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Search of Rational Solutions to Differential and Difference Systems by Means of Formal Series

S. A. Abramov

Computing Center, Russian Academy of Sciences, ul. Vavilova 40, Moscow, 119333 Russia

e-mail: sergeyabramov@mail.ru

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Abstract—Systems of linear ordinary differential and difference equations of arbitrary order with polynomial coefficients are considered. An algorithm for searching their rational solutions is suggested. The algorithm is based on the fact that these solutions are expanded into formal series that become polynomials after multiplication by a universal denominator. A combined algorithm relying on heuristics is also described. To reduce the amount of computation, at a certain moment, the latter selects between two algorithms, the new one and the standard algorithm based on the use of the universal denominator for changing unknowns and subsequent search for polynomial solutions.

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1. INTRODUCTION

Let K be a field of characteristic 0. The ring of polynomials and the field of rational functions of x are conventionally denoted as $K[x]$ and $K(x)$, respectively. The ring of formal Laurent series is denoted as $K((x))$. If R is a ring (in particular, field), then $\mathrm{Mat}_m(R)$ denotes the ring of square matrices of order m with entries from R.

We consider systems of the form

$$A_r(x)\xi^r y(x) + \ldots + A_1(x)\xi y(x) + A_0(x)y(x) = 0, \quad (1)$$

$\xi \in \left\{\frac{d}{dx}, E\right\}$, where E is the shift operator: $Ey(x) = y(x+1)$. Square matrices $A_i(x)$, $i = 0, 1, \ldots, r$, are of order m with entries from $K[x]$, with $A_r(x)$ and $A_0(x)$ being nonzero *leading* and *trailing* matrices, and $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions (T denotes transposition). The number r is called the *order* of the system. The system under study is assumed to have *full rank*; i.e., the equations of the system are independent over the ring of operators $K(x)[\xi]$. System (1) can be written in the form $L(y) = 0$, where L is the operator

$$A_r(x)\xi^r + \ldots + A_1(x)\xi + A_0(x). \quad (2)$$

Solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in K(x)^m$ of system (1) is called a *rational* solution. If $y(x) \in K[x]^m$, it is called a *polynomial* solution (a particular case of the rational solution).

Algorithms for searching all rational solutions to normal first-order systems of the form

$$\xi y(x) = A(x)y(x), \quad (3)$$

where $A(x) \in \mathrm{Mat}_m(K(x))$ is assumed to be invertible in the difference case, are well known (see, e.g., [1–5]). Algorithms from [1, 2, 4, 5] are based on finding a *universal denominator* of rational solutions to the original system (for brevity, we call it the universal denominator for the original system), i.e., a polynomial $U(x) \in K[x]$ such that, if the system has a rational solution $y(x) \in K(x)^m$, then it can be represented as $\frac{1}{U(x)}z(x)$, where $z(x) \in K[x]^m$. If a universal denominator is known, we can make the substitution

$$y(x) = \frac{1}{U(x)}z(x), \quad (4)$$

where $z(x) = (z_1, \ldots, z_m)^T$ is the vector of new unknowns, and, then, apply one of the algorithms for searching polynomial solutions (see, e.g., [1, 2, 6]).

The problem of searching rational solutions for full-rank systems (1) in the case where matrices $A_0(x)$ and $A_r(x)$ (one of them or both) may be singular were considered much less frequently. Nevertheless, appropriate algorithms were suggested in [7] (the differential case) and [8] (the difference case). These algorithms are also based on finding a universal denominator $U(x)$ that is used on the next stage for the substitution (4), which is followed by searching polynomial solutions to the new system obtained.

Below, we consider a different approach. It is based on expanding a general solution to the original system (1) into a series whose coefficients linearly depend on arbitrary constants. After multiplication by a universal denominator $U(x)$ found, the series corresponding to rational solutions turn to polynomials. To implement this general scheme in an algorithmic form, we consider formal series in terms of decreasing powers (this can also be viewed as expansion at point ∞). Each series of this kind contains only a finite number of powers of x with nonnegative exponents and, possibly, an infinite number of powers with negative ones. The greatest exponent of x with a nonzero coefficient (i.e., a coefficient given by a nonzero column) occurring in a series $y(x)$ is called the *degree* $\deg y(x)$ of the series $y(x)$. If $y(x)$ is a zero series, then we set $\deg y(x) = -\infty$.

In the case of a differential system

$$A_r(x)y^{(r)}(x) + \ldots + A_1(x)y'(x) + A_0(x)y(x) = 0, \quad (5)$$

the series under consideration belong to $K^m((x^{-1}))$. In the case of a difference system

$$A_r(x)y(x+r) + \ldots + A_1(x)y(x+1) + A_0(x)y(x) = 0, \quad (6)$$

series in terms of powers $x^{\underline{n}}$ (see, e.g., [9, Ch. 10]) are used:

$$x^{\underline{n}} = \begin{cases} x(x-1)\ldots(x-n+1), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \dfrac{1}{(x+1)(x+2)\ldots(x+|n|)}, & \text{if } n < 0. \end{cases} \quad (7)$$

We will use double-sided sequences of rational functions

$$(x^n)_{n \in \mathbb{Z}}, \quad (x^{\underline{n}})_{n \in \mathbb{Z}}$$

as bases for expanding solutions of systems in the differential and difference cases, respectively. Let coefficients of the expansion of a solution in the corresponding basis be $v(n) = (v_1(n), \ldots, v_m(n))^T$. Then, the sequence $v(n)$ satisfies the *induced recurrence* system

$$B_l(n)v(n+1) + B_{l-1}(n)v(n+l-1) + \ldots + B_t(n)v(n+t) = 0, \quad (8)$$

where l and t are integers such that $l \geq t$ and $B_t(n), \ldots, B_l(n) \in \mathrm{Mat}_m(K[n])$ [10].

We use the term "induced recurrence system," rather than the "induced difference system," in order to emphasize the special role of the induced systems. For brevity, we will refer to them as simply induced systems. The construction of these systems is discussed in Section 22. The number $l - t$ is called the *order* of system (8).

Considering initial terms whose coefficients satisfy the induced system (by convention, coefficients of sufficiently large powers are zero column vectors), we do not know for sure whether there exists a continuation of the given initial fragment to a formal series that is a solution to the original system or such a continuation is not possible. In what follows, we will carefully use the term *formal sum*, meaning the formal sum $\sum_{n=N}^{\infty} g(n)x^n$ in the differential case and $\sum_{n=N}^{\infty} g(n)x^{\underline{n}}$, $g(n) \in K^m$, in the difference case. Note that, for each particular formal sum, $g(n) = 0$ for sufficiently large n. Hence, each formal sum contains a finite number of nonzero terms. If $f(x)$ is a formal sum of the above-specified type and $g(n)$ is a sequence of its coefficients, then N is referred to as the *stopping point* of this sequence, which is denoted as $\mathrm{stp}\, g(n)$. The greatest n such that $g(n) \neq 0$ in a formal sum $f(x)$ is called the *degree* of $f(x)$, which is denoted as $\deg f(x)$. If a formal sum $f(x)$ contains only zero terms, then $\deg f(x) = -\infty$.

If a sequence $g(n)$ of coefficients of a formal sum satisfies the induced system, the sum is said to be *subordinate* to the original system.

The new algorithm constructs subordinate formal sums whose stopping points are selected in a special way. Generally, we cannot determine without considerable effort whether one or another formal sum can be continued to a formal series that is a solution to the original system. However, the "useful" formal sums selected by the algorithm admit such continuation. This is proved in Section 4.3.

The new algorithm does without substitutions of form (4). The induced system used by the algorithm in a quite general situation has typically a lesser order. Other things being equal, this is an undoubted advantage of the new algorithm. There may be a problem with the number of elements of the sequence to be found by means of the recurrence systems: the standard algorithm can sometimes do with a lesser number of elements.

In Section 6, a combined (in a sense, heuristic) algorithm is described. To reduce the amount of computation, this algorithm, based on analysis of intermediate results, makes a decision whether to apply the new algorithm or take advantage of the standard one, which is based on the use of a universal denominator for changing variables in the system and subsequent searching for polynomial solutions.

If the original system is not homogeneous and its right-hand side belongs to $K[x]^m$, then addition of the $(m+1)$-th component $y_{m+1}(x)$ identically equal to one to $y(x)$ transforms the original system to a homogeneous one (transformed matrices $A_0(x)$, $A_1(x)$, ..., $A_r(x)$ will belong to $\mathrm{Mat}_{m+1}(K[x])$). In what follows, we consider only homogeneous systems.

2. PRELIMINARIES

2.1. Embracing Systems

The leading matrix may be not invertible (singular) in $\mathrm{Mat}_m(K(x))$, which generates certain difficulties for finding solutions. Algorithms EG_δ and EG_σ make it possible to avoid difficulties of this kind. For any system S of form (1), algorithms EG_δ (in the differential case) and EG_σ (in the difference case) construct an *l-embracing* system $\bar{S}$ with a nonsingular leading matrix $\bar{A}_r(x)$. In this case, the set of solutions of system $\bar{S}$ contains all solutions of system S.

In the difference case, algorithm EG_σ can construct for system S, in addition to the l-embracing system, also a *t-embracing* system $\bar{\bar{S}}$ of the form (1) with a nonsingular trailing matrix $\bar{\bar{A}}_0(x)$, with the set of solutions of system $\bar{\bar{S}}$ containing all solutions of system S.

Besides, algorithm EG_σ finds a finite set $\mathscr{C}$ of *linear constraints*, i.e., linear relations with constant coefficients for a finite set of values $y_i(\eta + j)$, $i = 1, \dots, m$, $j = 0, 1, \dots, r$. Each linear constraint is obtained from one of the equations of the system at some stage of its transformation by substituting a particular value of the variable into this equation. We will use linear constraints when considering solutions in the form of sequences with elements from K^m. To this end, we may confine ourselves to linear constraints such that $h \in \mathbb{Z}$. The application of EG_σ results in a pair $(\tilde{S}, \mathscr{C})$ that has the same set of solutions as S has, with the leading and trailing matrices of system $\tilde{S}$ being nonsingular.

Algorithm EG_σ can be applied to both original difference system and the system that is induced for the original differential or difference system. A variant of algorithm EG_δ, which is more economical than that in [7] in terms of the number of operations, was proposed in [11, Section 4.2].

Remark 1. Let a system $\tilde{S}$ with a set $\mathscr{C}$ of linear constraints be given. If all its solutions belonging to the class of sequences of form $v(n) = (v_1(n), \dots, v_m(n))^T$ with values in K_m determined for $N \le n < \infty$ (N is fixed) are considered, then all linear constraints belonging to $\mathscr{C}$ containing some $v_i(\eta)$ ($\eta < \mathbb{N}$) with nonzero coefficients must be ignored.

2.2. Induced Recurrence Systems

In addition to what was said about induced systems in the Introduction, we note that the induced recurrence system is constructed from the original one by the transformations

$$x \longrightarrow E_n^{-1}, \quad \frac{d}{dx} \longrightarrow (n+1)E_n, \tag{9}$$

and

$$x \longrightarrow n + E_n^{-1}, \quad E \longrightarrow 1 + (n+1)E_n. \tag{10}$$

in the differential and difference cases, respectively [12–14]. These transformations are convenient to apply if operator (2) is written in the form of the operator matrix

$$\begin{pmatrix} L_{11} & \dots & L_{1m} \\ \dots & \dots & \dots \\ L_{m1} & \dots & L_{mm} \end{pmatrix}, \tag{11}$$

where $L_{ij} \in K[x, \xi]$, $i, j = 1, \dots, m$. Explicit formulas for transformations (9) and (10) applied to L_{ij} are given in [12, Sections 4.1 and 4.2].

If the original system is associated with an operator L of form (2) and system (8) is associated with the induced operator

$$R = B_l(n)E_n^l + B_{l-1}(n)E_n^{l-1} + \dots + B_t(n)E_n^t \tag{12}$$

(E_n denotes the shift by n: $E_n v(n) = v(n + 1)$), then the equality $\deg L(y) = n_0$ holds for some integer n_0 if and only if the sequence $v(n)$ of the coefficients of series $y(x)$ satisfies system $R(v) = 0$ for all $n \ge t + n_0$ (here, t is the same as in (12)).

Let $f(x)$ be a series or a formal sum and $g(n)$ be the corresponding sequence of the coefficients. Let $N_0 \in \mathbb{Z}$ and, if $f(x)$ is a formal sum, $N_0 \ge \operatorname{stp} g(n)$. Let $\lfloor f(x) \rfloor_{N_0}$ be a formal sum consisting of all terms occurring in $f(x)$ the power of x in which is not less than N_0.

The following proposition is easily derived from the above discussion.

Proposition 1. Let L be an operator of form (2), R be its induced operator (12), and integer t be the same as in operator R. Let $f(x)$ be a formal sum and $g(n)$ be its sequence of the coefficients. Then,

(i) if $N \ge \operatorname{stp} g(n)$, then $\deg L(f(x)) \le \deg L(\lfloor f(x) \rfloor_N)$;

(ii) the formal sum $f(x)$ is subordinate to system $L(y) = 0$ if and only if $L(f(x)) < -t + \operatorname{stp} g(n)$.

2.3. Indicial Equations

Recall that, for a nonzero element $a(x) = \sum a_i x^i$ from $K((x))$, its *valuation* $\operatorname{val}_x a(x)$ is defined as

$$\operatorname{val}_x a(x) = \min\{i : a_i \ne 0\}, \tag{13}$$

with $\operatorname{val}_x 0 = \infty$. In the scalar case, to obtain estimates of valuations of analytical solutions at singular points and bounds for degrees of polynomial solutions, the so-called *indicial* algebraic equations are used (see, e.g., [15, Ch. IV]). To determine similar bounds for the systems under consideration, some variants of the indicial equations are required. The induced recurrence systems introduced in Section 2.2 allow us to construct such equations.

If the leading matrix $B_l(n)$ of system (8) is singular, then one can construct the l-embracing system

$$\begin{aligned} &\bar{B}_l(n)v(n+l) + \bar{B}_{l-1}(n)v(n+l-1) + \dots \\ &\qquad + \bar{B}_t(n)v(n+t) = 0 \end{aligned} \tag{14}$$

for (8) by the EG_σ algorithm. In a similar way, one can construct the t-embracing system

$$\begin{aligned} &\bar{\bar{B}}_l(n)v(n+l) + \bar{\bar{B}}_{l-1}(n)v(n+l-1) + \dots \\ &\qquad + \bar{\bar{B}}_t(n)v(n+t) = 0. \end{aligned} \tag{15}$$

The bounds of interest can be found from certain algebraic equations:

(i) Let $y(x) \in K((x))^m$ be a solution of the differential system for which (8) is an induced recurrence system. Then, $\nu = \mathrm{val}_x y(x)$ satisfies the equation

$$\det \bar{B}_l(\nu - l) = 0. \quad (16)$$

(ii) Let $y(x) \in K[x]^m$ be a solution of the differential or difference system for which (8) is an induced recurrence system. Then, $\nu = \deg y(x)$ satisfies the equation

$$\det \bar{\bar{B}}_t(\nu - t) = 0 \quad (17)$$

(here, $\deg y(x) = \max_{i=1}^m \deg y_i(x)$ for $y(x) = (y_1(x), \ldots, y_m(x))^T \in K[x]^m$).

Using system (14) when n increases from $-\infty$, we can obtain a nonzero element from zero elements of the sequence only if the leading matrix of system (14) is singular, which yields (i). Assertion (ii) is proved similarly.

Equation (17) may be viewed as the indicial equation for the original system at point ∞, and the greatest nonnegative integer root of this equation yields the upper bound for exponents of the polynomial solutions. If equation (17) has no nonnegative integer roots, then the original system has no polynomial solutions. Similarly, in the differential case, equation (16) can be used for finding lower bounds of solution valuations at point 0. (Substitution of $x + \alpha$ for x into the original system transforms point α to point 0.)

As applied to (16) and (17), the term "indicial equation" is used in a conventional sense because, for example, the equations obtained in this way are not unique and depend on the constructed l- and t-embracing systems.

3. UNIVERSAL DENOMINATOR

The set of monic irreducible polynomials from K[x] will be denoted as Irr(K[x]).

3.1. Differential Case

For a differential system S of the considered form, one can find an l-embracing system $\bar{S}$. If solution of system S has singularity at point α, then α is a root of the determinant of the leading matrix of system $\bar{S}$. The ability to find a lower bound e_α of valuation of any solution of system S at a point α by means of the corresponding indicial equation makes it possible to construct a universal denominator. If the indicial equation corresponding to some α has no integer roots, then S has no rational solutions. Note that the bounds e_i are the same for all roots α_i of each factor of the left-hand side of the indicial equation, and the calculations rely on this [16]. Let polynomials $p_1(x), \ldots, p_s(x) \in$ Irr(K[x]) are made to correspond to nonnegative integers $e_1, \ldots, e_s$. Then, for the universal denominator, we can take the polynomial

$$U(x) = \prod_{i=1}^{s} p_i^{-e_i}(x). \quad (18)$$

In what follows, we will also need the polynomial

$$\tilde{U}(x) = \prod_{i=1}^{s} p_i^{-e_i + r}(x), \quad (19)$$

and the number

$$\rho = r \sum_{i=1}^{s} \deg p_i(x). \quad (20)$$

3.2. Difference Case

If $p(x) \in$ Irr(K[x]), $f(x) \in K[x]$, then $\mathrm{val}_{p(x)} f(x)$ is defined as the greatest $n \in \mathbb{N}$ such that $p^n(x) | f(x)$ ($\mathrm{val}_{p(x)} 0 = \infty$) and $\mathrm{val}_{p(x)} F(x) = \mathrm{val}_{p(x)} f(x) - \mathrm{val}_{p(x)} g(x)$ for $F(x) = \frac{f(x)}{g(x)}$, $f(x)$, $g(x) \in K[x]$. This definition agrees well with (13).

For $p(x) \in$ Irr(K[x]), $f(x) \in K[x] \backslash \{0\}$, we define the finite set

$$\mathcal{N}_{p(x)}(f(x)) = \{z \in \mathbb{Z} : p(x+k) | f(x)\};$$

for $\mathcal{N}_{p(x)}(f(x)) = \varnothing$, we set $\mathcal{N}_{p(x)}(f(x)) = -\infty$ and $\mathcal{N}_{p(x)}(f(x)) = +\infty$.

The denominator $\mathrm{den} F(x)$ of a rational function $F(x)$ is a polynomial of the least possible degree with the leading coefficient equal to one such that $F(x) = \frac{f(x)}{\mathrm{den} F(x)}$ for some polynomial $f(x)$. The denominator $\mathrm{den} A(x)$ of a matrix $A(x) \in \mathrm{Mat}_m(K(x))$ is the least common multiple of the denominators of entries of matrix $A(x)$. Let $\bar{A}_r(x)$ be a leading matrix of the l-embracing system and $\bar{A}_0(x)$ be a trailing matrix of the t-embracing system for (6). We set

$$V(x) = \mathrm{den} \bar{A}_r^{-1}(x - r), \quad W(x) = \mathrm{den} \bar{\bar{A}}_0^{-1}(x). \quad (21)$$

One of the algorithms for constructing a universal denominator consists of two steps. On the first step, it constructs the set $Q = \{$q1(), ..., $q_s(x)\}$, $s \geq 1$, of all elements from Irr(K[x]) such that

$$\min \mathcal{N}_{q_t}(V(x)) = 0, \quad \max \mathcal{N}_{q_t}(W(x)) \geq 0,$$

$t = 1, \ldots, s$. For each $q_t(x) \in Q$, the quantity

$$d_t = \max \mathcal{N}_{q_t}(W(x)).$$

is determined. The set of all $p(x)$ for which $p(x) = q_t(x + i)$ for some t, i, $1 \leq t \leq s$, $0 \leq i \leq d_t$, is denoted by M.

On the second step, the universal denominator

$$\prod_{p(x) \in M} p^{\gamma_{p(x)}}(x), \quad (22)$$

where

$$\gamma_{p(x)} = \min\left\{\sum_{n \in \mathbb{N}} \operatorname{val}_{p(x+n)} V(x),\right. \tag{23}$$
$$\left.\sum_{n \in \mathbb{N}} \operatorname{val}_{p(x-n)} W(x)\right\}.$$

is calculated. If we take into account that some (sometimes, many) $\gamma_{p(x)}$ for different $p(x)$ differing from one another by a shift by an integer coincide, the computation can considerably be sped up [4, 5, 8, 17].

In what follows, we will need the polynomial

$$\tilde{U}(x) = \operatorname{lcm}_{i=0}^{r} U(x+i), \tag{24}$$

where lcm stands for the least common multiple.

Proposition 2. Let

$$\mu_t = \max_{i=0}^{d_t} \operatorname{val}_{q_t(x+i)} U(x), \tag{25}$$

$t = 1, \ldots, s$, and

$$\rho = r\sum_{t=1}^{s} \mu_t \deg q_t(x). \tag{26}$$

Then, the equality

$$\deg \tilde{U}(x) = \deg U(x+j) + \rho \tag{27}$$

holds for all $j = 0, 1, \ldots, r$.

Proof. Structure of the universal denominator $U(x)$ obtained by the above-mentioned algorithm is such that, for $\tilde{U}(x) = \operatorname{lcm}_{j=0}^{r} U(x+j)$, we have

$$\sum_{i=0}^{d_i+r} \operatorname{val}_{q_t(x+i)} \tilde{U}(x) = \sum_{i=0}^{d_i} \operatorname{val}_{q_t(x+i)} U(x) + r\mu_t,$$

$t = 1, \ldots, s$. Therefore,

$$\deg \tilde{U}(x) = \deg U(x) + r\sum_{t=1}^{s} \mu_t \deg q_t(x),$$

and, since $\deg U(x) = \deg U(x+j)$ for any j, we have $\deg \tilde{U}(x) - \deg U(x+j) = r\sum_{t=1}^{s} \mu_t \deg q_t(x)$. Thus, equality (27) holds. □

Remark 2. Quantities $\mu_1, \ldots, \mu_s$ and ρ can be found when constructing $U(x)$ without additional computations.

Both in the differential and difference cases, we have

$$\deg \tilde{U}(x) - \deg U(x) = \rho. \tag{28}$$

4. SEARCH FOR NUMERATORS FOR KNOWN UNIVERSAL DENOMINATORS

4.1. Polynomial Solutions

First, we consider the well-known problem of searching polynomial solutions.

After the induced system is constructed and, if necessary, the corresponding t-embracing system for it is found, we determine the upper bound N of the exponents of all polynomial solutions as the greatest nonnegative integer root of equation (17). If there are no such roots, we conclude that the original system has no polynomial solutions. In some cases, the set of linear constraints obtained together with the t-embracing system gives us an opportunity to improve the estimate of bound N [7, Section 3], but we will not discuss this here. The polynomial solutions themselves can be found by the method of undetermined coefficients. However, currently, coefficients of the polynomial solutions can be found more efficiently by means of the recurrence system obtained (see, e.g., [6]).

Let us rewrite (15) as

$$\begin{aligned}\tilde{B}_t(n)v(n+t) &= -\tilde{B}_{t+1}(n)v(n+t+1) - \ldots \\ &- \tilde{B}_{l-1}(n)v(n+l-1) - \tilde{B}_l(n)v(n+1),\end{aligned} \tag{29}$$

Taking into account that $v(n) = 0$ for $n > N$, we will successively find

$$v(N),\ v(N-1),\ \ldots,\ v(0),\ v(-1),\ \ldots,\ v(-l+t).$$

To do this, we consider (29) for a fixed n as a system of linear algebraic equations in $v(n+1)$. For $n = N-t$, $N-t-1$, ..., solutions of such systems will contain constants the set of which will change when turning to the next n as long as the matrix on the left-hand side of the current system (29) is singular. On the one hand, the system must be compatible, which will give relations (linear algebraic equations) for earlier introduced constants; on the other hand, new constants come to existence, the number of which is equal to the difference of m and the rank of the matrix of the left-hand side. To the algebraic equations obtained, we add equations

$$v(-1) = 0,\ v(-2) = 0,\ \ldots,\ v(-l+t) = 0 \tag{30}$$

in the constants and the linear constraints with zeros substituted for the unknowns $v_i(\eta)$ for $\eta < 0$ and $\eta > N$. The resulting set of expressions $v(N)x^N + v(N-1)x^{N-1} + \ldots + v(0)$ is the set of all polynomial solutions of the original system (the constants occur linearly in $v(N)$, $v(N-1)$, ..., $v(0)$).

4.2. Change of Unknowns and the Order of the Induced System

We further assume that, for the original system $L(y) = 0$, the universal denominator $U(x)$ is constructed by formulas (18) and (22). We also assume that, in accordance with (19), (20), (24), and (26), $\tilde{U}(x)$ and ρ are found. Substitution of (4) into the original system $L(y) = 0$ and subsequent transition to the system with polynomial coefficients are achieved by going from the operator L to the operator

$$L_1 = \tilde{U}(x)L\frac{1}{U(x)}. \tag{31}$$

The induced recurrence operators for L and L_1 are denoted by R and R_1, respectively. The numbers t and l are determined based on the operator R (see (12)). Similar quantities related to R_1 are denoted as l_1 and t_1. The valuation of a matrix is assumed to be the least valuation of its entries; accordingly, the degree of a matrix is the greatest degree of its entries.

Proposition 3. The following relations hold:

(i) $\operatorname{ord} R_1 - \operatorname{ord} R \le \rho$;

(ii) $\operatorname{ord} R \le \operatorname{ord} L - t$ and $\operatorname{ord} R_1 \le \operatorname{ord} L - t_1$.

Proof. It will suffice to prove for scalar operators, i.e., for $m = 1$. In the general case, L can be represented as the operator matrix (11). Similar situation holds for R.

(i) One can check that, in the transition from L to L_1, each term of the operator L is transformed such that the order of the corresponding induced operator increases by not more than ρ.

(ii) We have $\operatorname{ord} R = \operatorname{ord} l - t$ and $\operatorname{ord} R = \operatorname{ord} l_1 - t_1$, with $l, l_1 \ge \operatorname{ord} R$. □

4.3. Multiplication by the Universal Denominator

The assertion of the following theorem is a key one for justification of the new algorithm for searching rational solutions.

Theorem 1. Let $f(x)$ be a formal sum that is subordinate to the system $L(y) = 0$, $v(n)$ be a sequence of its coefficients, and

$$\operatorname{stp} v(n) = -\operatorname{ord} L + t - \rho - \deg U(x). \qquad (32)$$

Let, in the product $U(x)f(x)$ written in terms of powers of x in the differential case and in terms of powers $x^{\underline{n}}$ in the difference case, all powers of x with the exponents $-1, \ldots, -\operatorname{ord} R - \rho$ have zero coefficients. Then, the original system $L(y) = 0$ has rational solution $\frac{1}{U(x)}\lfloor U(x)f(x) \rfloor_0$.

Proof. Note that $L_1(U(x)f(x)) = \tilde{U}(x)L(f(x))$. Since the formal sum $f(x)$ is subordinate to the original system, by Proposition 1(ii) and equality (32), we have $L(f(x)) < -\operatorname{ord} L - \rho - \deg U(x)$. Taking into account (28), we obtain

$$\deg L_1(U(x)f(x)) = \deg U(x) + \rho + L(f(x))$$
$$< \deg U(x) + \rho - \operatorname{ord} L - \rho - \deg U(x) = -\operatorname{ord} L.$$

By virtue of Proposition 3(ii), we have $-\operatorname{ord} L \le -t_1 - \operatorname{ord} R_1$ and

$$\deg L_1(U(x)f(x)) < -t_1 - \operatorname{ord} R_1. \qquad (33)$$

For the sequence $g(n)$ of the coefficients of the product $U(x)f(x)$, we have $\operatorname{stp} g(n) = \operatorname{stp} v(n) + \operatorname{val}_x U(x)$. Let us show that $-\operatorname{ord} R_1 \ge \operatorname{stp} g(n)$. To do this, it will suffice to prove that $-\operatorname{ord} R_1 \ge \operatorname{stp} v(n) + \deg U(x)$. The latter inequality follows from (32) and Proposition 3. From Proposition 1(i), it follows that

$$\deg L_1(\lfloor U(x)f(x) \rfloor_{-\operatorname{ord} R_1}) < -t_1 - \operatorname{ord} R_1.$$

By virtue of Proposition 1(ii), the coefficients of the formal sum $\lfloor U(x)f(x) \rfloor_{-\operatorname{ord} R_1}$ satisfy the recurrence system with the operator R_1. However, the last $\operatorname{ord} R_1$ coefficients of this formal sum are zeros, since the powers of x with the exponents $-1, \ldots, -\operatorname{ord} R - \rho$ have zero coefficients in $U(x)f(x)$ by assumption. By virtue of Proposition 3(ii), $\operatorname{ord} R_1 \le \operatorname{ord} R + \rho$. Hence, if we supplement $\lfloor U(x)f(x) \rfloor_0$ up to an infinite series by terms with zero coefficients, the entire sequence of coefficients of this series will satisfy the system with the operator R_1. Then, it follows that $\lfloor U(x)f(x) \rfloor_{-\operatorname{ord} R_1}$ is a polynomial solution of the system $F_1(y) = 0$. □

Remark 3. Construction of the set of all sequences with stopping point (32) satisfying the system $R(v) = 0$ requires less expenditures if the trailing matrix of the system is nonsingular. In the case of its singularity, if there arises a finite set $\mathscr{C}$ of linear constraints when transiting to the t-embracing system $\bar{\bar{R}}(v) = 0$, then, in accordance with Remark 1, certain elements of this set are deleted from it, with (32) playing role of N in Remark 1.

4.4. Termination of Search of Rational Solutions

Theorem 1 and Remark 3 show that, to find the numerators corresponding to the universal denominator $U(x)$, up to a certain moment, we can follow the same scheme as in the case of searching polynomial solution, which was discussed in Section 4.1. However, system (15) will now be the induced system for the original one: substitution (4) is not used. As before, system (29) is the t-embracing system for (15) written in the way convenient for our purposes. As before, we consider (29) as a system of linear algebraic equations in $v(n + t)$. Now, these systems are solved successively for $n = e^*, e^* - 1, \ldots, -\operatorname{ord} L + t - \rho - \deg U(x)$. Again, there will appear constants the set of which will change when the matrix on the left-hand side of a current system (29) is singular (there arise linear algebraic equations for the earlier introduced constants and new constants). The formal sum with the coefficients found is multiplied by $U(x)$. The coefficients of powers of x with the exponents $-1, -2, \ldots, -\operatorname{ord} R - \rho$ in the product obtained are set equal to zero. To the linear algebraic equations obtained, linear constraints (with zeros substituted for the unknowns $v_i(\eta)$ for $\eta < 0$ and $\eta > e^* + \deg U(x)$) that were not deleted in accordance with Remark 3 are added. Having solved the system of linear algebraic equations obtained, we finally find a set of constants that can take arbitrary values. The terms in which the exponents of x lie in the range from 0 to $e^* + \deg U(x)$ transformed in accordance with the solutions of this system give us the desired numerators of the rational solutions.

4.5. The Use of Recurrence Operator Instead of Multiplication by a Polynomial

Upon multiplication of a series or a formal sum $f(x)$ by a polynomial $U(x)$, the sequence of coefficients of $f(x)$ is transformed by applying a recurrence operator $S_{U(x)}$, which can be constructed based on the polynomial $U(x)$ by means of (9) in the differential case and (10) in the difference case.

In the approach to searching the numerators suggested in Section 4.4, the direct multiplication of the formal sum can be replaced by the application of a certain scalar recurrence operator to a finite sequence of coefficients of this sum (the operator is applied independently to all components of the vector). As will be shown in Section 6.1, this circumstance makes it possible to demonstrate advantages of the new algorithm compared to the standard one in certain cases.

5. STANDARD AND NEW ALGORITHMS FOR SEARCHING RATIONAL SOLUTIONS

For the sake of comparison of the standard and new algorithms, which will be performed in Section 6.1, we number steps of the new algorithm by numbers 1, 2, 3, ..., and those of the standard algorithm, by 1°, 2°, 3°, ... If the differential and difference cases are considered separately in the description of a certain step, they are denoted by the symbols d/dx and E, respectively.

5.1. Standard Algorithm

1°. (d/dx) By means of the EG_δ algorithm (Section 2.1), construct the l-embracing system for the original one and find potential singular "points" (irreducible polynomials) (see Section 3.1). For each of these "points," find (using shift, construction of the induced recurrence system with the help of EG_σ) the least integer root of the indicial polynomial corresponding to it. If one of these polynomials has no integer roots, then STOP. Otherwise, let $p_1(x)$, ..., $p_s(x)$ be candidates for singular "points corresponding to the least integer roots e_1, ..., e_s with negative values. Get the universal denominator by setting $U(x) = p_1^{-e_1}(x)\ldots p_s^{-e_s}(x)$.

(E) Apply EG_σ (Section 2.1) to the original system, find l- and t-embracing systems. Construct polynomials $V(x)$ and $W(x)$ and, based on them, a universal denominator $U(x)$ by means of one of the known algorithms, say, the algorithm from [5] (see Section 3.2).

2°. Perform substitution (4) into the original system $L(y) = 0$ and clear denominators turning to the system $L_1(y) = 0$ of form (31) with polynomial coefficients.

3°. For the system $L_1(y) = 0$ obtained, construct the induced system $R_1(v) = 0$, apply EG_σ to it, and get the recurrence system $\tilde{R}_1(v) = 0$ with a nonsingular trailing matrix. (This will possibly result in a finite set $\mathscr{C}$ of linear constraints.) If the determinant of this matrix has no nonnegative integer roots, then STOP. Otherwise, set $\bar{e}$ equal to the greatest of these roots.

4°. Find polynomial solutions of the system $L_1(y) = 0$ using $\bar{e}$ as the upper bound for the powers of these solutions (Section 4.1). This will yield numerators corresponding to the universal denominator $U(x)$.

5.2. New Algorithm

In addition to the differences of the new algorithm from the standard one that were mentioned in the Introduction, the former differs from the latter by a strategy of checking the absence of rational solutions it uses on the earlier stages of the computation [18, 19]. This check does not increase computational complexity of the algorithm.

1. (d/dx) The same constructions as on step 1° (d/dx) of the standard algorithm plus addition calculation of ρ (see (20)).

(E) Applying EG_σ to the original system, find l- and t-embracing systems. Construct polynomials $V(x)$ and $W(x)$ and, based on them, the universal denominator $U(x)$ by means of the algorithm from [5] (see Section 3.2) and, at the same time, get the value of ρ (see (26)).

2. For the original system, construct the induced recurrence system $R(v) = 0$. By means of EG_σ, find the t-embracing system $\tilde{R}_1(v) = 0$ for it with a nonsingular trailing matrix, having additionally obtained a finite set $\mathscr{C}$ of linear constraints. Find the indicial equation. If it has no nonnegative integer roots, then STOP. Otherwise, calculate the greatest integer root e^*. If $e^* + \deg U(x) < 0$, then STOP.

4. Find numerators corresponding to the universal denominator $U(x)$ following the procedure specified in Sections 4.4 and 4.5.

6. COMBINED ALGORITHM

6.1. Comparison of Computational Costs

To calculate the universal denominator $U(x)$, the standard and new algorithms perform the same actions (steps 1° and 1).

The standard algorithm performs substitution (4) into the system $L(y) = 0$ and clears denominators in the system obtained (step 2°). Then, it constructs the induced system $R_1(v) = 0$, applies EG_σ to it, and gets system $\tilde{R}_1(v) = 0$ with a nonsingular trailing matrix (step 3°). By means of this system, on step 4°, coefficients of those terms of the formal sum are constructed in which the exponents belong to the set

$$\{\bar{e}, \bar{e} - 1, \ldots, -\mathrm{ord}\,\bar{R}_1\}. \tag{34}$$

The new algorithm constructs on step 2 the recurrence system $\tilde{R}_1(v) = 0$ and, with the help of this system, on step 3, determines for the original system those terms of the corresponding formal sum in which the exponents belong to the set

$$\{e^*, e^* - 1, \ldots, -\operatorname{ord} L + t - \rho - \deg U(x)\}. \quad (35)$$

The standard algorithm uses $\bar{e}$ as an upper bound of the degrees of the polynomials that are numerators corresponding to the universal denominator $U(x)$. The new algorithm uses $e^* + \deg U(x)$ for this purpose. The equality

$$\bar{e} = e^* + \deg U(x), \quad (36)$$

holds, in particular, if e^* and $\bar{e}$ are exact upper bounds of the exponents of x in the solutions to the systems $L(y) = 0$ and $L_1(y) = 0$, respectively, in the form of series in terms of decreasing powers of x. Either of the numbers e^* and $\bar{e}$ may be greater than the corresponding upper bound, and we cannot calculate the deviations in advance. Much depends on the way the EG_σ algorithm was applied to the induced systems (this algorithm is based on eliminations similar to the Gauss ones, and, in certain situations, different choices of equations to perform eliminations are possible). Therefore, the assumption that equality (36) holds is quite reasonable when comparing expenditures of the standard and new algorithms (actually, both $e^* \geq \bar{e}$ and $e^* \leq \bar{e}$ is possible). According to Proposition 3, the number of elements of set (34) does not exceed the number of elements of set (35). However, when $l = \operatorname{ord} L$ and $\operatorname{ord} R_1 = \operatorname{ord} R + \rho$, these sets contain the same number of elements. These two equalities are often satisfied.

According to the new algorithm, the formal sum obtained should be multiplied by $U(x)$ on step 3. Suppose that

$$\rho \geq \deg U(x). \quad (37)$$

As shown in Section 4.5, the multiplication of the formal sum by $U(x)$ is equivalent to the application of the recurrence operator of the order equal to $\deg U(x)$ to the sequence of coefficients of this formal sum. Each term of the formal sum has a coefficient in the form of a vector of m components from the field K. The recurrence operator corresponding to the multiplication by $U(x)$ is written as a scalar operator. The cost of the construction of the recurrence system $R_1(v) = 0$ is not less than the total cost of the construction of the system $R(v) = 0$ and operator $S_{U(x)}$. The multiplication of a coefficient of such an operator by a vector from K^m requires m multiplications in the field K, and the multiplication of a matrix coefficient of the operator requires m^2 such multiplications and $m(m - 1)$ additions. The use of the new algorithm for the calculation of each term of the formal sum yields saving with the asymptotic estimate $\Omega(mu)$, where $u = \deg U(x)$.

The new algorithm does without substitution (4); the order of the induced system used by the algorithm is generally smaller. Other things being equal, this is an advantage of the new algorithm. A problem may arise with the number of elements of the sequence to be calculated by means of the recurrence system. The next subsection describes combination of the standard and new algorithm.

6.2. Condition Checked

As can be seen from the previous discussion, if condition (37) is fulfilled, we may without much risk to expect that step 3 will require less computation than step 4°. Step 2 seems to require less expenditures than the successive steps 2° and 3°. The following, partly heuristic, variant of the algorithm is suggested:

(1) Step 1.

(2) If (37) holds, then steps 2 and 3.

(3) Otherwise, steps 2°, 3°, and 4°.

In the search of the numerators corresponding to the universal denominator found, this combined algorithm chooses between the standard and new algorithms. One can try to replace the condition on which the choice depends by another one, which may result in a more efficient combination of the algorithms.

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