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Procedures for Searching Laurent and Regular Solutions of Linear Differential Equations with the Coefficients in the Form of Truncated Power Series

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Abstract—Linear ordinary differential equations whose coefficients are infinite (formal) power series given in a truncated form are considered. Computer algebra procedures (implemented in Maple) for constructing solutions of two forms are suggested. The procedures find the greatest number of series terms occurring in the solutions that can be found for the given truncated series—coefficients.

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1. INTRODUCTION

In [1, 2], algorithms were suggested for searching for the so-called Laurent and regular solutions (their definitions can be found in Section 2.2) of linear ordinary differential equations with infinite formal power series in the role of the coefficients. The question of how to represent infinite series is very important in computer algebra. In the given case, the series are specified in the truncated form, which means that the complete information about the equation is not available. Based on this incomplete information, the algorithms give maximum possible number of series terms occurring in the solutions. Results of experiments with preliminary (test) versions of procedures implementing these algorithms were reported in [1, 2]. By now, the procedures have been improved, and the interface and data representation have been unified. The improved procedures are discussed in this paper. These procedures can be accessed at

http://www.ccas.ru/ca/TruncatedSeries.

2. PRELIMINARIES

2.1. Equations, Operators, and Truncated Series

Let K be an algebraically closed number field. For the ring of polynomials in x over K, we use the standard notation $K[x]$. The ring of formal power series of x over K is denoted as $K[[x]]$, and the field of formal Laurent series, as $K((x))$. For a nonzero element $a(x) = \sum a_i x^i$ from $K((x))$, its *valuation* $\mathrm{val}_x a(x)$ is defined by the equality $\mathrm{val}_x a(x) = \min\{i \,|\, a_i \neq 0\}$, with $\mathrm{val}_x 0 = \infty$. Let $t \in \mathbb{Z} \cup \{-\infty\}$, the *t-truncation* $a^{\langle t \rangle}(x)$ is obtained by discarding all terms in $a(x)$ with the degree greater than t; if $t = -\infty$, then $a^{\langle t \rangle}(x) = 0$. The number t is referred to as the *truncation degree*.

In this paper, differential equations are written by means of the operation $\theta = x\frac{d}{dx}$ rather than the conventional differentiation operation $\frac{d}{dx}$ (the transition from one form of notation to another is easily performed). We consider equations of the form

$$a_r(x)\theta^r y + a_{r-1}(x)\theta^{r-1} y + \cdots + a_0(x) y = 0, \qquad (1)$$

where y is an unknown function of x. As for the *coefficients of the equation* $a_0(x), a_1(x), \ldots, a_r(x)$, we assume that $a_i(x) \in K[[x]]$, $i = 0, 1, \ldots, r$, with the *leading* coefficient $a_r(x)$ being not equal to zero. We also assume that valuation of at least one of the coefficients $a_0(x)$, $a_1(x), \ldots, a_r(x)$ is equal to zero.

Equation (1) can be written as $\mathcal{L}(y) = 0$, where operator $\mathcal{L}$ has the form

$$\sum_{i=0}^{r} a_i(x)\theta^i \in K[[x]][\theta], \qquad (2)$$

with r being the *order* of the operator $\mathcal{L}$.

In what follows, it is assumed that L is a differential operator with polynomial coefficients,

$$\sum_{i=0}^{r} a_i(x)\theta^i \in K[x][\theta], \quad (3)$$

and that there given nonnegative integers $t_0, t_1, \dots, t_r$ such that $t_i \geq \deg a_i(x)$, $i = 0, 1, \dots, r$. It is also assumed that $a_r(x) \neq 0$ and that valuation of at least one of the polynomials $a_0(x), a_1(x), \dots, a_r(x)$ is equal to zero:

$$\exists i : \mathrm{val} a_i(x) = 0. \quad (4)$$

Definition 1. The *prolongation* of operator L is any operator of the form

$$\tilde{L} = \sum_{i=0}^{r} \tilde{a}_i(x)\theta^i \in K[[x]][\theta],$$

such that $\tilde{a}_i(x) - a_i(x) = O(x^{t_i+1})$; i.e., $\mathrm{val}(\tilde{a}_i(x) - a_i(x)) > t_i$, $i = 0, 1, \dots, r$.

To the *truncated differential equation*

$$\sum_{i=0}^{r} (a_i(x) + O(x^{t_r+1}))\theta^i y = 0, \quad (5)$$

$t_i \geq \deg a_i(x)$, $i = 0, 1, \dots, r$, we make to correspond operator (3) and a set of numbers $t_0, t_1, \dots, t_r$. The prolongation of operator (3) in this case is also referred to as the *prolongation of equation* (5).

If L (or $\mathscr{L}$) is a differential operator, by a *solution of operator* L (or $\mathscr{L}$), we mean a solution of the equation $L(y) = 0$ ($\mathscr{L}(y) = 0$).

If L is a truncated variant of operator $\mathscr{L}$, then L and $L(y) = 0$ are *truncations* of the operator $\mathscr{L}$ and equation $\mathscr{L}(y) = 0$, respectively.

2.2. Laurent and Regular Solutions of Equations

A solution of a differential equation that is a formal Laurent series, is called a *Laurent solution*.

A *regular* solution has the form

$$y(x) = x^{\lambda} w(x), \quad (6)$$

where $\lambda \in K$, $w(x) \in K((x))[\ln x]$. Every solution of this form can be written as

$$x^{\lambda} \sum_{s=0}^{k} g_{k-s}(x) \frac{\ln^s x}{s!}, \quad (7)$$

where $k \in \mathbb{Z}_{\geq 0}$ and $g_s(x) \in K((x))$, $s = 0, 1, \dots, k$. In this case, we say that x^{λ} is a *power factor* of the solution $y(x)$. The set

$$x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_\rho} \quad (8)$$

is referred to as a *complete* set of power factors of regular solutions of equation $\mathscr{L}(y) = 0$ if,

– among the exponents of the elements in set (8), there are no those that differ from one another by an integer;

– each element x^{λ_i} of set (8) is a power factor for some nonzero regular solution of equation $\mathscr{L}(y) = 0$;

– for each nonzero regular solution of equation $\mathscr{L}(y) = 0$, set (8) contains a power factor for this solution.

Remark 1. According to the definition accepted in computer algebra (see, e.g., [3]), any linear combination over K of solutions of form (6) is also called a regular solution.

3. ALGORITHMS FOR CONSTRUCTING SOLUTIONS

3.1. Laurent Solutions

Let a differential equation $\mathscr{L}(y) = 0$ have nonzero Laurent solutions and $y(x) = \sum_{n=v}^{\infty} c_n x^n$ be a general Laurent solution, with the coefficients c_n containing arbitrary constants. The algorithm suggested in [4, Sect.6] constructs truncation of the general Laurent solution of any given degree m for $\mathscr{L}(y) = 0$: $y^{\langle m \rangle}(x) = \sum_{n=v}^{m} c_n x^n$.

It was shown in [1] that, for equation (5) with truncated coefficients, one can find truncations of maximally high degrees for Laurent solutions that are invariant with respect to all possible prolongations of the equation. An algorithm was suggested that receives an operator $L \in K[x][\theta]$ and nonnegative integers $t_0, t_1, \dots, t_r$, which have the same meaning as those in (3), at its input and constructs a finite set of expressions W of the form

$$y^{\langle m \rangle}(x, C_1, \dots, C_r) + O(x^{m+1}), \quad (9)$$

where $y(x, C_1, \dots, C_r) \in K[C_1, \dots, C_r]((x))$, that have the following properties (by solutions, we mean solutions belonging to $K((x))$):

• if (9) is an element of the set W, then, for any prolongation $\tilde{L}$ of the operator L, there is a solution $\tilde{y}(x)$ for which there exist $\tilde{C}_1, \dots, \tilde{C}_r \in K$ such that

$$\tilde{y}(x) = y^{\langle m \rangle}(x, \tilde{C}_1, \dots, \tilde{C}_r) + O(x^{m+1}),$$
$$\mathrm{val}\tilde{y}(x) = \mathrm{val} y(x, C_1, \dots, C_r);$$

• if $\tilde{y}(x)$ is a solution for some prolongation $\tilde{L}$ of the operator L and there exists an element (9) of the set W such that

$$\mathrm{val}\tilde{y}(x) = \mathrm{val} y(x, C_1, \dots, C_r), \quad (10)$$

then there exist $\tilde{C}_1, \dots, \tilde{C}_r \in K$ such that

$$\tilde{y}(x) = y^{\langle m \rangle}(x, \tilde{C}_1, \dots, \tilde{C}_r) + O(x^{m+1});$$

• values m are the greatest possible values related to each element of the set W in the specified way.

The set W includes all expressions of form (9) that possess these properties.

3.2. Regular Solutions

Let a differential equation $\mathcal{L}(y) = 0$ have nonzero solutions of form (7). The algorithms for constructing such solutions are discussed in [5–13]. By means of these algorithms, one can construct a truncation of a general regular solution for any given truncation degree m. That is, for all series occurring in the solution, the coefficients up to the degree m are calculated, which can contain arbitrary constants.

For equation (5) with truncated coefficients, the algorithm suggested in [2] constructs regular solutions with the maximally large truncations of the series occurring in them such that these solutions are invariant with respect to various possible prolongations of the equation. The input of the algorithm is the operator $L \in K[x][\theta]$ and nonnegative integers $t_0, t_1, \dots, t_r$. As a result of the algorithm application, the complete set (8) of the power factors of regular solutions becomes known, which is the same for all possible prolongations of the operator L. For each admissible factor x^{λ}, a finite set $W(\lambda)$ of expressions of the form

$$x^{\lambda} \sum_{s=0}^{k} \frac{\ln^s x}{s!} (g_{k-s}^{\langle m_s \rangle}(x, C_1, \dots, C_r) + O(x^{m_s+1})) \tag{11}$$

is constructed, where $g_s(x, C_1, \dots, C_r) \in K[C_1, \dots, C_r]((x))$ for $s = 0, 1, \dots, k$, that possess the following properties:

• if (11) is an element of the set $W(\lambda)$, then, for any prolongation $\tilde{L}$ of the operator L, there exists its solution $\tilde{y}(x) = x^{\lambda} \sum_{s=0}^{k} \tilde{g}_{k-s}(x) \frac{\ln^s x}{s!}$ for which there exist $\tilde{C}_1, \dots, \tilde{C}_r \in K$ such that

$$\tilde{g}_s(x) = g_s^{\langle m_s \rangle}(x, \tilde{C}_1, \dots, \tilde{C}_r) + O(x^{m_s+1}),$$
$$\operatorname{val} \tilde{g}_s(x) = \operatorname{val} g_s(x, C_1, \dots, C_r)$$

for $s = 0, 1, \dots, k$;

• if $\tilde{y}(x) = x^{\lambda} \sum_{s=0}^{k} \tilde{g}_{k-s}(x) \frac{\ln^s x}{s!}$ is a solution of some prolongation $\tilde{L}$ of the operator L and there exists an element (11) of the set $W(\lambda)$ such that

$$\operatorname{val} \tilde{g}_s(x) = \operatorname{val} g_s(x, C_1, \dots, C_r), \tag{12}$$

for $s = 0, 1, \dots, k$, then there exist $\tilde{C}_1, \dots, \tilde{C}_r \in K$ such that

$$\tilde{g}_s(x) = g_s^{\langle m_s \rangle}(x, \tilde{C}_1, \dots, \tilde{C}_r) + O(x^{m_s+1}),$$

for $s = 0, 1, \dots, k$;

• for each element of the set $W(\lambda)$, $m_0, m_1, \dots, m_k$ are the greatest possible values that are related to L in the specified way.

The set $W(\lambda)$ includes all expressions of form (11) that possess these properties; i.e., $W(\lambda)$ contains the complete list of formulas of form (11) that are invariant with respect to the prolongations of the operator L.

4. LITERALS

Let an operator L with polynomial coefficients of form (3) and a set of numbers $t_0, t_1, \dots, t_r$ be given, and let the coefficients of L have the form

$$a_i(x) = \sum_{j=0}^{t_i} a_{ij} x^j, \quad i = 0, 1, \dots, r$$

(if $t_i > d_i = \deg a_i(x)$, then $a_{ij} = 0$ for $j = d_i + 1, d_i + 2, \dots, t_i$). We say that coefficients $a_{i,j}$ are *not specified* if $j > t_i$ for $i = 0, 1, \dots, r$.

In addition to constructing truncations of (Laurent and regular) solutions that are invariant with respect to all prolongations of the equation $L(y) = 0$, the algorithm allows one to estimate the effect of unspecified coefficients on the subsequent terms of the series occurring in the solutions. For unspecified coefficients, the algorithm uses symbolic notation; they are further referred to as *literals*.

When considering a prolongation $\tilde{L}$ of an operator L, it may occur that $\tilde{L}$ has a Laurent solution $\tilde{y}(x)$ such that W does not contain expression (9) for which the equality of valuations (10) holds. The algorithm can determine what conditions on the unspecified coefficients are to be fulfilled in order that such an expression appears.

When considering a prolongation $\tilde{L}$ of an operator L, it may occur that $\tilde{L}$ has solution $\tilde{y}(x) = x^{\lambda} \sum_{s=0}^{k} \tilde{g}_{k-s}(x) \frac{\ln^s x}{s!}$ such that $W(\lambda)$ does not contain expression (11) for which the equality of valuations (12) holds. The algorithm can determine what conditions on the unspecified coefficients are to be fulfilled in order that such an expression appears.

For a truncated equation, the complete set of power factors is the same for all prolongations of the equation if free terms of all coefficients are known and at least one of them is not equal to zero. However, the maximal values of k in (7) can be different for different prolongations. The algorithm can determine what conditions on the unspecified coefficients are to be fulfilled in order that maximal values of k become invariant with respect to possible prolongations of the equation.

5. PROCEDURES FOR CONSTRUCTING SOLUTIONS

The algorithms for constructing solutions under consideration are implemented in the computer algebra system `Maple` ([14]) in the form of procedures of the `TruncatedSeries`[1] package. The package provides the user with two basic procedures:

• `LaurentSolution` – construction of Laurent solutions;

• `RegularSolution` – construction of regular solutions.

Preliminary implementations of these procedures were presented in [1, 2]. They are partially based on the implementations of the algorithms in the EG package [13].

5.1. Arguments and Result of Procedure Operation

Both procedures have the same arguments. The main arguments are as follows:

• The first argument is a differential equation of form (5), where $t_i \geq \deg a_i(x)$, $i = 0, 1, \ldots, r$. The application of θ^k to an unknown function $y(x)$ is written as `theta(y(x),x,k)`. Instead of the operator θ, one can also use an ordinary differentiation (operator $D = \frac{d}{dx}$); the application of the operator D^k to an unknown function $y(x)$ is specified in the standard (for Maple) form as `diff(y(x),x$k)`. The truncated coefficients of the equation are specified in the form of expressions $a_i(x) + O(x^{t_i+1})$, where $a_i(x)$ is a polynomial of degree t_i or less over the field of algebraic numbers, i.e., in the form similar to mathematical notation. Irrational algebraic numbers are represented in Maple as expressions `RootOf(p(_Z), index = k)`, where `p(_Z)` is an irreducible polynomial whose kth root is just the given algebraic number. For instance, $\mathrm{RootOf}(\_Z^2 - 2, index = 2) = -\sqrt{2}$.

• The second argument is an unknown function, e.g., `y(x)`.

The result of operation of the `LaurentSolution` procedure is the list of truncated Laurent solutions from the set W described in Section 3.1. Each element of the list has the form

$$c_{v_j}x^{v_j} + c_{v_j+1}x^{v_j+1} + \ldots + c_{m_j}x^{m_j} + O(x^{m_j+1}), \tag{13}$$

where v_j is a valuation for which there exists a Laurent solution for any prolongation of the given equation, m_j has the same meaning as that in Section 3.1, c_n are calculated coefficients of the Laurent solution that are linear combinations of arbitrary constants of the form $\_c_0, \_c_1, \ldots$

The result of operation of the `RegularSolution` procedure is the list of truncated regular solutions invariant with respect to prolongations of the coefficients of the given equation. The truncations contain arbitrary constants of the form $\_c_0, \_c_1, \ldots$

The following optional parameters can also be specified:

• `'output'='literal'` provides obtaining the result in the form of one truncation with literals rather than as a list of invariant truncations. All truncated series in the output have the form

$$c_v x^v + c_{v+1}x^{v+1} + \ldots + c_m x^m + O(x^{m+1}), \tag{14}$$

where $v = \min v_j$, $m = 1 + \max m_j$, and coefficients c_n contain literals; the literals are represented in the form $U_{[i,j]}$, which corresponds to an unspecified coefficient of x^j in the coefficient of θ^i in the original equation;

• `'degree'=n`, where `n` is an integer, provides obtaining truncations of the given degree. In this case, the coefficients of the truncations will possibly be expressed in terms of literals. The degrees of the constructed truncations may be greater than the given `n`: the number of the calculated coefficients should be at least sufficient for determining all possible valuations of the Laurent series occurring in the solution.

5.2. Examples of Construction of Laurent Solutions

1. Each of the following equations

$$\sin x\theta y(x) - x\cos x\, y(x) = 0, \tag{15}$$

$$(e^x - 1)\theta y(x) - xe^x y(x) = 0 \tag{16}$$

can be represented in the form

$$(x + O(x^2))\theta y(x) + (-x + O(x^2))y(x) = 0. \tag{17}$$

Let us apply the procedure to (17).

```
> eq1 := (x+O(x^2))*(theta(y(x),x,1))+
         (-x+O(x^2))*y(x);
```

$$eq1 := (x + O(x^2))\theta(y(x), x, 1) + (-x + O(x^2))y(x)$$

```
> LaurentSolution(eq1, y(x));
```

$$[x\_c_1 + O(x^2)].$$

Thus, there is only one invariant truncation of the solution with the valuation $v = 1$ and the truncation degree $m = 1$.

2. Let us apply the procedure to (17) once more having specified the desired truncation degree 2 by means of the option `'degree'=2`:

```
> LaurentSolution(eq1, y(x), 'degree'=2);
```

$$[x\_c_1 + x^2(-\_c_1U_{[0,2]} - \_c_1U_{[1,2]}) + O(x^3)].$$

As can be seen, the coefficient of the term of degree 2 depends on the literals; i.e., different prolongations of equation $eq1$ may have different coefficients of this term so that the invariant solution found earlier is the greatest possible one.

[1] The package and the `Maple` session with examples of use of the procedures described are available at the address http://www.ccas.ru/ca/TruncatedSeries

3. Let us add some terms corresponding to the coefficients of (15) to the coefficients of equation $eq1$. We will obtain a truncation of the solution up to the degree x^2, which corresponds to the expansion of function $\sin x$, which is a solution of (15), into the power series:

```
> eq2 := (x+O(x^3))*(theta(y(x),x,1))+
        (-x+x^3/2+O(x^4))*y(x);
```

$$eq2 := (x + O(x^3))\theta(y(x), x, 1) + \left(-x + \frac{x^3}{2} + O(x^4)\right)y(x)$$

```
> LaurentSolution(eq2, y(x));
```

$$[x\_c_1 + O(x^3)].$$

We, again, have $v = 1$, but $m = 2$. It is easy to check that the solution truncation found is a prolongation of the invariant truncation of the solution to equation $eq1$. It can be seen that the substitution of $U_{[0,2]} = 0$ and $U_{[1,2]} = 0$, which correspond to the coefficients added, into the above-found solution of equation $eq1$ truncated up to the degree $m = 2$ yields a truncated solution of equation $eq2$.

4. Now, let us add several terms corresponding to the coefficients of (16) to the coefficients of equation $eq1$. We will obtain a solution truncated up to degree x^2, which corresponds to the expansion of function $e^x - 1$, which is a solution of (16), into a power series.

```
> eq3 :=
(x+x^2/2+O(x^3))*theta(y(x),x,1)+
(-x-x^2-x^3/2+O(x^4))*y(x);
```

$$eq3 := \left(x + \frac{x^3}{2} + O(x^3)\right)\theta(y(x), x, 1) + \left(-x - x^2 - \frac{x^3}{2} + O(x^4)\right)y(x)$$

```
> LaurentSolution(eq3, y(x));
```

$$\left[x\_c_1 + \frac{x^2\_c_1}{2} + O(x^3)\right].$$

Hence, $v = 1$ and $m = 2$. Like in the previous case, it is easy to check that the truncated solution found is a prolongation of the truncated solution of equation $eq1$. It can be seen that, if we substitute $U_{[0,2]} = -1$ and $U_{[1,2]} = \frac{1}{2}$, which correspond to the coefficients added, into the solution of equation $eq1$ truncated up to degree 2, we will obtain a truncation of equation $eq3$.

Thus, different prolongations $eq2$ and $eq3$ of equation $eq1$ yield different invariant truncations of solutions. Note that, as expected, no new solutions appeared. The invariant truncations of solutions to equations $eq2$ and $eq3$ are prolongations of the invariant truncation of the solution to $eq1$ and correspond to its prolongation up to degree 2, which depends on literals, upon substitution of the appropriate coefficients of equations $eq2$ and $eq3$ for the literals.

5. For each of the equations $eq1$, $eq2$, and $eq3$, there exists only one value of the valuation for which Laurent solutions exist for any prolongation of the equation. Let us apply the procedure to the following equation:

```
> eq4 :=
(-1+x+O(x^2))*theta(y(x),x,2)+
(-2+O(x^2))*theta(y(x),x,1)+
(x+O(x^2))*y(x);
```

$$eq4 := (-1 + x + O(x^2))\theta(y(x), x, 2) + (-2 + O(x^2))\theta(y(x), x, 1) + (x + O(x^2))y(x)$$

```
> LaurentSolution(eq4, y(x));
```

$$\left[\_c_1 + \frac{x\_c_1}{3} + O(x^2)\right].$$

The answer obtained means that there exists only one invariant truncation of the solution with the valuation $v = 0$ and the truncation degree $m = 1$.

6. Let us apply the procedure to $eq4$ once more having specified the truncation degree of the solution equal to 3 by means of the option `'degree'=3`:

```
> LaurentSolution(eq4, y(x), 'degree'=3);
```

$$\left[\_c_1 + \frac{x\_c_1}{3} + x^2\left(\frac{1}{12}\_c_1 + \frac{1}{8}\_c_1U_{[0,2]}\right) + x^3\left(\frac{1}{45}\_c_1U_{[2,2]} + \frac{1}{36}\_c_1 + \frac{23}{360}\_c_1U_{[0,2]} + \frac{1}{45}\_c_1U_{[1,2]} + \frac{1}{15}U_{[0,3]}\_c_1\right) + O(x^4)\right].$$

7. Let us add several coefficients to those of equation $eq4$ and apply the procedure.

```
> eq5 :=
(-1+x+x^2+O(x^3))*theta(y(x),x,2)+
(-2+O(x^3))*theta(y(x),x,1)+
(x+6*x^2+O(x^4))*y(x);
```

$$eq5 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2) + (-2 + O(x^3))\theta(y(x), x, 1) + (x + 6x^2 + O(x^4))y(x)$$

```
> LaurentSolution(eq5, y(x));
```

$$\left[\frac{\_c_1}{x^2} - \frac{5\_c_1}{x} + \_c_2 + O(x), \_c_1 + \frac{x\_c_1}{3} + \frac{5x^2\_c_1}{6} + \frac{13x^3\_c_1}{30} + O(x^4)\right].$$

The answer obtained means that there exist two invariant solution truncations: one with the valuation $v_1 = 0$ and the truncation degree $m_1 = 3$, which is a prolongation of the earlier found truncation of the solution of equation *eq*4, and the other with the valuation $v_2 = -2$ and the truncation degree $m_2 = 0$, which is a new one.

8. Let us add several coefficients to those of equation *eq*4 in a different way and apply the procedure.

```
> eq6 := (-1+x+x^2+O(x^3))*theta(y(x),x,2)+
  (-2+x^2+O(x^3))*theta(y(x),x,1)+
  (x+6*x^2+O(x^4))*y(x);
```

$$eq6 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2)$$
$$+ (-2 + x^2 + O(x^3))\theta(y(x), x, 1)$$
$$+ (x + 6x^2 + O(x^4))y(x)$$

```
> LaurentSolution(eq6, y(x));
```

$$\left[\_c_1 + \frac{x\_c_1}{3} + \frac{5x^2\_c_1}{6} + \frac{41x^3\_c_1}{90} + O(x^4)\right].$$

The answer obtained means that there again exists only one invariant truncation of the solution with the valuation $v = 0$ and the truncation degree $m = 3$, which is a prolongation of the earlier found truncation of the solution of equation *eq*4.

It can be seen that different prolongations *eq*5 and *eq*6 of equation *eq*4 yield different invariant truncations. Note that there appeared a new invariant solution of equation *eq*5 with a different valuation. Note also that the second invariant truncation of equation *eq*5 and the invariant truncation of solution to *eq*6 are prolongations of the invariant truncation of solution to *eq*4 corresponding to its prolongation up to degree 3 in literals, with the substitution of the appropriate coefficients from equations *eq*5 and *eq*6 for the literals.

9. Let us verify whether it makes sense to consider the case of different $t_0, t_1, \ldots, t_r$ in (5) or we can confine ourselves to the case where these numbers are equal to each other. In other words, we are going to check whether the replacement of each t_i by $t = \min_{i=0}^{r} t_i$ in (5) can reduce accuracy of the result of the algorithm operation.

For the next equation, we obtain five initial terms of the solution:

```
> eq7 := (1+O(x))*(theta(y(x),x,1))+
         (x^4+O(x^5))*y(x);
```

$$eq7 := (1 + O(x))\,\theta(y(x), x, 1) + (x^4 + O(x^5))y(x)$$

```
> LaurentSolution(eq7, y(x));
```

$$\left[\_c_1 - \frac{c_1 x^4}{4} + O(x^5)\right].$$

If we take $t_0 = t_1 = 0$, then we obtain only one initial term of the solution:

```
> eq8 := (1+O(x))*(theta(y(x),x,1))+
         O(x)*y(x);
```

$$eq8 := (1 + O(x))\,\theta(y(x), x, 1) + O(x)y(x)$$

```
> LaurentSolution(eq8, y(x));
```

$$[\_c_1 + O(x)].$$

This shows that the replacement of each t_i with $t = \min_{i=0}^{r} t_i$ reduces accuracy of the algorithm operation. Hence, the efforts related to rejection of a priori assumption on equality of all t_i are not spent for nothing.

10. There are equations that have no nontrivial Laurent solutions for any prolongations:

```
> eq9 := (2+O(x))*(theta(y(x),x,1))+
         (1+O(x))*y(x);
```

$$eq9 := (2 + O(x))\,\theta(y(x), x, 1) + (1 + O(x))\,y(x)$$

```
> LaurentSolution(eq9, y(x));
```

$$[]$$

The answer—an empty list—means that there are no solutions for all prolongations of equation *eq*9.

11. The procedure can be applied to equations specified in terms of the differentiation operator $\frac{d}{dx}$.

```
> eq10 := (-x+x^2+x^3+O(x^4))*
          (diff(y(x),x,x))+
          (-3+x+O(x^2))*(diff(y(x),x))+
          O(x^3)*y(x);
```

$$eq10 := (-x + x^2 + x^3 + O(x^4))\left(\frac{d^2}{dx^2}y(x)\right)$$
$$+(-3 + x + O(x^2))\left(\frac{d}{dx}y(x)\right) + O(x^3)y(x)$$

```
> LaurentSolution(eq10, y(x));
```

$$[\_c_1 + O(x^4)].$$

If condition (4) is not fulfilled, invariant truncations of the Laurent solutions do not exist. In this case, the procedure returns `FAIL`. For the following equation given in terms of $\frac{d}{dx}$, the procedure constructs an equivalent equation in terms of θ and determines that condition (4) is not fulfilled; hence, invariant truncated solutions do not exist:

```
> eq11 := (x^2+O(x^3))*diff(y(x),x,x)+
          O(x)*diff(y(x),x)+(1+O(x))*y(x);
```

$$eq11 := (x^2 + O(x^3))\left(\frac{d^2}{dx^2}y(x)\right) + O(x)$$

$$\left(\frac{d}{dx}y(x)\right) + (1 + O(x))\,y(x)$$

```
> LaurentSolution(eq11, y(x)).
```

FAIL

5.3. Examples of Constructing Regular Solutions

1. Apply the procedure for searching regular solutions:

```
> eq12 :=
(-1+x+x^2+O(x^3))*theta(y(x),x,2)+
(-2+O(x^2))*theta(y(x),x,1)+
(O(x^4))*y(x);
```

$$eq12 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2)$$

$$+(-2 + O(x^2))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> RegularSolution(eq12, y(x));
```

$$[\_c_1 + O(x^4)].$$

2. Add one additional coefficient $U_{[1,2]} = 1$ to equation $eq12$ and apply the procedure once more:

```
> eq13 :=
(-1+x+x^2+O(x^3))*theta(y(x),x,2)+
(-2+x^2+O(x^3))*theta(y(x),x,1)+
O(x^4)*y(x);
```

$$eq13 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2)$$

$$+(-2 + x^2 + O(x^3))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> RegularSolution(eq13, y(x));
```

$$\left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^4)),\right.$$

$$\left.\_c_2 + O(x^4)\right].$$

It can be seen that, in this case, there appears the second truncation of the regular solution that contains logarithm.

3. Apply the procedure to the same equation with the option of the representation of the result in terms of literals:

```
> RegularSolution(eq13, y(x),
        'output'='literal');
```

$$\ln(x)\left(\_c_1 + \frac{1}{24}x^4\_c_1U_{[0,4]} + O(x^5)\right) - \frac{\_c_1}{x^2}$$

$$+\frac{4\_c_1}{x} + \_c_2 + x\left(\frac{2}{3}\_c_1U_{[1,3]} - \frac{4}{3}\_c_1U_{[2,3]}\right)$$

$$+ x^2\left(\frac{1}{4}\_c_1U_{[1,4]} - \frac{5}{12}\_c_1U_{[1,3]} + \frac{1}{3}\_c_1U_{[2,3]}\right.$$

$$\left.-\frac{1}{2}\_c_1U_{[2,4]} - \frac{1}{8}\_c_1U_{[0,4]} + \frac{1}{8}\_c_1\right)$$

$$+ x^3\left(\frac{2}{15}\_c_1U_{[2,4]} - \frac{1}{5}\_c_1U_{[1,4]} + \frac{2}{45}\_c_1U_{[1,3]}\right.$$

$$-\frac{4}{45}\_c_1U_{[2,3]} + \frac{2}{15}\_c_1U_{[1,5]} + \frac{7}{30}\_c_1U_{[0,4]}$$

$$\left.+\frac{1}{30}\_c_1 - \frac{4}{15}\_c_1U_{[2,5]} - \frac{1}{15}\_c_1U_{[0,5]}\right)$$

$$+ x^4\left(\frac{1}{24}U_{[0,4]}\_c_2 - \frac{3}{40}\_c_1U_{[2,4]} + \frac{7}{240}\_c_1U_{[1,4]}\right.$$

$$-\frac{7}{80}\_c_1U_{[1,3]} + \frac{1}{20}\_c_1U_{[2,3]} - \frac{7}{60}\_c_1U_{[1,5]}$$

$$+\frac{7}{180}\_c_1U_{[0,4]} + \frac{7}{160}\_c_1 + \frac{1}{15}\_c_1U_{[2,5]} + \frac{17}{120}\_c_1U_{[0,5]}$$

$$+\frac{1}{36}\_c_1U_{[1,3]}^2 - \frac{1}{36}U_{[1,3]}\_c_1U_{[2,3]} - \frac{1}{18}\_c_1U_{[2,3]}^2$$

$$\left.-\frac{1}{24}\_c_1U_{[0,6]} + \frac{1}{12}\_c_1U_{[1,6]} - \frac{1}{6}\_c_1U_{[2,6]}\right) + O(x^5).$$

4. Apply the procedure to the same equation with the option of specifying the truncation degree:

```
> RegularSolution(eq13, y(x), 'degree'=2);
```

$$\left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + x\left(\frac{2}{3}\_c_1U_{[1,3]} - \frac{4}{3}\_c_1U_{[2,3]}\right)\right.$$

$$+ x^2\left(\frac{1}{4}\_c_1U_{[1,4]} - \frac{5}{12}\_c_1U_{[1,3]}\right.$$

$$\left.+\frac{1}{3}\_c_1U_{[2,3]} - \frac{1}{8}\_c_1U_{[0,4]} - \frac{1}{2}\_c_1U_{[2,4]} + \frac{1}{8}\_c_1\right)$$

$$\left.+ O(x^3) + \ln(x)(\_c_1 + O(x^3)), \_c_2 + O(x^3)\right].$$

The answer shows that, in order to obtain the 2-truncation as a prolongation of the invariant truncation, it is required to specify $U_{[0,\,4]}$, $U_{[1,\,3]}$, $U_{[1,\,4]}$, $U_{[2,\,3]}$, $U_{[2,4]}$, i.e., the equation coefficients of x^4, $x^3\theta$, $x^4\theta$, $x^3\theta^2$, $x^4\theta^2$.

5. Apply the procedure to the same equation with both the option of representation of the result in terms of literals and the option specifying the truncation degree (the options can be used together):

```
> RegularSolution(eq13, y(x),
      'output'='literal', 'degree'=2);
```

$$\ln(x)(\_c_1 + O(x^3)) - \frac{\_c_1}{x^2}$$

$$+\frac{4\_c_1}{x} + \_c_2 + x\left(\frac{2}{3}\_c_1U_{[1,3]} - \frac{4}{3}\_c_1U_{[2,3]}\right)$$

$$+ x^2\left(\frac{1}{4}\_c_1U_{[1,4]} - \frac{5}{12}\_c_1U_{[1,3]} + \frac{1}{3}\_c_1U_{[2,3]}\right.$$

$$\left.-\frac{1}{2}\_c_1U_{[2,4]} - \frac{1}{8}\_c_1U_{[0,4]} + \frac{1}{8}\_c_1\right) + O(x^3).$$

6. Add one coefficient to equation $eq12$ in a different way: $U_{[1,2]} = 0$.

```
> eq14 :=
(-1+x+x^2+O(x^3))*theta(y(x),x,2)+
(-2+O(x^3))*theta(y(x),x,1)+
O(x^4)*y(x);
```

$$eq14 := (-1 + x + x^2 + O(x^3))\theta(y(x), x, 2)$$
$$+ (-2 + O(x^3))\theta(y(x), x, 1) + O(x^4)y(x)$$

```
> RegularSolution(eq14, y(x));
```

$$\left[\frac{\_c_1}{x^2} - \frac{4\_c_1}{x} + \_c_2 + O(x), \_c_2 + O(x^4)\right].$$

We see that, in this case, a new invariant regular solution—Laurent solution with valuation $v_2 = -2$ appears.

7. Apply the procedure to the equation:

```
> eq15 := (1+x^2+O(x^3))*theta(y(x),x,3)+
          (4-x+(1/2)*x^2+O(x^3))*
          theta(y(x),x,2)+
          (4-2*x+x^2+O(x^3))*
          theta(y(x),x,1)+
          O(x^3)*y(x);
```

$$eq15 := (1 + x^2 + O(x^3))\theta(y(x), x, 3)$$
$$+\left(4 - x + \frac{1}{2}x^2 + O(x^3)\right)\theta(y(x), x, 2)$$
$$+ (4 - 2x + x^2 + O(x^3))\theta(y(x), x, 1) + O(x^3)y(x)$$

```
> RegularSolution(eq15, y(x));
```

$$\left[\frac{\frac{21\_c_1}{16} + \frac{\_c_2}{2}}{x^2} + \frac{\_c_1}{x} + \_c_3 + O(x)\right.$$
$$+\ln(x)\left(\frac{1}{2}\frac{\_c_1}{x^2} + \_c_2 + O(x)\right.$$
$$+\ln(x)^2\left(\frac{1}{2}\_c_1 + O(x^3)\right),$$
$$\frac{1}{2}\frac{\_c_2}{x^2} + \_c_3 + O(x)$$
$$\left.+\ln(x)(\_c_2 + O(x^3)), \_c_3 + O(x^3)\right].$$

In this case, there are three different invariant truncations of regular solutions with different truncation degrees (the degree of logarithm is $k = 2$).

8. Equation

```
> eq16 := (-1+x+O(x^3))*theta(y(x),x,2)+
          (-1-x-(3/2)*x^2+O(x^3))*
          theta(y(x),x,1)+(3/4+(1/4)*x+
          (3/4)*x^2+O(x^3))*y(x);
```

$$eq16 := (-1 + x + O(x^3))\theta(y(x), x, 2)$$
$$+\left(-1 - x - \frac{3}{2}x^2 + O(x^3)\right)\theta(y(x), x, 1)$$
$$+\left(\frac{3}{4} + \frac{1}{4}x + \frac{3}{4}x^2 + O(x^3)\right)y(x)$$

```
> RegularSolution(eq16, y(x));
```

$$\left[\sqrt{x}\left(-\frac{2\_c_1}{x^2} + \frac{8\_c_1}{x} + \_c_2 + O(x)\right.\right.$$
$$\left.\left.+\ln(x)(\_c_1 + O(x^3))\right), \sqrt{x}(\_c_2 + O(x^3))\right].$$

In this case, we obtain the regular solution with a noninteger λ in the factor x^λ.

9. One more equation:

```
> eq17 := (1+O(x^2))*theta(y(x),x,3)+
          (1+2*x+O(x^2))*theta(y(x),x,2)+
          (2+x+O(x^2))*theta(y(x),x,1)+
          (2-x+O(x^2))*y(x);
```

$$eq17 := (1 + O(x^2))\theta(y(x), x, 3)$$
$$+ (1 + 2x + O(x^2))\theta(y(x), x, 2)$$
$$+ (2 + x + O(x^2))\theta(y(x), x, 1)$$
$$+ (2 - x + O(x^2))y(x)$$

```
> RegularSolution(eq17, y(x));
```

$$\left[\frac{\_c_1}{x} + O(x) + x^{\mathrm{RootOf}(\_Z^2+2, index=1)}\left(\_c_2\right.\right.$$
$$-\frac{1}{54}x(20 + 23\mathrm{RootOf}(\_Z^2 + 2, index = 1))\_c_2$$
$$\left.+O(x^2)\right) + x^{\mathrm{RootOf}(\_Z^2+2, index=2)}\left(\_c_3\right.$$
$$\left.\left.-\frac{1}{54}x(20 + 23\mathrm{RootOf}(\_Z^2 + 2, index = 2))\_c_3 + O(x^2)\right)\right].$$

In this case, all prolongations of the equation have three nonequivalent power factors with the exponents -1, $\sqrt{-2}$, $-\sqrt{-2}$, where $\sqrt{-2}$, $-\sqrt{-2}$ are represented by constructs $\mathrm{RootOf}(\_Z^2 + 2,\ index = 1)$ and $\mathrm{RootOf}(\_Z^2 + 2,\ index = 2)$.

10. The equation in terms of the differentiation operator $\frac{d}{dx}$:

```
> eq18 := (-x+x^2+x^3+O(x^4))*
          (diff(y(x), x, x))+
          (-3+x+2*x^2+O(x^3))*
          (diff(y(x), x))+
```

```
O(x^3)*y(x)
```

$$eq18 := (-x + x^2 + x^3 + O(x^4))\left(\frac{d^2}{dx^2}y(x)\right) + (-3 + x + 2x^2 + O(x^3))\left(\frac{d}{dx}y(x)\right) + O(x^3)y(x)$$

```
> RegularSolution(eq18, y(x));
```

$$\left[-\frac{\_c_1}{x^2} + \frac{4\_c_1}{x} + \_c_2 + O(x) + \ln(x)(\_c_1 + O(x^4)) \; \_c_2 + O(x^4)\right].$$

Having transformed the equation to that written in terms of θ, the procedure gets equation $eq13$. Therefore, the computation results coincide.

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