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Procedures for Constructing Truncated Solutions of Linear Differential Equations with Infinite and Truncated Power Series in the Role of Coefficients

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Abstract—In this paper, we propose a package for symbolic construction of exponential–logarithmic solutions to linear differential equations whose coefficients are represented in incomplete form, i.e., as power series for which only a finite number of initial terms is known. The series involved in the solutions are also represented in incomplete form. For each such series, the maximum possible number of initial terms is constructed, which are uniquely determined by known terms of coefficients of a given equation. Additionally, the truncation degree of each series in a solution should not exceed a value specified by the user. It ensures the termination of the computation even when any number of the terms of the series involved in the solution can be defined by known terms of the coefficients of a given equation.

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1. INTRODUCTION

In [1], an algorithm was proposed for constructing formal exponential–logarithmic solutions to linear differential equations whose coefficients are truncated power series of the form $a(x) + O(x^{t+1})$, where $a(x)$ is a polynomial and $t \geq \deg a(x)$. In addition to the algorithm, its preliminary implementation, which was carried out to test its performance, was described. The paper [1] continued the research started in [2, 3], where solutions to equations with truncated coefficients were also constructed.

As mentioned in [3], A.D. Bruno proposed [4] a method based on the Newton polygon that, for the series involved in solutions, finds any number of terms. Equations are generally nonlinear and defined using completely specified (explicit) analytic functions of one or several variables. Obviously, this is a different problem.

Below we consider the implementation of the algorithms proposed in [1–3] for a more general case where the equation coefficients can be both truncated and completely specified series. We propose a package of procedures with a unified interface, which is developed for the users who utilize computer algebra software in their work. As compared to their preliminary versions, these procedures return more information about the solutions, which is demonstrated in Section 6.

2. PRELIMINARIES

Suppose that K is an algebraically closed field of characteristic zero. Hereinafter, for a ring of polynomials in x over K, we use the standard notation $K[x]$. A ring of formal power series in x over K is denoted by $K[[x]]$, while a field of formal Laurent series is denoted by $K((x))$.

We assume that $\theta = x\dfrac{d}{dx}$. Suppose that we have a linear ordinary differential equation

$$a_r(x)\theta^r y + a_{r-1}(x)\theta^{r-1} y + \cdots + a_0(x)y = 0, \qquad (1)$$

where y is an unknown function. Equation (1) can be written as $L(y) = 0$, where the operator L has the form

$$a_r(x)\theta^r + a_{r-1}(x)\theta^{r-1} + \cdots + a_0(x). \qquad (2)$$

The case where the equation coefficients are represented as formal power series in x over K, $a_i(x) \in K[[x]]$ $(i = 0, \ldots, r)$, has been investigated pretty well.

Definition 1. Formal *exponential–logarithmic* solutions of Eq. (1) are solutions of the form

$$e^{Q(x)} x^{\lambda} w(x^{1/q}), \qquad (3)$$

where $Q(x) \in K[x^{-1/q}]$, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$,

$$w(x) = \sum_{s=0}^{m} w_s(x) \ln^s x,$$

$w_s(x) \in K((x))$, $s = 0, \ldots, m$, and $w_m(x) \neq 0$. In this case, $x^\lambda w(x^{1/q})$ is a *regular part*, $Q(x)$ is an *exponent of irregular part*, and q is a *ramification index* of solution (3).

For $q = 1$ and $Q(x) \in K$, solution (3) is called the *formal regular* solution; otherwise, it is called the *irregular* solution. For $q = 1$, $Q(x) \in K$, $\lambda \in \mathbb{Z}$, and $w(x) \in K((x))$, formal regular solution (3) is called the *Laurent* solution. Hereinafter, we omit "formal" for brevity.

Suppose that $a_r(x) \neq 0$, i.e., the order of the equation is r. It is well known (see, for example, [5, Ch. V], [6–8]) that there are r linearly independent solutions of form (3) for Eq. (1).

In [6–9], some algorithms that find the ramification index q and exponent $Q(x)$ for r linearly independent solutions of form (3) were proposed.

Definition 2. For a nonzero Laurent series $a(x) = \sum a_j x^j \in K((x))$, its *valuation* $\mathrm{val}\,a(x)$ is defined as $\min\{j \mid a_j \neq 0\}$, with $\mathrm{val}\,0 = \infty$. The valuation of operator (2) is considered equal to the minimum valuation of all its coefficients:

$$\mathrm{val}\,L = \min_{0 \le i \le r} \{\mathrm{val}\,a_i(x)\}.$$

Suppose that $\mathrm{val}\,L = 0$. It is well known (see, for example, [10]) that, to construct the ramification index q and exponent $Q(x)$ for all r linearly independent solutions of the equation $L(y) = 0$, it is sufficient to know $r\,\mathrm{val}\,a_r(x)$ values of the initial coefficients of all $a_i(x)$, $i = 0, 1, \ldots, r$.

Definition 3. Suppose that $a(x) = \sum a_j x^j \in K((x))$ and $t \in \mathbb{Z}$. If $t \ge \mathrm{val}\,a(x)$, then the expression

$$a^{\langle t \rangle}(x) = \sum_{j=\mathrm{val}\,a(x)}^{t} a_j x^j + O(x^{t+1})$$

is called the *t-truncation* of the series $a(x)$, the number t is called the *truncation degree of the series*, and $\mathrm{val}\,a(x)$ is called the *valuation of the truncation*. If $t < \mathrm{val}\,a(x)$ (in particular, if $a(x) = 0$), then $a^{\langle t \rangle}(x) = O(x^{t+1})$ is its t-truncation, the number t is the truncation degree, and $t + 1$ is the valuation of the truncation.

The algorithms proposed in [5, Ch. IV], [11], [12, Ch. II, VIII] can be used to construct the regular part of a solution with a specified truncation degree of the series involved in $w(x)$. For this purpose, it is also sufficient to know a certain finite number of initial coefficients for all $a_i(x)$ [13, Prop. 1].

Suppose that the coefficients of Eq. (1) are algorithmically defined formal power series, i.e., for any coefficient $a_i(x) = \sum_{j=0}^{\infty} a_{ij} x^j$, there is an algorithm Ξ_i that evaluates $\Xi_i(j) = a_{ij}$ for $j \in \mathbb{Z}_{\ge 0}$. We refer to these series as *completely specified*. In this case, for any nonnegative integer k, using the corresponding algorithms from [4–9, 11, 12], we can construct its *k-truncation* for each of r linearly independent solutions of form (3):

$$e^{Q(x)} x^\lambda w^{\langle k \rangle}(x^{1/q}),$$

where

$$w^{\langle k \rangle}(x) = \sum_{s=0}^{m} w_s^{\langle k \rangle}(x) \ln^s x$$

and $\mathrm{val}\,w_0(x) = 0$. To solve this problem for equations with polynomial coefficients, the Maple computer algebra system offers the `DEtools:-formal_sol` procedure. For an equation (as well as for systems of equations) whose coefficients are completely specified series, we can use procedures from the Maple package described in [14].

3. EQUATION WITH TRUNCATED COEFFICIENTS

Suppose that some coefficients of Eq. (1) are completely specified series and some of them are truncated series. Each truncated coefficient $a_i(x)$ is written as follows:

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij} x^j + O(x^{t_i+1}), \tag{4}$$

where $t_i \ge -1$ (for $t_i = -1$, the sum in (4) is zero).

Definition 4. A *prolongation* of Eq. (1) is an equation $\mathcal{L}(y) = 0$ with an operator

$$\mathcal{L} = \sum_{i=0}^{r} b_i(x)\theta^i \in K[[x]][\theta],$$

where

- $b_i(x) = a_i(x)$ for completely specified $a_i(x)$,
- $b_i^{\langle t_i \rangle}(x) = a_i(x)$ for truncated $a_i(x)$,

$i = 0, \ldots, r$.

Definition 5. Suppose that the equation $L(y) = 0$ with completely specified and truncated coefficients has form (1). The truncated Laurent series

$$w^{\langle k \rangle}(x) = \sum_{j=j_0}^{k} w_j x^j + O(x^{k+1}), \tag{5}$$

where $j_0, k \in \mathbb{Z}$, $k \ge j_0$, and $w_{j_0} \neq 0$, is called the *truncated Laurent solution* of the equation $L(y) = 0$ because any equation $\mathcal{L}(y) = 0$ that is a prolongation of $L(y) = 0$ has a solution $\tilde{w}(x) \in K((x))$ the truncation of which is (5):

$$\tilde{w}^{\langle k\rangle}(x) = w^{\langle k\rangle}(x).$$

The expression

$$x^{\lambda}\sum_{s=0}^{m} w_s^{\langle k_s\rangle}(x)\ln^s x, \tag{6}$$

in which

$$w_s^{\langle k_s\rangle}(x) = \sum_{j=j_s}^{k_s} w_{s,j}x^j + O(x^{k_s+1}), \tag{7}$$

$j_s, k_s \in \mathbb{Z}$, $k_s \geq j_s$, $s = 0,\ldots,m$, and $w_{m,j_m} \neq 0$, is called the *truncated regular solution* of the equation $L(y) = 0$ because any equation $\mathcal{L}(y) = 0$ that is a prolongation of $L(y) = 0$ has a regular solution $x^{\lambda}\tilde{w}(x)$ the truncation of which is (6):

$$\tilde{w}(x) = \sum_{s=0}^{m} \tilde{w}_s(x)\ln^s x, \tag{8}$$

where $\tilde{w}_s^{\langle k_s\rangle}(x) = w_s^{\langle k_s\rangle}(x)$, $s = 0, 1, \ldots, m$.

Suppose that $Q(x) \in K[x^{-1/q}]$, $q \in \mathbb{Z}_{>0}$, $\lambda \in K$, truncated series $\tilde{w}_s(x)$ have form (7), $s = 0,1,\ldots,m$, and $w_{m,j_m} \neq 0$. For the equation $L(y) = 0$, the expression

$$e^{Q(x)}x^{\lambda}\sum_{s=0}^{m} w_s^{\langle k_s\rangle}(x^{1/q})\ln^s x \tag{9}$$

is called the *solution with a truncated regular part* if any equation $\mathcal{L}(y) = 0$ that is a prolongation of $L(y) = 0$ has a solution $e^{Q(x)}x^{\lambda}\tilde{w}(x^{1/q})$ the truncation of which is (9), i.e., $\tilde{w}(x)$ has form (8) and $\tilde{w}_s^{\langle k_s\rangle}(x) = w_s^{\langle k_s\rangle}(x)$ ($s = 0, 1,\ldots,m$).

If all coefficients of the equation are completely specified series, then, as mentioned above, there are algorithms and their computer algebra implementations that construct solutions for any truncation.

In the case where all equation coefficients are truncated series, truncated solutions involve series the truncation degree of which cannot be an arbitrary number. In [1–3, 15], it was shown that truncated solutions with the maximum truncation degree of the series involved in these solutions can be constructed. The papers cited above describe algorithms for solving this problem and their Maple implementations.

For a "mixed" case, it was shown in [16, Prop. 1] that the problem of constructing truncated solutions with the maximum number of terms of series is algorithmically undecidable; moreover, a truncated solution of any degree does not always exist.

4. INDICIAL POLYNOMIAL

Suppose that the operator L of form (2) has truncated and completely specified coefficients and

$$\min_{0\leq i\leq r}\{\mathrm{val}\,a_i(x)\} \leq \min_{0\leq i\leq r} t_i, \tag{10}$$

where, for each completely specified coefficient $a_i(x)$, we assume that $t_i = \infty$. Let us denote $\gamma = \min_{0\leq i\leq r}\{\mathrm{val}\,a_i(x)\}$. The expression

$$I_L(n) = \sum_{i=0}^{r} a_{i,\gamma}n^i$$

is called the *indicial* polynomial of the equation $L(y) = 0$. Obviously, $I_L(n) \in K[n]$. It was shown in [2] that if $I_L(n)$ has integer roots, then, for $L(y) = 0$, there exist truncated Laurent solutions, in particular, solutions of form (5), where $j_0 = k$ and j_0 is the maximum integer root of $I_L(n)$. Otherwise, no prolongation of the equation $L(y) = 0$ has Laurent solutions.

It was shown in [3] that if $\deg I_L(n) > 0$, then, for $L(y) = 0$, there exist truncated regular solutions, in particular, solutions of form (6), where $m = 0$, $j_0 = 0$, $k_0 = 0$, and λ is the root of $I_L(n)$ such that $I_L(\lambda + n) \neq 0$ for all $n \in \mathbb{Z}_{>0}$. In the absence of roots, i.e., if $I_L(n) \in K\setminus\{0\}$, no prolongation of the equation $L(y) = 0$ has regular solutions.

If condition (10) does not hold, then there are no indicial polynomials for $L(y) = 0$. In [1, Prop. 2], it was shown that, in this case, there are no truncated Laurent and regular solutions for $L(y) = 0$. However, if $\mathrm{val}\,a_i(x) = \gamma$ holds for certain i ($0 < i \leq r$), then each prolongation has at least i linearly independent regular solutions.

5. EQUATION THRESHOLD

In [16], the concept of an equation threshold was introduced in connection with the construction of truncated Laurent solutions. We extend this concept to the cases of truncated regular solutions and solutions with truncated regular parts.

Definition 6. Suppose that the equation $L(y) = 0$ with completely specified and truncated coefficients has form (1). We consider the set N of all integers k such that $L(y) = 0$ has a solution with truncated regular part (9) for $j_m = 0$, with the maximum truncation degree of the series involved in the solution being $k = \max\{k_0, k_1,\ldots,k_m\}$. Suppose that the set N is not empty and has a maximum element. This element is called the *threshold* of the original equation for exponential-logarithmic solutions. If the set N includes arbitrarily large integers, then the equation threshold is assumed to be ∞. If this set is empty and condition (10) holds, then the threshold is assumed to be $-\infty$. For an empty set N, when (10) does not hold, we assume that the threshold is not determined.

The condition $j_m = 0$ in this definition ensures the uniqueness of the equation threshold.

The concepts of equation thresholds for regular solutions and Laurent solutions are introduced in a similar way.

Suppose that, for $L(y) = 0$, there is an indicial polynomial $I_L(n)$. If $I_L(n)$ has integer roots, then the equation threshold for Laurent solutions is determined and its value (integer or ∞) is not less than the maximum integer root of $I_L(n)$. Under the same condition, the equation threshold for regular solutions and solutions with truncated regular parts is a nonnegative integer or ∞.

Proposition 1. There is no algorithm that, given an arbitrary equation $L(y) = 0$ of form (1) with completely specified and truncated coefficients, can determine whether its threshold is finite or infinite for Laurent solutions, regular solutions, and exponential-logarithmic solutions.

Proof. In [16, Prop. 1], this proposition was proved for the threshold of Laurent solutions. The fact that the Laurent solution is a particular case of regular and exponential-logarithmic solutions implies this general statement.

Proposition 2. Suppose that $L(y) = 0$ is an equation of form (1) and $d \in \mathbb{Z}$. It can be algorithmically checked if the threshold is defined for this equation and its value is less than d; if it is less, then the threshold h of this equation can be found. In addition, all solutions with truncated regular parts (in particular, truncated Laurent and truncated regular solutions) the truncation degrees of which do not exceed h can be constructed.

Proof. For Laurent truncated solutions, this statement was proved in [16, Prop. 1] for the case where condition (10) holds. If (10) does not hold, then, for $L(y) = 0$, there are both prolongations that have Laurent solutions and prolongations that do not have them, i.e., the threshold is not determined.

For regular truncated solutions under condition (10), the proof is similar to the case of Laurent truncated solutions. If (10) does not hold, then there are no truncated regular solutions (see [1, Prop. 2]); moreover, if some prolongation of the equation $L(y) = 0$ does not have regular solutions, then there is a prolongation for which regular solutions exist.

In [1], an algorithm for constructing all exponents $Q(x)$ and corresponding ramification indices q for which solutions with truncated regular parts exist was described. For each constructed $Q(x)$ and q, we obtain a new equation $L_1(z) = 0$ by substituting

$$y(x) = e^{Q(t^q)}z(t), \quad x = t^q,$$

into the equation $L(y) = 0$. Each coefficient of the new equation can be either a completely specified or truncated series. For $L_1(z) = 0$, we solve the problem of finding the threshold for truncated regular solutions at certain d. If there is no indicial polynomial for the new equation (i.e., the threshold is not determined), then we assume that $h(Q,q) = FAIL$. If d exceeds the threshold, then the threshold is determined and $h(Q,q)$ is set equal to it; otherwise, $h(Q,q) = d$.

For the original equation, we also find the threshold for its regular solutions. We denote the result by $h(0, 1)$. Thus, $h(0, 1)$ is equal to $FAIL$, d, or threshold for regular solutions (integer or $-\infty$).

If, for certain $Q(x)$ and q (including $Q(x) = 0$ and $q = 1$), we obtain $h(Q,q) = d$, then the threshold for solutions with truncated regular parts is not found. If, for all $Q(x)$ and q (including $Q(x) = 0$ and $q = 1$), we obtain $h(Q,q) = FAIL$, then the threshold of the original equation for solutions with truncated regular parts is not determined. Otherwise, the threshold is the maximum value of those $h(Q,q)$ that are not equal to $FAIL$.

6. SOLUTION WITH A TRUNCATED EXPONENT OF THE IRREGULAR PART

Definition 7. Suppose that the equation $L(y) = 0$ with completely specified and truncated coefficients has form (1). Assume that

$$Q(x) = \frac{\varepsilon_1}{x^{\kappa_1}} + \dots + \frac{\varepsilon_\sigma}{x^{\kappa_\sigma}},$$

where $\kappa_1 > \ldots > \kappa_\sigma > 0$, $\kappa_i \in \mathbb{Q}$, and $\varepsilon_i \in K\backslash\{0\}$, $i = 1,\ldots,\sigma$. The expression

$$e^{Q(x)}Y(x) \tag{11}$$

is called the *solution with a truncated exponent* for the equation $L(y) = 0$ because any equation $\mathscr{L}(y) = 0$ that is a prolongation of $L(y) = 0$ has a irregular solution

$$e^{Q(x)+\tilde{Q}(x)}x^{\lambda}\sum_{s=0}^{m} w_s(x^{1/q}) \tag{12}$$

such that

$$\tilde{Q}(x) = \frac{\varepsilon_{\sigma+1}}{x^{\kappa_{\sigma+1}}} + \dots + \frac{\varepsilon_{\sigma+\varsigma}}{x^{\kappa_{\sigma+\varsigma}}}, \tag{13}$$

where $\kappa_\sigma > \kappa_{\sigma+1} > \ldots > \kappa_{\sigma+\varsigma} > 0$, $\kappa_i \in \mathbb{Q}$, and $\varepsilon_i \in K\backslash\{0\}$, $i = \sigma+1,\ldots,\sigma+\varsigma$.

In [1], an algorithm (implemented in Maple) that constructs solutions of form (11) with the maximum number of terms in $Q(x)$ for equations of form (1) was described. In addition, the following information invariant to the prolongations of the equation can be obtained:

• for all prolongations of a truncated equation, there is a solution (12) with $\varsigma = 0$, i.e., its exponent $Q(x)$ is completely constructed, the ramification index q is also invariant to the prolongations of this equation, but its regular part does not contain an invariant exponent λ; this truncated solution is written as follows:

$$e^{Q(x)} y_{reg}(x^{1/q});$$

• for all prolongations of a truncated equation, there is a solution (12) with $\varsigma \geq 1$, and the value $\kappa_{\sigma+1}$ is invariant to the prolongations of this equation, while $\varepsilon_{\sigma+1}$ is not invariant to the prolongations of this equation; this truncated solution is written as follows:

$$e^{Q(x)} y_{irr(\kappa_{\sigma+1})}(x);$$

• for all prolongations of a truncated equation, there is a solution (12) with $\varsigma \geq 1$, and the value $\kappa_{\sigma+1}$ is not invariant to the prolongations of this equation; this truncated solution is written as follows:

$$e^{Q(x)} y_{irr}(x).$$

If there is a prolongation of the equation $L(y) = 0$ that has solution (12) with the exponent $Q(x)$ ($\varsigma = 0$), as well as there is a prolongation of this equation that does not have this solution ($\varsigma \geq 1$), then this truncated solution uses notation (11).

7. PROCEDURES FOR CONSTRUCTION OF SOLUTIONS

The algorithms for constructing truncated solutions are implemented in Maple 2020 [17] as procedures of the `TruncatedSeries` package.[1] The `LaurentSolution` (construction of Laurent solutions) and `RegularSolution` (construction of regular solutions) procedures for equations all coefficients of which are truncated series were described in [15]. For this case, in [1], a preliminary implementation of the `FormalSolution` procedure (construction of exponential-logarithmic solutions of form (11)) was described. An extension of the `LaurentSolution` procedure to the "mixed" case was proposed in [16].

The main arguments of the procedures are

(1) a homogeneous linear ordinary differential equation with truncated or completely specified series as coefficients;

(2) an unknown function, e.g., `y(x)`.

The equation must be written using the operator $\theta = x\dfrac{d}{dx}$ or operator $\dfrac{d}{dx}$. The application of θ^i to $y(x)$ is written in the `TruncatedSeries` package as `theta(y(x),x,i)`. The ith derivative of the function $y(x)$ uses the standard Maple notation: `diff(y(x),x$i)`.

The truncated coefficients of the equation are represented as expressions `a(x)+O(x^(t+1))`, where `a(x)` is a polynomial of a degree not higher than `t` over $\mathbb{A}$ (i.e., over the field of algebraic numbers).

A completely specified series $\sum_{i=0}^{\infty} a_i x^i$ can be written either as a polynomial (i.e., $a_i = 0$, beginning with certain i) or as a sum of a polynomial (the initial terms of the series) and infinite power sum. For its representation, the Maple construction

```
Sum(f(i)*x^i, i = i0..infinity)
```

is used, where `f(i)` is an expression for computing the coefficient of the series at x^i, `i` is a name (summation index), and `i0` is a non-negative integer.

For a correct operation of the procedures from our package, the irrational algebraic numbers involved in the value of `f(i)` for all integer values of i, which are greater than or equal to `i0`, must be represented using the Maple construction

```
RootOf(p(_Z), index = k),
```

where `p(_Z)` is an irreducible polynomial the `k`th root of which is an algebraic number. For instance, $\mathrm{RootOf}(\_Z^2 - 2, index = 2)$ represents the number $-\sqrt{2}$. The same applies to the irrational numbers in the polynomials that define the coefficients of the equation.

The following optional arguments (options) can also be set:

• `'top'=d`, where `d` is an integer for which it is required to determine whether it exceeds the equation threshold or not (by default, if all coefficients of the equation are truncated series, then `d` is equal to the threshold; otherwise, d is assumed equal to the lower boundary of the equation threshold, with its value being specified individually for each procedure);

• `'threshold'='h'`, where `h` is the name of a variable to which either the threshold value is assigned (if it can be determined) or the Maple constant `FAIL` if the threshold is not determined or exceeds `d`.

7.1. Laurent Solutions

The `LaurentSolution` procedure outputs

• a list of truncated Laurent solutions with different valuations: the list contains all truncated Laurent solutions invariant to the prolongations of the input equation, with the number of computed coefficients in each truncation being the maximum possible number taking into account `d`;

• an empty list if there are no Laurent solutions for all prolongations of the equation;

• the Maple constant `FAIL` otherwise.

[1] The package and Maple sessions with usage examples are available at http://www.ccas.ru/ca/TruncatedSeries.

Each truncated solution in the list returned by `LaurentSolution` is represented in form (5), where w_j (computed coefficients of a truncated Laurent solution) are linear combinations (over $\mathbb{A}$) of arbitrary constants $\_c_1, \_c_2, \ldots$. All elements of the list have different valuations: an arbitrary constant (which is a coefficient at the power equal to the valuation of a certain list element) cannot be zero in the corresponding truncated solution.

By h^* we denote the maximum integer root of the indicial polynomial or $-\infty$ if there are no integer roots. As mentioned in Section 4, the equation threshold for Laurent solutions is not less than h^*. If the `'top'` option is off or it is used to set `d` lower than h^*, then `d` is assumed equal to h^*.

The truncation degrees for different elements of the list may be different because the maximum number of coefficients invariant to the prolongations of the equation may differ for different valuations. The truncation degree for each element does not exceed `d` and equation threshold for Laurent solutions; however, it is not less than h^*.

7.2. Regular Solutions

The `RegularSolution` procedure outputs

• a list of truncated regular solutions with different sets of valuations involved in solutions for truncated Laurent series: the list contains all truncated regular solutions invariant to the prolongations of the input equation, with the number of computed coefficients in each Laurent series being the maximum possible number taking into account `d`;

• an empty list if there are no truncated regular solutions for all prolongations of the equation;

• the Maple constant `FAIL` otherwise.

Each truncated solution y_i from the list returned by `RegularSolution` is a finite sum $y_{i1} + \ldots + y_{ig}$ of expressions of the form

$$y_{ij} = x^{\lambda_j} \sum_{s=0}^{m_{i,j}} w_{i,j,s}(x) \ln^s x,$$

where $\lambda_1, \ldots, \lambda_g$ are all different roots of $I_L(n)$ such that $I_L(\lambda_j + n) \neq 0$ for all $n \in \mathbb{Z}_{>0}$, while $w_{i,j,s}(x)$ are truncated Laurent series ($j = 1, \ldots, g$ and $s = 0, \ldots, m_{i,j}$). In each pair of different truncated solutions y_{i_1} and y_{i_2} from the list, for each λ_j, it is either $m_{i_1,j} \neq m_{i_2,j}$ or, for certain s $(0 \leq s \leq m_{i_1,j})$,

$$\mathrm{val} w_{i_1,j,s}(x) \neq \mathrm{val} w_{i_2,j,s}(x).$$

The coefficients of all truncated series are linear combinations (over $\mathbb{A}$) of arbitrary constants $\_c_1, \_c_2, \ldots$. An arbitrary constant, which is a coefficient at the power equal to the valuation of a certain series involved in a list element, cannot be zero in the corresponding truncated solution.

If the indicial polynomial has roots, then we assume that $h^* = 0$; otherwise, $h^* = -\infty$. As mentioned in Section 4, the equation threshold for regular solutions is not less than h^*. If the `'top'` option is off or it is used to set `d` lower than h^*, then `d` is assumed equal to h^*.

The truncation degrees for the series in different elements of the list may be different because the maximum number of coefficients invariant to the prolongations of the equation may differ for different valuations. The truncation degree for each series in the solution does not exceed `d` and equation threshold for regular solutions; however, it is not less than h^*.

7.3. Formal Solutions

The `FormalSolution` procedure outputs

• a list of truncated solutions;

• the Maple constant `FAIL` if there are no truncated solutions.

Each truncated solution from the list returned by `FormalSolution` is a finite sum of expressions of the form $e^Q Y$, where $Q \in \mathbb{A}[x^{-1/q}]$ $(q \in \mathbb{Z}_{>0})$; in this case, it is possible that $Q = 0$ and $q = 1$. For Y, there are following variants:

• the truncated regular part of a solution includes arbitrary constants $\_c_1, \_c_2, \ldots$;

• $y_{reg,i}(x^{1/q})$ (where $q \in \mathbb{Z}_{>0}$), $y_{irr(\kappa),i}(x)$ (where $\kappa \in \mathbb{Q}$), $y_{irr,i}(x)$, or $y_i(x)$, where $i \in \mathbb{Z}_{>0}$, for solutions with truncated exponents of the types listed in Section 6.

If the equation has a solution with the exponent Q, ramification index q, and truncated regular part Y, then, to construct this regular part, the `RegularSolution` procedure is applied to an equation obtained by substituting $y(x) = e^{Q(t^q)} z(t)$, $x = t^q$, into the original equation. For each new equation, `d` is determined depending on whether all its coefficients are truncated or not. All elements of the list returned by `RegularSolution` are involved in the result of `FormalSolution`.

7.4. Examples

Let us consider some examples of using the procedures described above.

The equation

$$(-1 + x + x^2 + O(x^3))\theta^2 y - (2 + O(x^3))\theta y + \left(\sum_{i=4}^{\infty} (i^2 + 2i + 1 - (i+1)^2) x^i \right) y = 0$$

is written in Maple as follows:

```
>eq1 := (-1+x+x^2+O(x^3))
  *theta(y(x), x, 2)-
  (2+O(x^3))*theta(y(x), x, 1)+
  Sum((i^2+2*i+1-(i+1)^2)*x^i,
  i=4..infinity)*y(x):
```

This equation has both truncated coefficients and coefficient specified algorithmically using infinite sum. Let us apply the `LaurenSolution` procedure to `eq1` while specifying the corresponding options:

```
> LaurentSolution(eq1, y(x),
          'top'=7, 'threshold'='h');
```

$$\left[\frac{\_c_1}{x^2}-\frac{4\_c_1}{x}+\_c_2+O(x), \_c_2+O(x^8)\right]$$

The result indicates that, for all prolongations of the equation, there are Laurent solutions with valuations −2 and 0. The truncation degrees differ for different valuations. Note that the `RegularSolution` procedure, once applied to this equation, returns the same result, i.e., in that case, all truncated regular solutions are truncated Laurent solutions.

Let us check the threshold value

```
>h;
```

FAIL

This means that the value set using the `'top'` option does not exceed the threshold. A similar result is obtained for any value of this option because, in that case, the equation threshold is ∞, and any finite value of this option does not exceed the threshold. In addition, it can be seen that, for valuation −2, the maximum possible degree of truncation is already reached because it is lower than the value specified using the `'top'` option and it remains constant when the value of this option is increased.

The equation

$$(x^7+x^5)\frac{d^4y}{dx^4}+\left(18x^4+\sum_{k=7}^{\infty}\frac{(-1)^k x^k}{k!}\right)\frac{d^3y}{dx^3}$$
$$+(96x^3+O(x^6))\frac{d^2y}{dx^2}$$
$$+(168x^2+O(x^5))\frac{dy}{dx}+72xy=0$$

is written in Maple as follows:

```
> eq2 := (x^7+x^5)*diff(y(x), x$4) +
         (18*x^4 + Sum((-1)^k*x^k/k!,
         k=7..infinity))*diff(y(x), x$3) +
         (96*x^3 + O(x^6))*diff(y(x), x$2)+
         (168*x^2 + O(x^5))*diff(y(x), x) +
         72*x*y(x):
```

This equation includes both truncated coefficients and coefficient defined algorithmically (both as a polynomial and as an infinite sum). Note that this equation is written using standard Maple differentiation. It should also be noted that, if the minimum valuation of the equation coefficients is $\gamma > 0$ (for `eq2`, $\gamma = 1$) and $t_i \geq \gamma$ for each truncated coefficient $a_i(x)$, then the package procedures divide the equation by x^γ and replace each t_i with $t_i - \gamma$. Let us apply the `RegularSolution` procedure to `eq2` while specifying the corresponding options:

```
> RegularSolution(eq2, y(x),
          'top'=3, 'threshold'='h');
```

$$\left[\frac{1}{x^2}\left(-\frac{1}{420}\frac{\_c_2}{x^2}+\frac{\_c_3}{x}+\_c_4+O(x)\right.\right.$$
$$\left.\left.+\ln(x)\left(\frac{\_c_1}{x}+\_c_2-30x\_c_1+O(x^2)\right)\right),\right.$$

$$\frac{1}{x^2}\left(-\frac{1}{420}\frac{\_c_2}{x^2}+\frac{\_c_3}{x}+\_c_4+O(x)\right.$$
$$\left.+\ln(x)\left(\_c_2-\frac{5}{3}x^2\_c_2+O(x^3)\right)\right),$$

$$\frac{1}{x^2}\left(\frac{\_c_3}{x}+\_c_4+x\left(-30\_c_3+\frac{197}{2}\_c_1\right)+O(x^2)\right.$$
$$\left.+\ln(x)\left(\frac{\_c_1}{x}-30x\_c_1+O(x^2)\right)\right),$$

$$\frac{1}{x^2}\left(\frac{\_c_3}{x}+\_c_4-30x\_c_3+O(x^2)\right),$$

$$\frac{1}{x^2}\left(\_c_4+\frac{197}{2}x\_c_1+O(x^2)\right.$$
$$\left.+\ln(x)\left(\frac{\_c_1}{x}-30x\_c_1+O(x^2)\right)\right),$$

$$\frac{1}{x^2}\left(\_c_4-\frac{5}{3}x^2\_c_4+O(x^3)\right),$$

$$\frac{1}{x^2}\left(\frac{197}{2}x\_c_1+O(x^2)\right.$$
$$\left.\left.+\ln(x)\left(\frac{\_c_1}{x}-30x\_c_1+O(x^2)\right)\right)\right]$$

The result indicates that there are regular solutions for all prolongations of the equation. Five elements of the list correspond to the solutions with $\ln x$ and different valuations of the Laurent series involved in the solutions. The other two elements of the list do not contain $\ln x$ and are Laurent solutions with different valuations (these solutions can be found using `LaurentSolution`). It can be seen that, for different valuations, the truncated series involved in the solutions may have different degrees of truncation.

Let us check the threshold value

```
> h;
```

$$2$$

As it should be, the found threshold does not exceed the value specified using the `'top'` option. It can be seen that it is reached in the solutions that include series with zero valuations.

The equation

$$(x^4 + O(x^7))\theta^3 y + (3x + O(x))\theta^2 y + \left(1 + \sum_{i=1}^{\infty} i x^i\right)\theta y = 0$$

is written in Maple as follows:

```
> eq3 := (x^4 + O(x^7))*theta(y(x), x, 3) +
        (3*x + O(x^5))*theta(y(x), x, 2) +
       (1 + Sum(i*x^i, i = 1 .. infinity))*
           theta(y(x), x, 1):
```

Let us apply the FormalSolution procedure to `eq3` without the `'top'` option:

```
>FormalSolution(eq3, y(x), 'threshold'='h');
```

$$\left[\_c_1 + O(x) + e^{\frac{1}{3x}} x^{2/3}\left(\_c_2 + \frac{35}{27}\_c_2 x + \frac{8947}{1458}\_c_2 x^2 + O(x^3)\right) + e^{\frac{1}{x^3} - \frac{1}{3x}} y_{reg}(x)\right]$$

The result indicates that, for all prolongations of the equation, there is a one-dimensional space of Laurent solutions of the form $(\_c_1 + O(x))$, where $\_c_1 \in \mathbb{A}$. The check of the equation threshold value

```
> h;
```

FAIL

shows that the threshold is not reached for Laurent solutions with their truncation degree being set by default: `d = 0`.

In addition, `eq3` has a solution with a truncated regular part for which a larger number of terms are computed because substituting $y(x) = e^{1/(3x)} z(x)$ into `eq3` results in an equation all coefficients of which are truncated series. For the new equation, by default, d is set equal to the threshold, which is a finite number.

Another truncated solution does not include series. All prolongations of `eq3` have solutions of form (12), where

$$Q(x) = \frac{1}{x^3} - \frac{1}{3x},$$

$\tilde{Q}(x) = 0$, $q = 1$, but the values of λ differ for different prolongations, which is why the regular part in the result is denoted by $y_{reg}(x)$.

More examples of using the procedures from the `TruncatedSeries` package can be found in [1–3, 14–16, 18].

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