmatrix} x^{2r} g_{2r}(x) & 0 \\ 0 & 1 \end{pmatrix} y^{(2r)} + \cdots + \begin{pmatrix} x g_1(x) & 0 \\ 0 & 0 \end{pmatrix} y' + \begin{pmatrix} g_0(x) & 0 \\ 0 & 0 \end{pmatrix} y = 0,$$

$y = (y_1, y_2)^T$, the series $g_0(x), g_1(x), \ldots, g_{2r}(x)$ are as in (4). The leading matrix of S is not zero even when $g_{2r}(x)$ is zero power series.

The system S is equivalent to the system

$$M(y_1) = 0, \ \ y_2^{(2r)} = 0, \tag{9}$$

where M is as in (8). Thus, the proper formal solutions of S which have the ramification index r are exactly those solutions that have the form $(f(x), 0)^T$, where $f(x)$ is a proper formal solution of $M(y_1) = 0$ having the ramification index r.

If we had an algorithm for computing $N_r(S)$ when it is known that $N_r(S) \neq 0$, we would be able to check whether $N_r(S) = 2r$ and, using Lemma 1(c), the zero testing for an arbitrary computable series $s(x) = \sum_{i=1}^{\infty} s_i x^i$ would be possible. However, this problem is undecidable due to classical results by A.Turing (Turing (1936); Martin-Löf (1970)). Contradiction. □

This enables us to obtain a strengthening of the statement which has been proven in Abramov & Barkatou (2014) and says that we are not able to compute algorithmically the dimension of the space of all formal solutions although we can construct a basis for the subspace of regular solutions. As a direct consequence of Proposition 2 we obtain:

Proposition 3. *It is impossible to compute algorithmically the dimension of the formal solutions space of an arbitrary constructive linear differential system even when we know, in addition to the system, the list of all values of the ramification indices.*

One more consequence of Proposition 2:

Proposition 4. *There exists no algorithm which, given $r \in \mathbb{Z}_{>0}$ and a constructive full rank system S having a proper formal solution of ramification index r, constructs the list of all polynomials Q corresponding to solutions of S which are of the form (2).*

Proof. Indeed, if all the exponential parts which relate to a given ramification index are known then using for each pair (r, e^Q) a corresponding substitution (changing $y(x)$) in the original system, we obtain each time a constructive system for which we can compute the dimension of its regular solutions — for this, we can use the algorithm from Abramov & Khmelnov (2014). After using all the pairs we would know the the maximal number of linearly independent proper formal solutions of S having ramification index r. This contradicts to Proposition 2. □

It is well to bear in mind, that those algorithmically undecidable problems that were discussed in the present and previous sections are decidable for scalar equations of known arbitrary order as well as for first order systems of the form (1) where m is arbitrary, if coefficients are Laurent series represented algorithmically and having known valuations. In those cases, the bases of the spaces of formal solutions can be constructed Barkatou (1997, 1988); Lutz & Schäfke (1985); Pflügel (2000).

4. Constructing Formal Solutions of Given Ramification Index

Thus, by Proposition 1, we are not able to test algorithmically the existence of a solution having the ramification which is equal to a given integer r. However, if we know in advance that such solutions exist then we can construct such a solution. We prove the following general statement.

Proposition 5. *There exists an algorithm which, given a constructive system S and integers r, d such that the inequality $N_r(S) \geqslant d$ is guaranteed to hold, computes d linearly independent proper formal solutions of ramification index r.*

Proof. An algorithm which generates a sequence of linearly independent proper formal solutions of a given constructive system S has been described in Ryabenko (2015) by A.Ryabenko. In fact, that algorithm computes step-by-step truncated versions $S^{\langle k \rangle}$, $k = 1, 2, \dots$, of S (in $S^{\langle k \rangle}$ one preserves k initial terms of the power series which are the coefficients of the system S). For those of such systems with polynomial coefficients which are of full rank (this is recognizable since the coefficients of $S^{\langle k \rangle}$ are polynomials), the algorithm finds the ramification indices and exponential parts of their irregular solutions. Each computed pair is used for the corresponding substitution (changing $y(x)$) in the original system. For the obtained constructive system, the search for regular solutions is performed by the algorithm from Abramov & Khmelnov (2014). In some moment the generated sequence contains the maximal possible number of linearly independent proper formal solutions, this is proven in Ryabenko (2015) using results from Abramov et al. (2013); Lutz & Schäfke (1985). However, this algorithm (we will refer to it as the "generating algorithm") will not terminate by itself: after constructing the maximal number of linearly independent proper formal solutions, the algorithm tries (with no result) to find a new solution which is linearly independent with respect to the constructed solutions. The given positive integer d is such that $N_r(S) \geqslant d$, and this enables us to stop the generating algorithm at the moment when d linearly independent proper formal solutions have been computed. □

As a consequence, there exists an algorithm which, given a constructive system S and the set of all the pairs $(r, N_r(S))$, where r is a positive integer and $N_r(S) > 0$, constructs a basis for the space of formal solutions of S. However, the set of such pairs cannot be computed algorithmically.

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